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Robust Design Optimization in Computational Mechanics

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The motivation of this paper is to propose a methodology for analyzing the robust design optimization problem of complex dynamical systems excited by deterministic loads but taking into account model uncertainties and data uncertainties with an adapted nonparametric probabilistic approach, whereas only data uncertainties are generally considered in the literature by using a parametric probabilistic approach. The possible designs are represented by a numerical finite element model whose design parameters are deterministic and belong to an admissible set. The optimization problem is formulated for the stochastic system as the minimization of a cost function associated with the random response of the stochastic system including the variability of the stochastic system induced by uncertainties and the bias corresponding to the distance of the mean random response to a given target. The gradient and the Hessian of the cost function with respect to the design parameters are explicitly calculated. The complete theory and a numerical application are presented. [DOI: 10.1115/1.2775493]

Keywords: robust design, model uncertainties, structural dynamics

1 Introduction

It is known that the accuracy of the dynamical responses predictions in computational mechanics is mainly related to the level of model and data uncertainties. For this reason, probabilistic models of random uncertainties are implemented in the numerical simulation models in order to improve the predictions in computational dynamics. In this probabilistic context, it is necessary to distinguish the mean dynamical system corresponding to a nominal mechanical model from the stochastic dynamical system corresponding to a more realistic model of the real dynamical system manufactured from the design. These past decades, design optimization has become a major challenge in the industrial technologies such as aerospace, aeronautics, automotive, and nuclear industries. In the early works, the design optimization problem was studied from a deterministic point of view, neglecting the presence of uncertainties in the numerical model used for the optimization problem. In this case, the deterministic design optimization problem yields an optimal design whose responses satisfy for the best given target (performance objective), see, for instance, Ref. [1]. This optimal design can then be used in order to construct the stochastic dynamical system for analyzing a posteriori the effects of uncertainties on the dynamical response.

It should be noted that data uncertainties can clearly be taken into account by the parametric probabilistic approach whose stochastic finite element methods (see, for instance, Refs. [2–5]) and other theoretical and numerical methods (see, for instance, Refs. [6,7]) constitute very efficient tools in computational mechanics. Nevertheless, such a parametric probabilistic approach does not allow model uncertainties to be taken into account [8]. More recently, a nonparametric probabilistic approach [9–11] has been proposed to take into account model uncertainties.

These past decades, various researches have been carried out in order to include the effects of uncertainties in the design optimization problem [12] called the robust design. In the context of mechanical engineering, the robust design leads to solve a nonlinear constrained optimization problem with numerical models, which are little sensitive to uncertainties in the vicinity of the

design point (see, for instance, Refs. [13,14] for the early works and Refs. [15–20] for the most recent advances concerning this research area). The main difficulty of such robust design optimization problems concerns the probabilistic model of uncertainties. It should be noted that the relevance of the probabilistic model used is an important factor for the robust design optimization problem. The use of an erroneous probabilistic model yields an erroneous optimal design. However, there is no reason for that the responses of the real dynamical system, which is manufactured from this erroneous optimal design, correspond to the performance objective. Until now, most of the published papers concerning robust design have been carried out in the context of static performances using parametric probabilistic models for modeling data uncertainties in the mechanical system. The robust design in the dynamic field is relatively recent [18,19]. It should be noted that the dynamical systems can be very sensitive to model uncertainties and not only sensitive to data uncertainties. Moreover, this sensitivity generally increases with the complexity of the dynamical system. In any case, all the works published until now concern robust design with respect to data uncertainties and not with respect to model uncertainties.

The motivation of this paper is to propose a methodology for analyzing the robust design optimization problem of complex dynamical systems excited by deterministic loads but taking into account model uncertainties and data uncertainties with the nonparametric probabilistic approach introduced above, whereas only data uncertainties are generally considered in the literature by using a parametric probabilistic approach. Several experimental validations [21–24] and numerical validations [11,25–27] have proved the capability of the nonparametric probabilistic approach to take into account model uncertainties and data uncertainties. A complete theory concerning the robust design optimization problem of complex dynamical systems excited by deterministic loads is presented in this paper. The nonparametric probabilistic model is used for modeling nonhomogeneous model and data uncertainties. The mean reduced matrix model of the design is constructed by using substructuring techniques (see, for instance, Refs. [28–34]). The design parameters are deterministic and belong to an admissible set. The cost function used to formulate the robust design optimization problem is then defined as a function of the design parameter. Concerning the formulation of the cost function, the performance objective includes not only the target but also the robustness with respect to model uncertainties and data uncertain-

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ties. More precisely, the cost function is defined as a linear combination of the normalized variance of the stochastic system related to the variability of the system induced by uncertainties and of the bias corresponding to the distance between the mean random response of the stochastic system and a given target. The norm and the distance used are related to the square integrable norm over a given frequency band of analysis. The gradient and the Hessian of the cost function with respect to the design parameter are algebraically calculated that is useful for improving the performance of the optimization algorithm.

In Sec. 2, the set of mean reduced matrix models related to the set of all the feasible designs is constructed by using the Benfield and Hruda substructuring technique [30]. In Sec. 3, the design optimization problem is formulated assuming no uncertainties in the model of the dynamical system. Section 4 is devoted to the implementation of the nonparametric probabilistic approach for model and data uncertainties. Section 5 concerns the construction of the cost function describing the target and the sensitivity of the dynamical system to uncertainties (performance objectives) in order to formulate the robust design optimization problem with respect to model and data uncertainties. Finally, a numerical application is presented in Sec. 6.

2 Mean Dynamical System

The dynamical system is made up of a given master system (a structure) coupled with a subsystem (a substructure), which has to be designed. The dynamical system is assumed to be linear and slightly damped. The equations are discretized by the finite element method and are written in the frequency domain. The frequency band of analysis is denoted by B . It is assumed that the master system has no rigid body displacements and that the subsystem is free with r rigid body modes. Let $\mathbf{p}=(p_1, \dots, p_s)$ be the $\mathbb{C}^s$ vector of the design parameters (geometry, elasticity properties, boundary conditions, etc.). The vector of the design parameters belongs to an admissible set $\mathcal{P}$ defined by the set of constraints prescribed by the design. For $\mathbf{p}$ fixed in $\mathcal{P}$ and for ω fixed in B , the equation of the mean dynamical system is written as

$$\{[\underline{A}^1(\omega)] + [\underline{A}^2(\mathbf{p}, \omega)]\} \underline{\mathbf{u}}(\mathbf{p}, \omega) = \mathbf{f}(\mathbf{p}, \omega) \quad (1)$$

in which $\underline{\mathbf{u}}(\mathbf{p}, \omega)$ is the $\mathbb{C}^n$ vector of the n DOF and $\mathbf{f}(\mathbf{p}, \omega)$ is the $\mathbb{C}^n$ vector induced by the external forces. In Eq. (1), the symmetric $n \times n$ complex matrices $[\underline{A}^1(\omega)]$ and $[\underline{A}^2(\mathbf{p}, \omega)]$ are the dynamical stiffness matrices of the master system and of the subsystem. It is assumed that vector $\mathbf{f}(\mathbf{p}, \omega)$ and matrix $[\underline{A}^2(\mathbf{p}, \omega)]$ are affine mappings of the design parameter $\mathbf{p}=(p_1, \dots, p_s)$ and are written as

$$\mathbf{f}(\mathbf{p}, \omega) = \mathbf{f}^0(\omega) + \sum_{i=1}^s p_i \mathbf{f}^i(\omega) \quad (2)$$

$$[\underline{A}^2(\mathbf{p}, \omega)] = [\underline{A}^{2,0}(\omega)] + \sum_{i=1}^s p_i [\underline{A}^{2,i}(\omega)] \quad (3)$$

Note that such an assumption allows a large class of design problems to be studied. For instance, let us consider an Euler beam with rectangular section $S=bh$ and bending inertia $I=bh^3/12=Sh^2/12$. It is assumed that S has a fixed value and that the optimization variables are the mass density per unit length ρ and the section height h . Then the design parameter can be chosen as $\mathbf{p}=(p_1, p_2)$ with $p_1=\rho$ and $p_2=h^2$, which satisfies the linear assumption. Such a linear assumption is useful for optimization algorithm because the gradient of vector $\mathbf{f}(\mathbf{p}, \omega)$ and the gradient of matrix $[\underline{A}^2(\mathbf{p}, \omega)]$ are calculated once and do not depend on $\mathbf{p}$. In addition, the Hessian of $\mathbf{f}(\mathbf{p}, \omega)$ and $[\underline{A}^2(\mathbf{p}, \omega)]$ are zero. It should be noted that the theory presented in this paper is also valuable when this linear assumption is removed, but then requires the numerical calculation of the gradient and the Hessian of $\mathbf{f}(\mathbf{p}, \omega)$ and $[\underline{A}^2(\mathbf{p}, \omega)]$ for each $\mathbf{p}$ in $\mathcal{P}$. Let n_1 and n_2 be the number of the

internal DOF of the master system and the subsystem. Let n_Σ be the number of coupling interface DOF. We then have $n=n_1+n_\Sigma+n_2$. The bloc decomposition of $\underline{\mathbf{u}}(\mathbf{p}, \omega)$, $\mathbf{f}(\mathbf{p}, \omega)$, $[\underline{A}^1(\omega)]$, and $[\underline{A}^2(\mathbf{p}, \omega)]$ with respect to n_1 , n_Σ , and n_2 is given by

$$\underline{\mathbf{u}}(\mathbf{p}, \omega) = \begin{bmatrix} \underline{\mathbf{u}}^1(\mathbf{p}, \omega) \\ \underline{\mathbf{u}}_\Sigma(\mathbf{p}, \omega) \\ \underline{\mathbf{u}}^2(\mathbf{p}, \omega) \end{bmatrix} \quad \mathbf{f}(\mathbf{p}, \omega) = \begin{bmatrix} \mathbf{f}^1(\mathbf{p}, \omega) \\ \mathbf{f}_\Sigma(\mathbf{p}, \omega) \\ \mathbf{f}^2(\mathbf{p}, \omega) \end{bmatrix} \quad (4)$$

$$[\underline{A}^1(\omega)] = \begin{bmatrix} [\underline{A}_{ii}^1(\omega)] & [\underline{A}_{i\Sigma}^1(\omega)] & [0] \\ [\underline{A}_{\Sigma i}^1(\omega)]^T & [\underline{A}_{\Sigma\Sigma}^1(\omega)] & [0] \\ [0] & [0] & [0] \end{bmatrix}$$

$$[\underline{A}^2(\mathbf{p}, \omega)] = \begin{bmatrix} [0] & [0] & [0] \\ [0] & [\underline{A}_{\Sigma\Sigma}^2(\mathbf{p}, \omega)] & [\underline{A}_{\Sigma i}^2(\mathbf{p}, \omega)]^T \\ [0] & [\underline{A}_{i\Sigma}^2(\mathbf{p}, \omega)] & [\underline{A}_{ii}^2(\mathbf{p}, \omega)] \end{bmatrix} \quad (5)$$

In this paper, the Benfield and Hruda dynamic substructuring method [30] is used and is briefly summarized below (note that any other substructuring method could be used (see, for instance, Refs. [28,29,31–34])). First, the Craig and Bampton method [28] is applied to the master system with N_1 elastic modes (with fixed coupling interface) stored in the $n_1 \times N_1$ real matrix $[\Phi^1]$. The static boundary functions related to this coupling interface are stored in the $n_1 \times n_\Sigma$ real matrix $[\underline{S}^1]$. Second, the mean reduced matrices of the master system are assembled with the mean finite element matrices of the subsystem. Third, the submatrix of the coupled system (master system with subsystem) corresponding to the DOF of the subsystem is extracted. The eigenmodes of this submatrix corresponding to the N_2 lowest eigenfrequencies are stored in the real $(n_\Sigma+n_2) \times N_2$ matrix $[\Phi^2(\mathbf{p})]$. In the robust design optimization context, the probabilistic model of uncertainties must be independent of the value of the design parameter $\mathbf{p}$. This implies that the projection basis for the subsystem must be independent of $\mathbf{p}$. Consequently, a numerical value $\mathbf{p}_0$ of $\mathbf{p}$ is chosen as an initial design value. Let V_{N_2} be the subspace of $\mathbb{R}^{n_\Sigma+n_2}$ spanned by the N_2 columns of $[\Phi^2(\mathbf{p}_0)]$. The value of N_2 must be chosen such that $[\underline{\mathbf{u}}_\Sigma(\mathbf{p}, \omega), \underline{\mathbf{u}}^2(\mathbf{p}, \omega)]$ belongs to V_{N_2} for all $\mathbf{p}$ in $\mathcal{P}$ for a given tolerance of this approximation. It should be noted that N_2 cannot be selected a priori but has to be defined during the computation studying the convergence with respect to N_2 . Clearly, V_{N_2} converges to $\mathbb{R}^{n_\Sigma+n_2}$ when N_2 goes to n_2+n_Σ , for all $\mathbf{p}$ in $\mathcal{P}$. Nevertheless, the reduction will be efficient if $N_2 \ll n_2+n_\Sigma$. It will be the case if the set $\mathcal{P}$ of the possible designs is not too “big” and for n_2+n_Σ very large. If n_2+n_Σ is small (for instance, a few dozens of degrees of freedom), we will have $N_2 \approx n_2+n_\Sigma$ but it is not a difficulty. If n_2+n_Σ is large (for instance, several thousands or ten thousands of degrees of freedom) and if the convergence is not obtained for $N_2 \ll n_2+n_\Sigma$, this means that the projection basis has to be constructed for each $\mathbf{p}$ in $\mathcal{P}$. For such a case, all the developments presented in this paper can be used except the explicit calculation of the gradient and the Hessian of the cost function, which then have to be numerically calculated (if used in the optimization algorithm). Finally, the projection basis corresponding to the Benfield and Hruda dynamic substructuring method is then written as

$$\begin{bmatrix} \underline{\mathbf{u}}^1(\mathbf{p}, \omega) \\ \underline{\mathbf{u}}_\Sigma(\mathbf{p}, \omega) \\ \underline{\mathbf{u}}^2(\mathbf{p}, \omega) \end{bmatrix} = [\underline{H}(\mathbf{p}_0)] \begin{bmatrix} \underline{\mathbf{q}}^1(\mathbf{p}, \omega) \\ \underline{\mathbf{q}}^2(\mathbf{p}, \omega) \end{bmatrix} \quad (6)$$

$$[H(\mathbf{p}_0)] = \begin{bmatrix} [\Phi^1] & [\underline{S}^1][\Phi_s^2(\mathbf{p}_0)] \\ [0] & [\Phi_s^2(\mathbf{p}_0)] \\ [0] & [\Phi_s^2(\mathbf{p}_0)] \end{bmatrix}$$

in which the matrices $[\Phi_s^2(\mathbf{p}_0)]$ and $[\Phi_r^2(\mathbf{p}_0)]$ correspond to the bloc decomposition of matrix $[\Phi^2(\mathbf{p}_0)]$ with respect to the n_s coupling interface DOF and the n_2 internal DOF of the subsystem. Projecting Eq. (1) in using Eq. (6) yields the mean reduced matrix equation

$$([A_{\text{red}}^1(\omega)] + [A_{\text{red}}^2(\mathbf{p}, \omega)]) \begin{bmatrix} \underline{\mathbf{q}}^1(\mathbf{p}, \omega) \\ \underline{\mathbf{q}}^2(\mathbf{p}, \omega) \end{bmatrix} = \mathbf{f}_{\text{red}}(\mathbf{p}, \omega) \quad (7)$$

in which $\mathbf{f}_{\text{red}}(\mathbf{p}, \omega) = [H(\mathbf{p}_0)]^T \mathbf{f}(\mathbf{p}, \omega)$ and where $[A_{\text{red}}^1(\omega)]$ and $[A_{\text{red}}^2(\mathbf{p}, \omega)]$ are the symmetric $N \times N$ complex matrices in which $N = N_1 + N_2$ using the bloc decomposition with respect to the N_1 and N_2 reduced coordinates. Note that $[A_{\text{red}}^1(\omega)]$ is a full matrix and that $[A_{\text{red}}^2(\mathbf{p}, \omega)]$ is written as

$$[A_{\text{red}}^2(\mathbf{p}, \omega)] = \begin{bmatrix} [0] & [0] \\ [0] & [A_s^2(\mathbf{p}, \omega)] \end{bmatrix} \quad (8)$$

in which

$$[A_s^2(\mathbf{p}, \omega)] = [[0]^T \quad [\Phi^2(\mathbf{p}_0)]^T] [A^2(\mathbf{p}, \omega)] \begin{bmatrix} [0] \\ [\Phi^2(\mathbf{p}_0)] \end{bmatrix} \quad (9)$$

3 Formulation for the Design Optimization Problem With a Numerical Model With No Uncertainties

In this section, we remind a formulation to solve the design optimization problem assuming that there are no uncertainties. This formulation will be used to compare the solution of this deterministic design optimization problem with the robust design optimization solution presented in Sec. 5. Let $\mathbf{w}(\mathbf{p}, \omega)$ be the vector in C^k of the observations of the mean dynamical system, defined as a function of the displacement vector $\mathbf{u}(\mathbf{p}, \omega)$ such that

$$\mathbf{w}(\mathbf{p}, \omega) = \mathbf{b}_\omega(\mathbf{u}(\mathbf{p}, \omega)) \quad (10)$$

where $\mathbf{b}_\omega$ is a given function from C^n into C^k depending on the frequency ω . The performance objectives for the observations in the frequency band $B_1 \subset B$ will be defined as the “target.” This target is then represented by the function $\omega \mapsto \mathbf{g}(\omega)$ from B_1 into C^k . The cost function $j(\mathbf{p})$ is formulated as a distance between the target $\mathbf{g}$ and the observation $\mathbf{w}(\mathbf{p}, \cdot)$. It is a function of design parameter $\mathbf{p}$ and is written as

$$j(\mathbf{p}) = \frac{\|\mathbf{w}(\mathbf{p}, \cdot) - \mathbf{g}\|_{B_1}^2}{\|\mathbf{g}\|_{B_1}^2} \quad (11)$$

in which $\|\mathbf{g}\|_{B_1}^2 = \int_{B_1} \|\mathbf{g}(\omega)\|^2 d\omega$ with $\|\mathbf{g}(\omega)\|$ the Hermitian norm of $\mathbf{g}(\omega)$. The design optimization problem is formulated as the minimization of the cost function $j(\mathbf{p})$ with respect to the design parameter $\mathbf{p}$ in the admissible set $\mathcal{P}$: find $\mathbf{p}^D$ in $\mathcal{P}$ such that $j(\mathbf{p}^D) \leq j(\mathbf{p})$ for all $\mathbf{p}$ in $\mathcal{P}$. Such an optimization problem can be solved numerically by using the sequential quadratic optimization algorithm [35,36].

4 Stochastic Dynamical System With Model and Data Uncertainties

As explained in the Introduction, the robust design optimization problem is formulated with respect to the model uncertainties and data uncertainties existing in the mean model of the dynamical system. In this section, we then introduce this nonparametric probabilistic approach of uncertainties [9–11]. It is assumed that the mean model of the master system and subsystem contain model uncertainties and data uncertainties. The level of uncertain-

ties of these two systems is a priori different and will be then characterized by different values of the dispersion parameters defined below.

Let $[M_{\text{red}}^1]$, $[D_{\text{red}}^1]$, $[K_{\text{red}}^1]$ and $[M_s^2(\mathbf{p})]$, $[D_s^2(\mathbf{p})]$, $[K_s^2(\mathbf{p})]$ be the mean reduced mass, and damping, stiffness matrices of the mean master system and of the mean subsystem, respectively. The dynamic stiffness reduced matrices are such that $[A_{\text{red}}^1(\omega)] = -\omega^2[M_{\text{red}}^1] + i\omega[D_{\text{red}}^1] + [K_{\text{red}}^1]$ and $[A_s^2(\mathbf{p}, \omega)] = -\omega^2[M_s^2(\mathbf{p})] + i\omega[D_s^2(\mathbf{p})] + [K_s^2(\mathbf{p})]$. The matrices $[M_{\text{red}}^1]$, $[D_{\text{red}}^1]$, $[K_{\text{red}}^1]$ are positive symmetric $N \times N$ matrices whose rank is $N_1 = N - n_2$ (which is assumed to be positive), whereas $[M_s^2(\mathbf{p})]$ is a positive-definite symmetric $N_2 \times N_2$ matrix and $[D_s^2(\mathbf{p})]$, $[K_s^2(\mathbf{p})]$ are positive symmetric $N_2 \times N_2$ matrices whose rank is $N_2 - r$. The methodology of the nonparametric probabilistic approach consists in replacing the matrices $[M_{\text{red}}^1]$, $[D_{\text{red}}^1]$, $[K_{\text{red}}^1]$ and $[M_s^2(\mathbf{p})]$, $[D_s^2(\mathbf{p})]$, $[K_s^2(\mathbf{p})]$ by the random matrices $[\mathbf{M}_{\text{red}}^1]$, $[\mathbf{D}_{\text{red}}^1]$, $[\mathbf{K}_{\text{red}}^1]$ and $[\mathbf{M}_s^2(\mathbf{p})]$, $[\mathbf{D}_s^2(\mathbf{p})]$, $[\mathbf{K}_s^2(\mathbf{p})]$ such that $\mathcal{E}\{[\mathbf{M}_{\text{red}}^1]\} = [M_{\text{red}}^1]$, $\mathcal{E}\{[\mathbf{D}_{\text{red}}^1]\} = [D_{\text{red}}^1]$, $\mathcal{E}\{[\mathbf{K}_{\text{red}}^1]\} = [K_{\text{red}}^1]$ and $\mathcal{E}\{[\mathbf{M}_s^2(\mathbf{p})]\} = [M_s^2(\mathbf{p})]$, $\mathcal{E}\{[\mathbf{D}_s^2(\mathbf{p})]\} = [D_s^2(\mathbf{p})]$, $\mathcal{E}\{[\mathbf{K}_s^2(\mathbf{p})]\} = [K_s^2(\mathbf{p})]$ in which $\mathcal{E}$ is the mathematical expectation.

The probability model for each one of these random matrices is briefly recalled below. Let $[\mathbf{E}^i(\mathbf{p})]$, $i = \{1, 2\}$ be the positive symmetric $n \times n$ real matrix of rank m representing one of the matrices of the set $\{[\mathbf{M}_{\text{red}}^1], [\mathbf{D}_{\text{red}}^1], [\mathbf{K}_{\text{red}}^1]\}$ when $i=1$ or of the set $\{[\mathbf{M}_s^2(\mathbf{p})], [\mathbf{D}_s^2(\mathbf{p})], [\mathbf{K}_s^2(\mathbf{p})]\}$ when $i=2$. Using the nonparametric probabilistic approach, the matrix $[\mathbf{E}^i(\mathbf{p})]$ is replaced by the random matrix $[\mathbf{E}^i(\mathbf{p})]$ such that

$$[\mathbf{E}^i(\mathbf{p})] = [\underline{L}_E^i(\mathbf{p})]^T [\mathbf{G}_E^i] [\underline{L}_E^i(\mathbf{p})] \quad (12)$$

in which $[\underline{L}_E^i(\mathbf{p})]$ is an $m \times n$ rectangular real matrix such that $[\underline{L}_E^i(\mathbf{p})] = [\underline{L}_E^i(\mathbf{p})]^T [\underline{L}_E^i(\mathbf{p})]$ and where $[\mathbf{G}_E^i]$ is a random matrix with value in the set of all the positive-definite symmetric $m \times m$ matrices. The probability model of random matrix $[\mathbf{G}_E^i]$ is constructed by using the maximum entropy principle with the available information. The dispersion of the random matrix $[\mathbf{G}_E^i]$ is controlled by one real positive parameter δ_E^i called the dispersion parameter. This means that the dispersion parameters related to random matrices $[\mathbf{G}_M^1]$, $[\mathbf{G}_D^1]$, $[\mathbf{G}_K^1]$ and $[\mathbf{G}_M^2]$, $[\mathbf{G}_D^2]$, $[\mathbf{G}_K^2]$ are δ_M^1 , δ_D^1 , δ_K^1 and δ_M^2 , δ_D^2 , δ_K^2 . It should be noted that as a result of this theory, these six random matrices are independent random matrices. All the details concerning the construction of this probability model can be found in Refs. [9–11]. The algebraic representation of random matrix $[\mathbf{G}_E^i]$ adapted to the Monte Carlo numerical simulation is briefly recalled. Random matrix $[\mathbf{G}_E^i]$ is written as $[\mathbf{G}_E^i] = [\underline{L}_{G_E^i}]^T [\underline{L}_{G_E^i}]$ in which $[\underline{L}_{G_E^i}]$ is an $m \times m$ real upper triangular random matrix such that (1) random variables $\{[\underline{L}_{G_E^i}]_{jj'}, j \leq j'\}$ are independent; (2) for $j < j'$, real-valued random variable $[\underline{L}_{G_E^i}]_{jj'}$ can be written as $[\underline{L}_{G_E^i}]_{jj'} = \sigma_m U_{jj'}$ in which $\sigma_m = \delta_E^i(m+1)^{-1/2}$ and where $U_{jj'}$ is a real-valued Gaussian random variable with zero mean and variance equal to 1; (3) for $j = j'$, positive-valued random variable $[\underline{L}_{G_E^i}]_{jj}$ can be written as $[\underline{L}_{G_E^i}]_{jj} = \sigma_m \sqrt{2} V_j$ in which σ_m is defined above and where V_j is a positive-valued gamma random variable whose probability density function $p_{V_j}(v)$ with respect to dv is written as

$$p_{V_j}(v) = 1_{R^+}(v) \frac{1}{\Gamma((m+1)/(2(\delta_E^i)^2) + (1-j)/2)} v^{(m+1)/(2(\delta_E^i)^2) - (1+j)/2} e^{-v} \quad (13)$$

In coherence with the notation of Sec. 2, let $\mathbf{U}(\mathbf{p}, \omega)$ be the C^n -valued random vector of the displacement whose bloc decomposition (similar to Eq. (4)) is written as $\mathbf{U}(\mathbf{p}, \omega) = [\mathbf{U}^1(\mathbf{p}, \omega), \mathbf{U}_s(\mathbf{p}, \omega), \mathbf{U}^2(\mathbf{p}, \omega)]$. Then the equations of the sto-

chastic reduced system corresponding to the nonparametric probabilistic model of uncertainties are

$$\begin{bmatrix} \mathbf{U}^1(\mathbf{p}, \omega) \\ \mathbf{U}_s^1(\mathbf{p}, \omega) \\ \mathbf{U}^2(\mathbf{p}, \omega) \end{bmatrix} = [\mathbf{H}(\mathbf{p}_0)] \begin{bmatrix} \mathbf{Q}^1(\mathbf{p}, \omega) \\ \mathbf{Q}^2(\mathbf{p}, \omega) \end{bmatrix} \quad (14)$$

where $\mathbf{Q}(\mathbf{p}, \omega) = [\mathbf{Q}^1(\mathbf{p}, \omega), \mathbf{Q}^2(\mathbf{p}, \omega)]$ is the $\mathbb{C}^N$ -valued random vector of the generalized coordinates, solution of the random matrix equation that

$$([\mathbf{A}_{\text{red}}^1(\omega)] + [\mathbf{A}_{\text{red}}^2(\mathbf{p}, \omega)])\mathbf{Q}(\mathbf{p}, \omega) = \mathbf{f}_{\text{red}}(\mathbf{p}, \omega) \quad (15)$$

in which the matrix $[\mathbf{A}_{\text{red}}^1(\omega)]$ is such that $[\mathbf{A}_{\text{red}}^1(\omega)] = -\omega^2[\mathbf{M}_{\text{red}}^1] + i\omega[\mathbf{D}_{\text{red}}^1] + [\mathbf{K}_{\text{red}}^1]$ and where the matrix $[\mathbf{A}_{\text{red}}^2(\mathbf{p}, \omega)]$ is such that

$$[\mathbf{A}_{\text{red}}^2(\mathbf{p}, \omega)] = \begin{bmatrix} [0] & [0] \\ [0] & [\mathbf{A}_s^2(\mathbf{p}, \omega)] \end{bmatrix} \quad (16)$$

with

$$[\mathbf{A}_s^2(\mathbf{p}, \omega)] = -\omega^2[\mathbf{M}_s^2(\mathbf{p})] + i\omega[\mathbf{D}_s^2(\mathbf{p})] + [\mathbf{K}_s^2(\mathbf{p})] \quad (17)$$

5 Formulation of the Robust Design Optimization Problem With Respect to Model Uncertainties and Data Uncertainties

In this section, the robust design optimization problem is formulated with respect to model uncertainties and data uncertainties using the nonparametric probabilistic approach described in Sec. 4. The robust design optimization problem deals with the minimization of a cost function with respect to the design parameter. The cost function is constructed with an uncertain numerical model. Contrary to the design optimization problem described in Sec. 3, the cost function is not defined for the performance of the mean dynamical system but is defined with respect to the performance of the stochastic dynamical system representing the real manufactured system. For the robust problem, the performance objectives are double: (1) minimizing the distance between the mean value of the stochastic observation and the target and (2) minimizing the sensitivity of the stochastic observation with respect to model uncertainties and data uncertainties. The solution of this robust design optimization problem yields an optimal value of the design parameter, which corresponds to an optimal dynamical system from which the real manufactured system fulfills the performance objectives.

In coherence with the notation introduced in Sec. 3, let $\mathbf{W}(\mathbf{p}, \omega)$ be the $\mathbb{C}^k$ -valued random variable modeling the random observation of the stochastic dynamical system. It is defined as a function of the random displacement vector $\mathbf{U}(\mathbf{p}, \omega)$ such that

$$\mathbf{W}(\mathbf{p}, \omega) = \mathbf{b}_\omega(\mathbf{U}(\mathbf{p}, \omega)) \quad (18)$$

in which $\mathbf{b}_\omega$ is the function introduced in Eq. (10). The cost function is then defined by

$$j(\mathbf{p}, \alpha) = \frac{1}{\|\mathbf{g}\|_{\mathbb{B}_1}^2} (\alpha \|\boldsymbol{\mu}(\mathbf{p}, \cdot) - \mathbf{g}\|_{\mathbb{B}_1}^2 + (1 - \alpha) \sigma^2(\mathbf{p})) \quad (19)$$

in which $\boldsymbol{\mu}(\mathbf{p}, \omega)$ is the mean value of random vector $\mathbf{W}(\mathbf{p}, \omega)$ such that $\boldsymbol{\mu}(\mathbf{p}, \omega) = \mathcal{E}\{\mathbf{W}(\mathbf{p}, \omega)\}$ and where $\sigma^2(\mathbf{p}) = \mathcal{E}\{\|\mathbf{W}(\mathbf{p}, \cdot) - \boldsymbol{\mu}(\mathbf{p}, \cdot)\|_{\mathbb{B}_1}^2\}$. In Eq. (19), the scalar α belonging to $]0, 1/2]$ is the weighting factor, which has to be adjusted in order to favorize the robustness objective performance (2) with respect to the target objective performance (1). It should be noted that cost function $j(\mathbf{p}, \alpha)$ can be rewritten as

$$j(\mathbf{p}, \alpha) = \alpha j_1(\mathbf{p}) + (1 - 2\alpha) j_2(\mathbf{p}) \quad (20)$$

in which $j_1(\mathbf{p}) = \|\mathbf{g}\|_{\mathbb{B}_1}^{-2} \mathcal{E}\{\|\mathbf{W}(\mathbf{p}, \cdot) - \mathbf{g}\|_{\mathbb{B}_1}^2\}$ and where $j_2(\mathbf{p}) = \|\mathbf{g}\|_{\mathbb{B}_1}^{-2} \sigma^2(\mathbf{p})$. Note that $j_1(\mathbf{p})$ is related to a distance between the stochastic observation and the target and allows the target perfor-

mance objective (1) and the robustness performance objective (2) to be simultaneously achieved with equal weight. The quantity $j_2(\mathbf{p})$ is a penalty term for favorizing the robustness performance objective (2) when α decreases.

The robust design optimization problem is formulated as a multiobjective optimization problem, which consists to minimize the cost function $\mathbf{p} \mapsto j(\mathbf{p}, \alpha)$ with respect to the admissible set $\mathcal{P}$ of the design parameter $\mathbf{p}$. For given dispersion parameters $\delta_M^1, \delta_D^1, \delta_K^1$ and $\delta_M^2, \delta_D^2, \delta_K^2$ and for a given value of $\alpha \in]0, 1/2]$, the robust design optimization problem is written as follows: find $\mathbf{p}^{\text{RD}}$ in $\mathcal{P}$ such that

$$j(\mathbf{p}^{\text{RD}}, \alpha) \leq j(\mathbf{p}, \alpha) \quad \text{for all } \mathbf{p} \text{ in } \mathcal{P} \quad (21)$$

The value of the weighting factor α characterizes the importance of each performance objective with respect to the other one and is adjusted in order to obtain the better solution. Since the paper deals with robust design optimization, the case for which the target objective performance (1) would be favorized with respect to the robust objective performance (2) is not considered. For this reason, the weighting factor is chosen such that $\alpha \leq 1/2$. When $\alpha = 1/2$, the weight of the performance objectives (1) and (2) are the same. For small values of α , the performance objective related to the robustness with respect to model and data uncertainties becomes more important with respect to the performance objective related to the target. In the formulation of the cost function $j(\mathbf{p}, \alpha)$ given by Eq. (19), a target has been introduced. Then the cost function given by Eq. (19) is different from most of the formulations encountered in the literature [15,18,19]. In addition, since the normalization does not change the optimization problem, it should be noted that the formulation used is coherent with the usual formulation of the robust design optimization problem [18–20] when the target is not taken into account ($\mathbf{g} = 0$) and for the monodimensional case ($k = 1$). Note that the definition of the robust design optimization problem is coherent with respect to the deterministic design optimization problem given in Sec. 3. Since the value $\alpha = 0$ is excluded, the formulation for robust design optimization is coherent with the formulation of design optimization when the stochastic dynamical system tends to be deterministic, that is to say when $\lim_{\|\delta\| \rightarrow 0} \mathbf{p}^{\text{RD}} = \mathbf{p}^{\text{D}}$ in which $\delta = (\delta_M^1, \delta_D^1, \delta_K^1, \delta_M^2, \delta_D^2, \delta_K^2)$. Finally, the robust design optimization problem is solved by using the sequential quadratic optimization algorithm [35,36] coupled with the Monte Carlo numerical simulation. In addition, it should be noted that the random germs of the random matrices do not depend on the design parameter $\mathbf{p}$. Consequently, the gradient and the Hessian can algebraically be constructed (see the Appendix), which improves the precision of the optimization algorithm.

6 Numerical Application

The objective of this application is the robust design optimization of dynamic absorbers in order to reduce the vibration level of a heterogeneous dynamical system in a given narrow-frequency band and for a given deterministic excitation. The heterogeneous dynamical system is constituted of a homogeneous plate coupled with lumped masses and oscillators. Only the vibration level of the homogeneous plate has to be reduced. The dynamic absorbers are identical and each one is constituted of oscillators in parallel at different eigenfrequencies. Two cases are investigated. The first case concerns robust design optimization with respect to a single design parameter. A more complicated case involving a multidimensional design parameter is then considered.

6.1 Mean Finite Element Model of the Dynamical System.

The dynamical system is constituted of a master system corresponding to the heterogeneous dynamical system and of a subsystem made up of the dynamic absorbers. The frequency band of interest is [350, 750] Hz but the frequency band of analysis for

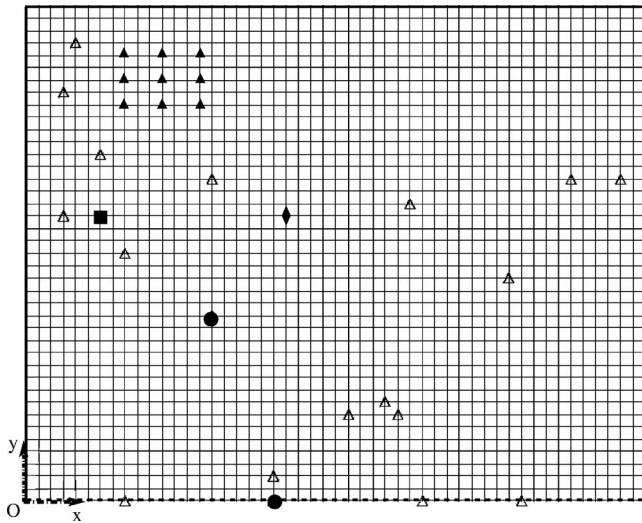


Fig. 1 Finite element mesh of the dynamical system: attached spring (■), attached lumped mass (●), attached set of three single DOF linear oscillators (Δ), vibration absorbers (▲), excitation node (◆), simply supported boundary (thick line), and free boundary (thick dashed line)

optimal design is [500,600] Hz. Consequently, the target will be specified in this frequency band in order to optimize the absorbers in this band.

The mean master system is the heterogeneous system made up of a plate with two attached lumped masses, one attached spring and 51 attached single DOF linear oscillators. The plate is a thin plate in bending mode and is located in the plane (Ox,Oy) of a Cartesian coordinate system $(Oxyz)$. The out-of-plane displacements are only considered. The plate is made of a homogeneous, isotropic elastic material with mass density of 7800 kg m^{-3} , Poisson ratio of 0.29, and Young modulus of $2 \times 10^{11} \text{ N m}^{-2}$. The plate has constant thickness of $0.4 \times 10^{-3} \text{ m}$, length of 0.5 m, and width of 0.4 m. The plate is simply supported on three edges and is free on the fourth edge corresponding to $y=0$. The mean finite element model of the plate is constituted of 2000 bending plate elements (with four nodes) and is shown in Fig. 1. A damping model is added to the plate corresponding to a hysteretic model with a mean loss factor of 0.02. The two lumped masses have masses of 4 kg and 1 kg, located at points $(0.15, 0.15, 0)$ and $(0.2, 0, 0)$, respectively (see Fig. 1). The attached spring has stiffness of $2.388 \times 10^{11} \text{ N m}^{-1}$ and is located at point $(0.06, 0.23, 0)$ (see Fig. 1). The attached oscillators are grouped by sets of three oscillators (see Fig. 1). The eigenfrequencies of these oscillators are in the band [350,750] Hz.

The mean subsystem to be optimized is constituted of nine identical vibration absorbers, each one being made up of five single DOF linear oscillators in parallel (see Fig. 1). The eigenfrequencies of the five oscillators are fixed, are equal to 560 Hz, 565 Hz, 570 Hz, 575 Hz, and 580 Hz, and have been chosen to form an absorber around the main resonance occurring at 571 Hz in the narrow-frequency band [500,600] Hz. Note that these values have been deduced from a sensitivity analysis performed with the mean model in order to obtain a significant reduction of the vibration levels. The critical damping rate is the same for the five oscillators and is 0.01. In a first case, it is assumed that the five oscillators of a vibration absorber have the same mass m , which has to be optimized. The total mass of the subsystem is then defined by $m=45m$. The design parameter is m . In a second case, it is assumed that the mass of each oscillator constituting a vibration absorber can be different. Denoting as m_i the mass of the oscillator number i of each vibration absorber constituting the subsystem, the design parameter is the $\mathbb{R}^5$ -vector $\mathbf{m}$ in which $\mathbf{m}$

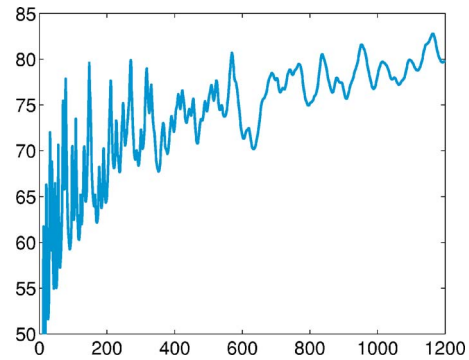


Fig. 2 Reference observation of the mean master system. Graph of function $\nu \mapsto 20 \log_{10}(w^{\text{master}}(2\pi\nu))$. Horizontal axis is the frequency ν in Hz.

$= (9m_1, 9m_2, 9m_3, 9m_4, 9m_5)$. Since the eigenfrequency and the critical damping of the oscillators of the vibration absorbers are fixed, the mass, damping, and stiffness matrices of the subsystem are affine functions of the design parameter $\mathbf{m}$. In the present case, the excitation does not depend on $\mathbf{m}$.

The finite element model of the mean dynamical system (master system coupled with the five vibration absorbers) is thus constituted of $n=6106$ DOFs with $n_1=6052$ internal DOFs of the mean master system, $n_2=45$ internal DOFs for the mean subsystem and $n_\Sigma=9$ coupling interface DOFs. The mean dynamical system is submitted to a given deterministic unit transverse load constant in frequency band [5,1200] Hz with amplitude 1 (see Fig. 1). The observation chosen for the dynamic analysis is the signal energy related to the out-of-plane accelerations of the plate. Consequently, the real-valued function b_ω introduced in Eq. (9) is such that $w(\mathbf{p}, \omega) = b_\omega(\mathbf{u}(\mathbf{p}, \omega)) = \omega^2 \|\mathbf{u}^{\text{plate}}(\mathbf{p}, \omega)\|$ in which $\mathbf{u}^{\text{plate}}(\mathbf{p}, \omega)$ is the complex vector constituted of the 1960 out-of-plane displacements of the plate.

6.2 Reference Solution for the Master System. The reference observation in the master system is $w^{\text{master}}(\omega) = \omega^2 \|\mathbf{u}_{\text{unc}}^{\text{plate}}(\mathbf{p}, \omega)\|$, in which $\mathbf{u}_{\text{unc}}^{\text{plate}}$ is the plate response corresponding to the mean model of the master system (uncoupled with the absorbers of the subsystem). Figure 2 displays the graph of $\omega \mapsto 20 \log_{10}(w^{\text{master}}(\omega))$. In Fig. 2, it can be seen that the level of the reference solution for the mean master system is lower than 77.5 dB in the frequency band [350,750] Hz except for one single peak whose resonance occurs at 571 Hz with level 80.5 dB, i.e., 3 dB more.

6.3 Estimation of the Numerical Parameters for the Robust Design Optimization Problem. Let $\{W^{\text{master}}(\theta_j, \omega), j=1, \dots, n_s\}$ be the n_s independent realizations of random variable $W^{\text{master}}(\omega)$. The robust optimization problem needs to solve the stochastic reduced equation, Eq. (15). The numerical parameters are then the dimension N of the reduced dynamical system and the number n_s of realizations used in the Monte Carlo numerical simulation. Therefore, a convergence analysis has to be performed with respect to N and n_s for the stochastic reduced system. The computation is performed for the dispersion parameters of the master system such that $\delta_M^1 = \delta_D^1 = \delta_K^1 = 0.05$. A stochastic convergence analysis with $N_2 = n_2 + n_\Sigma = 54$ is carried out in order to define the number N_1 of modes to be kept in the modal reduction and the number n_s of realizations. The mean-square convergence is analyzed by studying the function $(n_s, N_1) \mapsto \text{Conv}(n_s, N_1)$ defined by

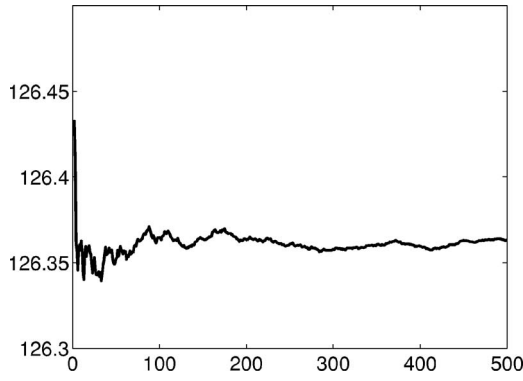


Fig. 3 Convergence analysis: graph of function $n_s \mapsto 20 \log_{10}(\text{Conv}(n_s, N_1))$ for the stochastic master system with $N_1=300$. Horizontal axis is n_s .

$$\text{Conv}(n_s, N_1) = \left\{ \frac{1}{n_s} \sum_{j=1}^{n_s} \int_{\mathcal{B}} \|W^{\text{master}}(\theta_j, \omega)\|^2 d\omega^{1/2} \right\}^2 \quad (22)$$

in which $W^{\text{master}}(\theta_j, \omega)$ is calculated with a reduced model of dimension $N=N_1+N_2$. Figure 3 displays the graph of $n_s \mapsto 20 \log_{10}(\text{Conv}(n_s, N_1))$ for $N_1=300$. It can be seen that a reasonable convergence is reached for $n_s=300$. Figure 4 displays the graph $N_1 \mapsto 20 \log_{10}(\text{Conv}(n_s, N_1))$ for $n_s=300$. Convergence is reached for $N_1=225$.

6.4 Target and Its Comparison With the Reference Observation. As explained above, the robust design optimization is carried out over the frequency band $\mathcal{B}_1=[500, 600]$ Hz. In band $\mathcal{B}_1$, there are three main resonances. The higher resonance occurs at 571 Hz with a level of 80.5 dB (see Fig. 5). There are two secondary resonances with a smaller level. The target is specified in order that the two secondary responses be not modified and that to reduce strongly the higher resonance. Figure 5 shows the target $\omega \mapsto 20 \log_{10}(g(\omega))$ related to the reference observation $\omega \mapsto 20 \log_{10}(w^{\text{master}}(\omega))$ defined in Sec. 6.2.

6.5 Case 1. Robust Design Optimization With Respect to a Single Design Parameter. In the present case, the masses of the oscillators of each vibration absorber constituting the subsystem are assumed to be identical. The design parameter is the total mass m of the subsystem. The robust design optimization is carried out with $\delta_M^l = \delta_D^l = \delta_K^l = 0.05$ for the master system and with $\delta_M^2 = \delta_D^2 = \delta_K^2 = 0$ for the subsystem (no uncertainties in the subsystem). Note that the subsystem is assumed to be deterministic for conve-

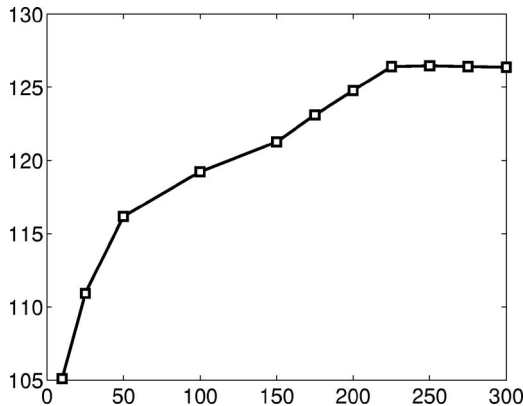


Fig. 4 Convergence analysis: graph of function $N_1 \mapsto 20 \log_{10}(\text{Conv}(n_s, N_1))$ for the stochastic master system with $n_s=300$. Horizontal axis is N_1 .

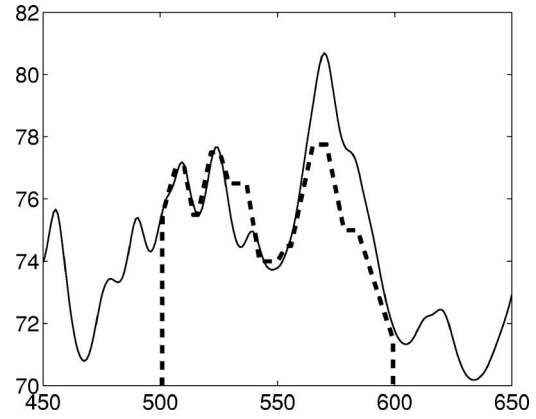


Fig. 5 Definition of the target $\nu \mapsto 20 \log_{10}(g(2\pi\nu))$ (thick dashed line). Comparison with the reference observation $\nu \mapsto 20 \log_{10}(w^{\text{master}}(2\pi\nu))$ (thin solid line) in the frequency band $\mathcal{B}_1=[500, 600]$ Hz (horizontal axis).

nience, the generalization to uncertain subsystems being straightforward. It has been verified that the numerical parameters identified from the convergence analysis of the stochastic master system also yield convergent results for the stochastic reduced system. Consequently, the robust design optimization is carried out with $N_1=225$, $N_2=54$, and $n_s=300$. The admissible set for design parameter m is defined such that $m \in [4.5 \times 10^{-7}, 1.8 \times 10^{-3}]$ kg. We are interested in comparing the design optimization (no uncertainties) with the robust design optimization (with uncertainties) for a weighting factor α , which is chosen as 0.5. The design optimization yields optimal design parameters $m^D = 2.6 \times 10^{-4}$ kg and $m^{\text{RD}} = 8.12 \times 10^{-4}$ kg. A stochastic dynamical analysis of each one of the two optimal designs is then carried out in order to analyze the sensitivity of these two optimal designs with respect to model and data uncertainties. Let $\mu^D(\omega)$, $\mu^{\text{RD}}(\omega)$ and $\sigma^D(\omega)$, $\sigma^{\text{RD}}(\omega)$ be the mean values and the standard deviations of the random observations $W^D(\omega)$ and $W^{\text{RD}}(\omega)$ defined by $W^D(\omega) = W(m^D, \omega)$, $W^{\text{RD}}(\omega) = W(m^{\text{RD}}, \omega)$. We obtain $j(m^D, \frac{1}{2}) = 7.9835 \times 10^{-3}$ with $\|\sigma^D\|_{\mathcal{B}_1} = 6.1924 \times 10^{-2} \|g\|_{\mathcal{B}_1}$ and $j(m^{\text{RD}}, \frac{1}{2}) = 7.2899 \times 10^{-3}$ with $\|\sigma^{\text{RD}}\|_{\mathcal{B}_1} = 5.8034 \times 10^{-2} \|g\|_{\mathcal{B}_1}$. Clearly, the robust design optimization yields the optimal design point whether it is with respect to the target performance objective or with respect to the robustness performance objective.

Below, the sensitivity of the robust design optimization is analyzed with respect to the weighting factor $\alpha \in]0, 1/2]$. From Fig. 6, it can be seen that the mass of the robust design decreases with the weighting factor α . For this application, the sensitivities of the cost function and of the normalized standard deviation $\|\sigma^{\text{RD}}\|_{\mathcal{B}_1} / \|\sigma^D\|_{\mathcal{B}_1}$ are sufficiently weak with respect to the weighting factor (lower than 1.5%) to be neglected. For this reason, all the results concerning the robust design optimization are presented for $\alpha=1/2$.

Similar to the stochastic case, let $w^D(\omega) = w(m^D, \omega)$ and $w^{\text{RD}}(\omega) = w(m^{\text{RD}}, \omega)$. Figure 7 displays the comparison of the target with the response of the mean models corresponding to the design optimization point and corresponding to the robust design optimization point. It can be seen from this figure that the responses of the mean model, corresponding to these two design points are different. In particular, the deterministic design point yields a response of the mean model, which is closer to the target than to the robust design point. Nevertheless, this kind of information is not essential (see below). For all ω fixed in $\mathcal{B}$, the confidence interval of the random variable $W^{\text{RD}}(\omega)$ is constructed for a probability level $P_c=0.95$ using the sample quantiles [37]. Let $\bar{W}^{\text{RD}}(\theta_1, \omega) \leq \dots \leq \bar{W}^{\text{RD}}(\theta_{n_s}, \omega)$ be the order statistics. The up-

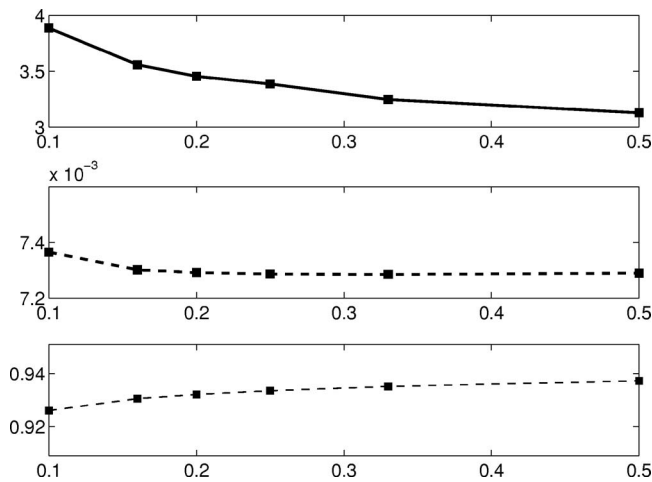


Fig. 6 Case 1. Sensitivity of the robust design optimization with respect to the weighting factor $\alpha \in [0, 0.5]$: graph of $\alpha \mapsto m^{\text{RD}}/m^{\text{D}}$ (thick line), graph of $\alpha \mapsto j(m^{\text{RD}}, \alpha)$ (thick dashed line), and graph of $\alpha \mapsto \|\sigma^{\text{RD}}\|_{B_1}/\|\sigma^{\text{D}}\|_{B_1}$ (thin dashed line). Horizontal axis is α .

per interval $w^+(\omega)$ and the lower interval $w^-(\omega)$ delimiting the confidence interval with probability level P_c are then given by

$$w^+(\omega) = \tilde{W}^{\text{RD}}(\theta_{k_+}, \omega) \quad k_+ = \text{fix}(0.5n_s(1 + P_c)) \quad (23)$$

$$w^-(\omega) = \tilde{W}^{\text{RD}}(\theta_{k_-}, \omega) \quad k_- = \text{fix}(0.5n_s(1 - P_c)) \quad (24)$$

in which $\text{fix}(x)$ is the integer part of real x . Figure 8 compares the reference solution (response of the mean master system) $\omega \mapsto 20 \log_{10}(w^{\text{master}}(\omega))$ with the confidence region of the robust design optimization. In particular, the resonance of the reference solution occurring at frequency 571 Hz has been reduced (about of 3 dB or 4 dB) by the robust design optimization process. It can be seen that the response of the mean master system belongs to the confidence region of the response of the stochastic system corresponding to the robust design optimization except in the frequency band B_1 for which the target is active. Figures 9 and 10 compare the reference solution $\omega \mapsto 20 \log_{10}(w^{\text{master}}(\omega))$ with the confidence regions of the random responses $\omega \mapsto 20 \log_{10}(W^{\text{D}}(\omega))$ corresponding to the design optimization and

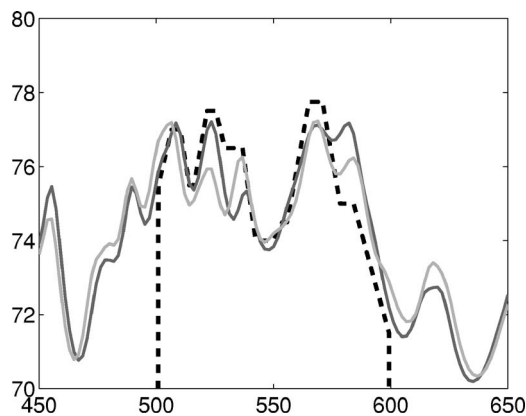


Fig. 7 Case 1. Comparison of the target $\nu \mapsto 20 \log_{10}(g(2\pi\nu))$ (thick dashed line) with the response of the mean model corresponding to the design optimization $\nu \mapsto 20 \log_{10}(w^{\text{D}}(2\pi\nu))$ (thin dark gray line) and corresponding to the robust design optimization $\nu \mapsto 20 \log_{10}(w^{\text{RD}}(2\pi\nu))$ (thin light gray line) for $\alpha = 1/2$. Horizontal axis is the frequency ν in Hz.

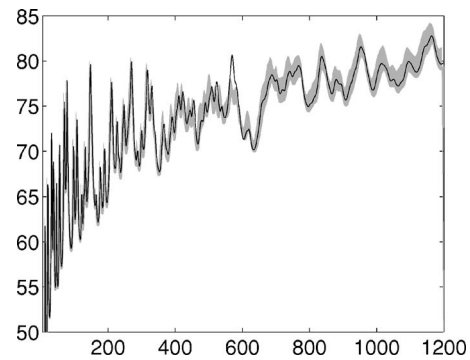


Fig. 8 Case 1. Comparison of the reference observation $\nu \mapsto 20 \log_{10}(w^{\text{master}}(2\pi\nu))$ (thin solid line) with the confidence region (light gray region) of random response for the robust design, over the band $B=[5, 1200]$ Hz (horizontal axis) and for $\alpha=1/2$ and a probability level $P_c=0.95$. Horizontal axis is the frequency ν in Hz.

$\omega \mapsto 20 \log_{10}(W^{\text{RD}}(\omega))$ corresponding to the robust design optimization for a probability level $P_c=0.95$ in the frequency band B_1 . In Fig. 10, there are five resonances, which occur at frequencies 508 Hz, 524 Hz, 539 Hz, 571 Hz, and 583 Hz. In Fig. 10, for peaks 1 and 4, it can be seen that the robust design optimization yields similar results to the design optimization. For peaks 2 and 5, the robust design optimization yields lower response levels. Moreover, the confidence region is particularly narrow in the frequency band [550, 600] Hz, which means that the optimum design is more robust with respect to model and data uncertainties than in

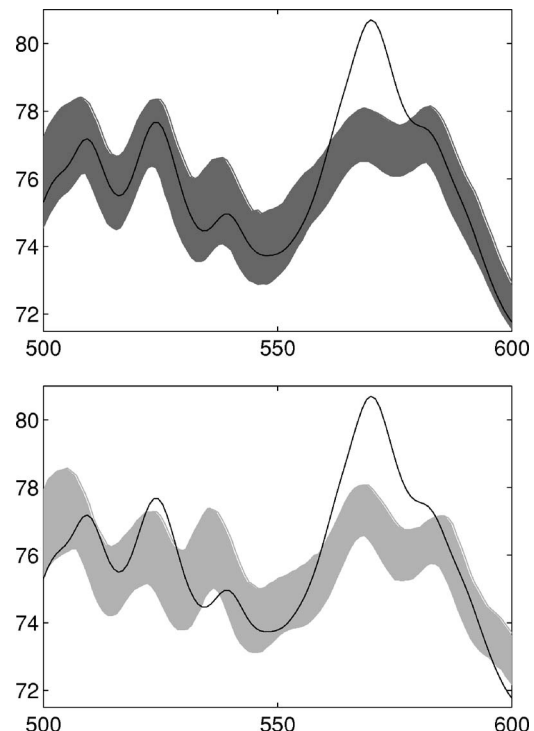


Fig. 9 Case 1. Comparison of the reference solution $\nu \mapsto 20 \log_{10}(w^{\text{master}}(2\pi\nu))$ (thin solid line) with the confidence region (dark gray region) of the random response $\nu \mapsto 20 \log_{10}(W^{\text{D}}(2\pi\nu))$ corresponding to the design optimization and with the confidence region (light gray region) of the random response $\nu \mapsto 20 \log_{10}(W^{\text{RD}}(2\pi\nu))$ corresponding to the robust design optimization. Horizontal axis is the frequency ν in Hz.

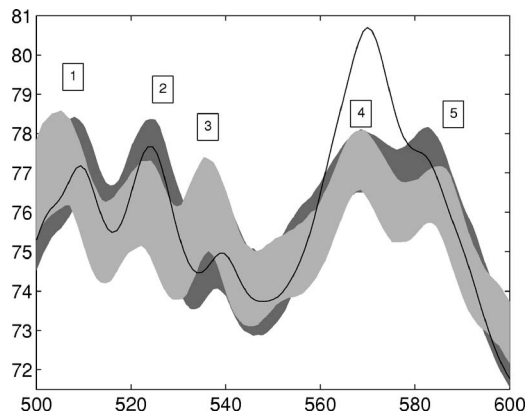


Fig. 10 Case 1. Comparison of the reference solution $\nu \mapsto 20 \log_{10}(w^{\text{master}}(2\pi\nu))$ (thin solid line) with the two confidence regions defined in Fig. 9.

the frequency band [500,550] Hz. Finally, a sensitivity analysis with respect to the level of uncertainty in the dynamical system is carried out. Figure 11 displays the graph of $\delta \mapsto m^{\text{RD}}(\delta)$ for $\alpha=0.1$ ($\circ$), for $\alpha=0.25$ ($\diamond$), and for $\alpha=0.5$ ($\square$).

6.6 Case 2. Robust Design Optimization With Respect to a Multidimensional Design Parameter. In this section, no assumption is made on the mass of the oscillators of each vibration absorbers. Denoting as m_i the mass of the oscillator number i , the design parameter $\mathbf{m}$ is the $\mathbb{R}^5$ vector $\mathbf{m} = (9m_1, 9m_2, 9m_3, 9m_4, 9m_5)$. The admissible set for design parameter $\mathbf{m}$ is such that $\mathbf{m}$ belongs to the five-dimensional hypercube defined by $\mathbf{m} \in [9 \times 10^{-8}, 3.6 \times 10^{-4}]^5$. First, the design optimization (no uncertainties) is carried out for several initial values of $\mathbf{m}$, which are randomly chosen in order to obtain the best local optimum. The design optimization yields an optimal design parameter $\mathbf{m}^D = (4995, 9, 9, 9, 3024) \times 10^{-8}$. This optimal design parameter is then used as an initial value for the robust design optimization problem. The weighting factor α is set to $\alpha=0.5$. Two subcases for which the subsystem is deterministic are presented; (1) a low level of uncertainty in the master system ($\delta_M^l = \delta_D^l = \delta_K^l = 0.05$) yielding $\mathbf{m}^{\text{RD}} = (6894, 9, 9, 9, 3582) \times 10^{-8}$; (2) a medium level of uncertainty in the master system ($\delta_M^l = \delta_D^l = \delta_K^l = 0.15$) yielding $\mathbf{m}^{\text{RD}} = (1422, 9, 9, 9, 1062) \times 10^{-8}$. For each case, Figs. 12 and 13 compare the reference solution $\omega \mapsto 20 \log_{10}(w^{\text{master}}(\omega))$ with the

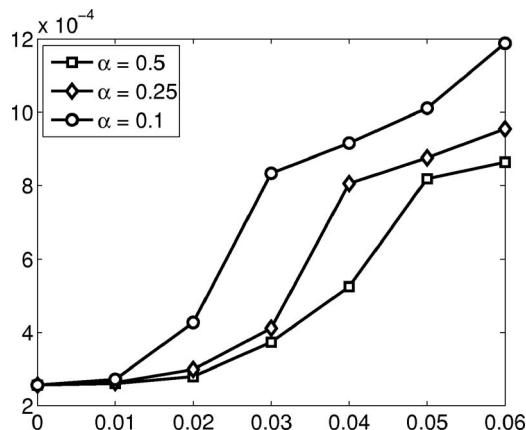


Fig. 11 Case 1. Sensitivity analysis for the robust design optimization: Graph of function $\delta \mapsto m^{\text{RD}}(\delta)$ for $\alpha=0.5$ (thin line with $\square$), for $\alpha=0.25$ (thin line with $\diamond$), and for $\alpha=0.1$ (thin line with $\circ$). Horizontal axis is δ .

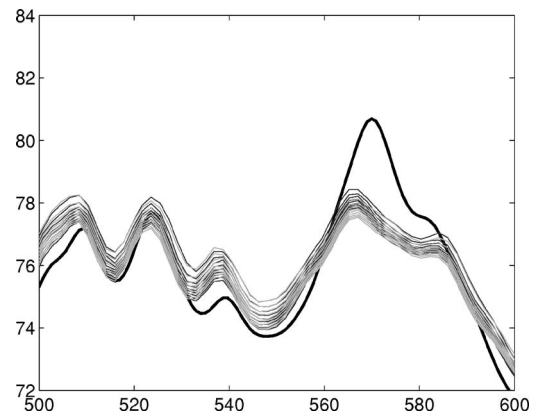


Fig. 12 Case 2. Low uncertainty level. Comparison of the reference solution $\nu \mapsto 20 \log_{10}(w^{\text{master}}(2\pi\nu))$ (thick solid line) with the 10th, 20th, 30th, 40th, 50th, 60th, 70th, 80th, and 90th quantiles of the random response $\nu \mapsto 20 \log_{10}(W^D(2\pi\nu))$ (thin black lines) corresponding to the design optimization and of the random response $\nu \mapsto 20 \log_{10}(W^{\text{RD}}(2\pi\nu))$ (thin gray lines) corresponding to the robust design optimization. Horizontal axis is the frequency ν in Hz.

10th, 20th, 30th, 40th, 50th, 60th, 70th, 80th, and 90th quantiles of the random responses $\omega \mapsto 20 \log_{10}(W^D(\omega))$ corresponding to the design optimization and of the random responses $\omega \mapsto 20 \log_{10}(W^{\text{RD}}(\omega))$ corresponding to the robust design optimization in the frequency band B_1 . For the low uncertainty subcase, it can be seen in Fig. 12 that robust design optimization yields similar results to design optimization except for peak 4 for which the response level is lower that corresponds to the target specification. For the medium uncertainty subcase, Fig. 13 shows that robust design optimization yields similar results to design optimization except for peak 4 for which the response level is lower and for peak 5 for which the response level is higher. In this case, the difference between the two cost functions is very small which means that, for the dynamical system under consideration, the robust design optimization does not improve the design optimization for a higher level of uncertainty. Note that design optimization shift the position of peaks 4 and 5 to the left and to the right, respectively. Finally, it should be noted that both design optimi-

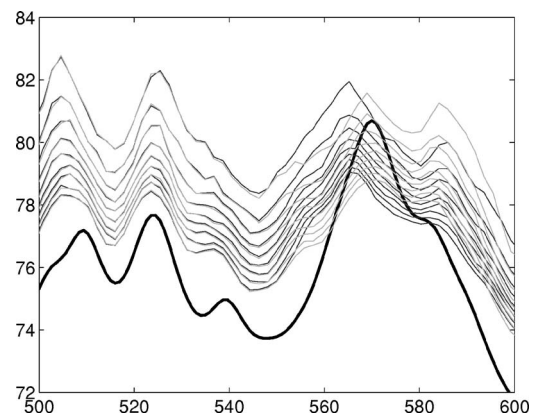


Fig. 13 Case 2. Medium uncertainty level. Comparison of the reference solution $\nu \mapsto 20 \log_{10}(w^{\text{master}}(2\pi\nu))$ (thick solid line) with the 10th, 20th, 30th, 40th, 50th, 60th, 70th, 80th, and 90th quantiles of the random response $\nu \mapsto 20 \log_{10}(W^D(2\pi\nu))$ (thin black lines) corresponding to the design optimization and of the random response $\nu \mapsto 20 \log_{10}(W^{\text{RD}}(2\pi\nu))$ (thin gray lines) corresponding to the robust design optimization. Horizontal axis is the frequency ν in Hz.

zation and robust design optimization have been solved by using a sequential quadratic optimization algorithm. The calculations have been made (1) in supplying the gradients of the cost function to the algorithm and (2) in supplying the gradient and the Hessian of the cost function to the algorithm using the numerical method proposed in the Appendix. Method (2) allows a CPU time gain about 30% to be obtained.

7 Conclusion

In this paper, we have presented an approach that allows the robust design optimization problem to be formulated and solved in presence of model uncertainties. Model uncertainties are taken into account with a nonparametric probabilistic approach and a target is introduced in the cost function. Thanks to an adapted algebraic development and under several assumptions, the numerical optimization problem can be solved with accuracy and with a low numerical cost. The proposed robust design formulation is general enough and the proposed approach can be used in other cases for analyzing complex dynamical systems in computational mechanics. Concerning the limitations of the proposed method, it should be noted that the gradient and the Hessian algebraically calculated can be used if the dynamical stiffness operator and the input force are affine mappings of the vector-valued design parameter and if the projection basis can be constructed independently of the vector-valued design parameter and has reasonable dimension. Note that if these conditions are not satisfied, the theory presented can be used with an optimization algorithm for which either the gradient and Hessian are not given or the gradient and Hessian are numerically calculated. The case of random loads is not investigated in this work but the extension is straightforward. In addition, this paper is focused on robust design with respect to model uncertainties. With respect to potential difficulties induced by the optimization algorithm when the dimension of the vector-valued design parameter is high, the proposed method does not introduce new difficulties with respect to the state of the art. The method has been validated for the five-dimensional case. As a final comment, the two numerical examples presented in this paper show that there are differences between design optimization and robust design optimization for the structural dynamics problem considered.

Acknowledgment

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Appendix: Numerical Analysis Related to the Optimization

In this Appendix, we summarize the algebraic development allowing the gradient and the Hessian of cost function $j(\mathbf{p}, \alpha)$ related to the robust design optimization problem to be computed. Such an algebraic development allows the precision in the optimization algorithm to be increased. The developments presented in this Appendix are valuable under the following assumptions: (1) the dynamic stiffness operator and the force vector are affine functions of the design parameter, and (2) the projection basis related to the dynamical substructuring reduction method does not depend of the design parameter.

1 Notation

Let $[A]$ and $[B]$ be $m \times n$ and $p \times q$ complex matrices. The operator $\star$ is introduced such that $[A] \star [B]$ is the $mp \times nq$ complex matrix defined by

$$[A] \star [B] = \begin{bmatrix} [A]_{11} & [B] & \dots & [A]_{1n} & [B] \\ \vdots & & \ddots & \vdots & \\ [A]_{m1} & [B] & \dots & [A]_{mn} & [B] \end{bmatrix} \quad (A1)$$

Let $\mathbf{b}(\mathbf{p})$ be a vector in $\mathbb{C}^n$ and let $[A(\mathbf{p})]$ be an $m \times n$ complex matrix, which depends on the vector $\mathbf{p} = (p_1, \dots, p_s)$ in $\mathbb{C}^s$. Below, when no confusion is possible, the argument $\mathbf{p}$ is removed in order to simplify the writing.

The vector $\mathbf{g}_b(\mathbf{p})$ is the vector in $\mathbb{C}^{ns}$ defined by

$$\mathbf{g}_b(\mathbf{p}) = \{\partial_{p_1} \mathbf{b}, \dots, \partial_{p_s} \mathbf{b}\}$$

The vector $\mathbf{h}_b(\mathbf{p})$ is the vector in $\mathbb{C}^{ns(s+1)/2}$ defined by

$$\mathbf{h}_b(\mathbf{p}) = \{\partial_{p_1}^2 \mathbf{b}, \dots, \partial_{p_1 p_s}^2 \mathbf{b}, \partial_{p_2}^2 \mathbf{b}, \dots, \partial_{p_2 p_s}^2 \mathbf{b}, \dots, \partial_{p_s}^2 \mathbf{b}\}$$

The vector $\mathbf{i}_{[A]b}$ is the vector in $\mathbb{C}^{ms}$ defined by

$$\mathbf{i}_{[A]b} = \{(\partial_{p_1} [A]) \mathbf{b}, \dots, (\partial_{p_s} [A]) \mathbf{b}\}$$

The vector $\mathbf{j}_{[A]b}(\mathbf{p})$ is the vector in $\mathbb{C}^{ms(s+1)/2}$ defined by

$$\mathbf{j}_{[A]b} = \{(\partial_{p_1} [A]) \partial_{p_1} \mathbf{b}, \dots, (\partial_{p_1} [A]) \partial_{p_s} \mathbf{b}, (\partial_{p_2} [A]) \partial_{p_2} \mathbf{b}, \dots, (\partial_{p_2} [A]) \partial_{p_s} \mathbf{b}, \dots, (\partial_{p_s} [A]) \partial_{p_s} \mathbf{b}\}$$

The vector $\mathbf{k}_{[A]b}(\mathbf{p})$ is the vector in $\mathbb{C}^{ms(s+1)/2}$ defined by

$$\mathbf{k}_{[A]b} = \{(\partial_{p_1}^2 [A]) \mathbf{b}, \dots, (\partial_{p_1 p_s}^2 [A]) \mathbf{b}, (\partial_{p_2}^2 [A]) \mathbf{b}, \dots, (\partial_{p_2 p_s}^2 [A]) \mathbf{b}, \dots, (\partial_{p_s}^2 [A]) \mathbf{b}\}$$

2 Gradient and Hessian of the Cost Function for the Stochastic Dynamical System

By using Eqs. (20), the first-order and the second-order partial derivatives of the cost function $j(\mathbf{p}, \alpha)$ with respect to $\mathbf{p}$ are written as

$$\partial_{p_i} j(\mathbf{p}, \alpha) = \frac{1}{\|\mathbf{g}\|_{B_1}} \mathcal{E} \left\{ 2 \operatorname{Re} \int_{B_1} \alpha (\mathbf{W}(\mathbf{p}, \omega) - \mathbf{g}(\omega))^* \partial_{p_i} \mathbf{W}(\mathbf{p}, \omega) + (1 - 2\alpha) (\mathbf{W}(\mathbf{p}, \omega) - \boldsymbol{\mu}(\mathbf{p}, \omega))^* \partial_{p_i} (\mathbf{W}(\mathbf{p}, \omega) - \boldsymbol{\mu}(\mathbf{p}, \omega)) d\omega \right\} \quad (A2)$$

$$\begin{aligned} \partial_{p_i p_j}^2 j(\mathbf{p}, \alpha) = \frac{1}{\|\mathbf{g}\|_{B_1}} \mathcal{E} \left\{ 2 \operatorname{Re} \int_{B_1} \alpha (\mathbf{W}(\mathbf{p}, \omega) - \mathbf{g}(\omega))^* \partial_{p_i p_j}^2 \mathbf{W}(\mathbf{p}, \omega) \right. \\ + \alpha \partial_{p_j} \mathbf{W}(\mathbf{p}, \omega)^* \partial_{p_i} \mathbf{W}(\mathbf{p}, \omega) + (1 - 2\alpha) (\mathbf{W}(\mathbf{p}, \omega) - \boldsymbol{\mu}(\mathbf{p}, \omega))^* \partial_{p_i p_j}^2 (\mathbf{W}(\mathbf{p}, \omega) - \boldsymbol{\mu}(\mathbf{p}, \omega)) + (1 \\ - 2\alpha) \partial_{p_j} (\mathbf{W}(\mathbf{p}, \omega) - \mathbf{g}(\omega))^* \partial_{p_i} (\mathbf{W}(\mathbf{p}, \omega) - \boldsymbol{\mu}(\mathbf{p}, \omega)) d\omega \left. \right\} \quad (A3) \end{aligned}$$

with summation over indices k , in which $\mathcal{E}$ is the mathematical expectation, Re is the real part of a complex number, and where α

starred vector is its transconjugate. By using Eqs. (14) and (18), it can easily be shown that the $\mathbb{C}^k$ -valued random vectors $\mathbf{W}(\mathbf{p}, \omega)$, $\partial_{p_i} \mathbf{W}(\mathbf{p}, \omega)$ for $i \in \{1, \dots, s\}$, $\partial_{p_i p_j}^2 \mathbf{W}(\mathbf{p}, \omega)$ for $1 \leq i \leq j \leq s$ can be calculated from the $\mathbb{C}^N$ -valued random vectors $\mathbf{Q}(\mathbf{p}, \omega)$, $\partial_{p_i} \mathbf{Q}(\mathbf{p}, \omega)$ for $i \in \{1, \dots, s\}$, $\partial_{p_i p_j}^2 \mathbf{Q}(\mathbf{p}, \omega)$ for $1 \leq i \leq j \leq s$.

3 Gradient and Hessian of the Random Vector of the Random Reduced Coordinates

Let $\mathbf{Q}(\mathbf{p}, \omega)$ be the $\mathbb{C}^{NS}$ -valued random vector defined by

$$\mathbf{Q}(\mathbf{p}, \omega) = \begin{bmatrix} \mathbf{Q}(\mathbf{p}, \omega) \\ \mathbf{g}_{\mathbf{Q}}(\mathbf{p}, \omega) \\ \mathbf{h}_{\mathbf{Q}}(\mathbf{p}, \omega) \end{bmatrix} \quad (\text{A4})$$

in which $S=1+3s/2+5s^2/2$ and N is the dimension of vector $\mathbf{Q}(\mathbf{p}, \omega)$. Let $[\mathcal{A}_{\text{red}}(\mathbf{p}, \omega)]$ be the random matrix with values in the set of all the symmetric $NS \times NS$ complex matrices and let $\mathcal{F}_{\text{red}}(\mathbf{p}, \omega)$ be the $\mathbb{C}^{NS}$ -valued random vector such that

$$[\mathcal{A}_{\text{red}}(\mathbf{p}, \omega)] = [I_S] \star ([\mathbf{A}_{\text{red}}^1(\omega)] + [\mathbf{A}_{\text{red}}^2(\mathbf{p}, \omega)]) \quad (\text{A5})$$

$$\mathcal{F}_{\text{red}}(\mathbf{p}, \omega) = \begin{bmatrix} \mathbf{f}_{\text{red}}(\mathbf{p}, \omega) \\ \mathbf{g}_{\text{red}}^{\mathbf{f}}(\mathbf{p}, \omega) - \mathbf{i}_{[\mathbf{A}_{\text{red}}^2]} \mathbf{Q}(\mathbf{p}, \omega) \\ -\mathbf{j}_{[\mathbf{A}_{\text{red}}^2]} \mathbf{Q}(\mathbf{p}, \omega) - \mathbf{k}_{[\mathbf{A}_{\text{red}}^2]} \mathbf{Q}(\mathbf{p}, \omega) \end{bmatrix} \quad (\text{A6})$$

in which $[I_S]$ is the $S \times S$ identity matrix. From Eq. (15) and taking into account that $[\mathbf{A}_{\text{red}}^1(\omega)]$ is independent of $\mathbf{p}$, it can be shown that the random vector $\mathbf{Q}(\mathbf{p}, \omega)$ is solution of the random matrix equation

$$[\mathcal{A}_{\text{red}}(\mathbf{p}, \omega)] \mathbf{Q}(\mathbf{p}, \omega) = \mathcal{F}_{\text{red}}(\mathbf{p}, \omega) \quad (\text{A7})$$

It should be noted that the calculation of $\mathbf{i}_{[\mathbf{A}_{\text{red}}^2]} \mathbf{Q}(\mathbf{p}, \omega)$, $\mathbf{j}_{[\mathbf{A}_{\text{red}}^2]} \mathbf{Q}(\mathbf{p}, \omega)$, and $\mathbf{k}_{[\mathbf{A}_{\text{red}}^2]} \mathbf{Q}(\mathbf{p}, \omega)$ requires the calculation of the partial derivatives $\partial_{p_i} [\mathbf{A}_{\text{red}}^2(\mathbf{p}, \omega)]$ for $i = \{1, \dots, s\}$ and $\partial_{p_i p_j}^2 [\mathbf{A}_{\text{red}}^2(\mathbf{p}, \omega)]$ for $1 \leq i \leq j \leq s$. This calculation is carried out in the next section.

4 Gradient of the Random Reduced Dynamical Stiffness Matrix for the Stochastic Dynamical System

The method allowing the partial derivative $\partial_{p_i} [\mathbf{A}_{\text{red}}^2(\mathbf{p}, \omega)]$ to be algebraically calculated for $i = \{1, \dots, s\}$ is presented below. From Eqs. (16) and (17), it can be seen that the calculation of $\partial_{p_i} [\mathbf{A}_{\text{red}}^2(\mathbf{p}, \omega)]$ requires the calculation of $\partial_{p_i} [\mathbf{M}_s^2(\mathbf{p})]$, $\partial_{p_i} [\mathbf{D}_s^2(\mathbf{p})]$, and $\partial_{p_i} [\mathbf{K}_s^2(\mathbf{p})]$. Rewriting Eq. (12) for matrices $[\mathbf{M}_s^2(\mathbf{p})]$, $[\mathbf{D}_s^2(\mathbf{p})]$, and $[\mathbf{K}_s^2(\mathbf{p})]$ yields

$$[\mathbf{E}_s^2(\mathbf{p})] = [\mathbf{L}_E^2(\mathbf{p})]^T [\mathbf{G}_E^2] [\mathbf{L}_E^2(\mathbf{p})] \quad (\text{A8})$$

in which $[\mathbf{E}_s^2(\mathbf{p})]$ denotes $\{[\mathbf{M}_s^2(\mathbf{p})], [\mathbf{D}_s^2(\mathbf{p})], [\mathbf{K}_s^2(\mathbf{p})], [\mathbf{L}_E^2(\mathbf{p})]\}$ denotes $\{[\mathbf{L}_M^2(\mathbf{p})], [\mathbf{L}_D^2(\mathbf{p})], [\mathbf{L}_K^2(\mathbf{p})]\}$, and $[\mathbf{G}_E^2]$ denotes $\{[\mathbf{G}_M^2], [\mathbf{G}_D^2], [\mathbf{G}_K^2]\}$. In Eq. (A8), the matrix $[\mathbf{G}_M^2]$ is a random matrix with values in the set of all the symmetric $N_2 \times N_2$ positive definite matrices and matrices $[\mathbf{G}_D^2]$, $[\mathbf{G}_K^2]$ are random matrices with values in the set of all the symmetric $(N_2 - r) \times (N_2 - r)$ positive definite matrices. Matrix $[\mathbf{L}_E^2(\mathbf{p})]$ is such that

$$[\mathbf{E}_s^2(\mathbf{p})] = [\mathbf{L}_E^2(\mathbf{p})]^T [\mathbf{L}_E^2(\mathbf{p})] \quad (\text{A9})$$

in which $[\mathbf{L}_M^2(\mathbf{p})]$ is a $N_2 \times N_2$ upper triangular real matrix obtained by Cholesky factorization and where $[\mathbf{L}_D^2(\mathbf{p})]$, $[\mathbf{L}_K^2(\mathbf{p})]$ are $N_2 \times (N_2 - r)$ real matrices. In addition, according to Eq. (3), the matrix $[\mathbf{E}_s^2(\mathbf{p})]$ can be written as

$$[\mathbf{E}_s^2(\mathbf{p})] = [\mathbf{E}_s^{2,0}] + \sum_{i=1}^s p_i [\mathbf{E}_s^{2,i}] \quad (\text{A10})$$

Taking the partial derivative of Eq. (A8) with respect to p_i , the matrix $\partial_{p_i} [\mathbf{E}_s^2(\mathbf{p})]$ is given by

$$\partial_{p_i} [\mathbf{E}_s^2(\mathbf{p})] = \{\partial_{p_i} [\mathbf{L}_E^2(\mathbf{p})]^T\} [\mathbf{G}_E^2] [\mathbf{L}_E^2(\mathbf{p})] + [\mathbf{L}_E^2(\mathbf{p})]^T \{\partial_{p_i} [\mathbf{G}_E^2]\} [\mathbf{L}_E^2(\mathbf{p})] \quad (\text{A11})$$

Using Eqs. (A9) and (A10), the matrix $\partial_{p_i} [\mathbf{L}_E^2(\mathbf{p})]$ is computed in solving the following matrix equation:

$$[\mathbf{E}_s^{2,i}] = \{\partial_{p_i} [\mathbf{L}_E^2(\mathbf{p})]^T\} [\mathbf{L}_E^2(\mathbf{p})] + [\mathbf{L}_E^2(\mathbf{p})]^T \{\partial_{p_i} [\mathbf{L}_E^2(\mathbf{p})]\} \quad (\text{A12})$$

5 Hessian of the Random Reduced Dynamical Stiffness Matrix for the Stochastic Dynamical System

The method allowing the partial derivative $\partial_{p_i p_j}^2 [\mathbf{A}_{\text{red}}^2(\mathbf{p}, \omega)]$ to be algebraically calculated for $1 \leq i \leq j \leq s$ is presented. Taking the partial derivative of Eq. (A11) with respect to p_j , the matrix $\partial_{p_i p_j}^2 [\mathbf{E}_s^2(\mathbf{p})]$ is given by

$$\begin{aligned} \partial_{p_i p_j}^2 [\mathbf{E}_s^2(\mathbf{p})] &= \{\partial_{p_i p_j}^2 [\mathbf{L}_E^2(\mathbf{p})]^T\} [\mathbf{G}_E^2] [\mathbf{L}_E^2(\mathbf{p})] + \{\partial_{p_i} [\mathbf{L}_E^2(\mathbf{p})]^T\} \{\partial_{p_j} [\mathbf{G}_E^2]\} [\mathbf{L}_E^2(\mathbf{p})] \\ &\quad \times \{\partial_{p_j} [\mathbf{L}_E^2(\mathbf{p})]\} + \{\partial_{p_j} [\mathbf{L}_E^2(\mathbf{p})]^T\} \{\partial_{p_i} [\mathbf{G}_E^2]\} [\mathbf{L}_E^2(\mathbf{p})] \\ &\quad + [\mathbf{L}_E^2(\mathbf{p})]^T \{\partial_{p_i p_j}^2 [\mathbf{G}_E^2]\} [\mathbf{L}_E^2(\mathbf{p})] \end{aligned} \quad (\text{A13})$$

Taking the partial derivative of Eq. (A12), the matrix $\partial_{p_i p_j}^2 [\mathbf{L}_E^2(\mathbf{p})]$ is computed by solving the following matrix equation:

$$\begin{aligned} &-\{\partial_{p_i} [\mathbf{L}_E^2(\mathbf{p})]^T\} \{\partial_{p_j} [\mathbf{L}_E^2(\mathbf{p})]\} - \{\partial_{p_j} [\mathbf{L}_E^2(\mathbf{p})]^T\} \{\partial_{p_i} [\mathbf{L}_E^2(\mathbf{p})]\} \\ &= \{\partial_{p_i p_j}^2 [\mathbf{L}_E^2(\mathbf{p})]^T\} [\mathbf{L}_E^2(\mathbf{p})] + [\mathbf{L}_E^2(\mathbf{p})]^T \{\partial_{p_i p_j}^2 [\mathbf{L}_E^2(\mathbf{p})]\} \end{aligned} \quad (\text{A14})$$

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The Support Reaction of a Simply Supported and Uniformly Loaded Thin Circular Aeolotropic Plate

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A previous analytical solution of the deflection of a thin circular aeolotropic plate, with simply supported edge and uniform lateral load, has been used to derive approximate series expressions for the plate support reaction, which are directly applicable in practice. The support reaction, which has been calculated for some typical anisotropic materials of varying degree of anisotropy, varies significantly along the plate perimeter and strongly anisotropic materials require in general a higher order series solution. Certain solution constants of previous deflection approximations were not found to harmonize and are therefore recalculated. [DOI: 10.1115/1.2775494]

Keywords: plate, anisotropic, simply supported, circular

1 Introduction

The Kirchhoff bending theory applied to a thin circular anisotropic plate, clamped on the boundary, and carrying a uniform lateral load is surprisingly simple and straightforward, see, e.g., Timoshenko and Woinowsky-Krieger [1]. This is also true for an elliptic plate in which the material and geometric principal axes coincide. The analysis of a simply supported circular anisotropic plate, on the other hand, is much more complicated. Okubo [2] applied the complex variable method of Morkovin [3] to derive approximate expressions for the deflection of a circular plate and related items, e.g., bending moments and their extrema, bending stresses, etc., but he did not solve for the support reaction, which would have completed the analysis of the plate problem considered. The aim of the present work is to fill this gap and to this end approximate expressions of the support reaction of a thin circular aeolotropic plate, with simply supported edge and uniform lateral load, have been derived. For any set of plate characteristics (i.e., bending rigidities, plate parameters, and a uniform load), the support reaction can be calculated through numerical evaluation of an explicit expression. There are two steps in the solution procedure: In the first, Okubo's plate problem is solved in order to obtain a set of plate parameters and in the second step, the present work, these parameters and the set of plate characteristics are used to obtain the support reaction. The entire procedure assumes familiarity with Okubo's theory.

The solution procedure can in practice be applied to, e.g., anisotropic structural components in the form of an isotropic plate reinforced with parallel stiffeners, as such components with reasonable approximation can be analyzed within the theory of anisotropic thin plates. An engineering problem of this kind originally initiated this work and a satisfactory solution was found through application of the present procedure. The support reaction is dependent on position and therefore the extreme values and their position on the plate perimeter were of primary interest in the case concerned.

The work of Okubo [2] seems to have gone unnoticed for some time but a recent application can be found in Ref. [4] where stresses in circular aeolotropic plates, subjected to a uniform lateral load, have been calculated and a later extension in Ref. [5] where the deflection of an elliptic plate has been rederived, con-

trasting a previous solution by Ohasi [6]. In the course of the analysis, it was further found that certain results in Ref. [2] could not be reproduced, in that different sets of solution constants were not found to harmonize. The results of the pertinent recalculations are given in the Appendix.

2 Plate Equation and Its Solution

For a material with three planes of elastic symmetry, the plate equation [1] takes the form

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2(D_2 + D_4) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_3 \frac{\partial^4 w}{\partial y^4} = q \quad (1)$$

in Okubo's notation. Here x, y are Cartesian coordinates, w the deflection of the plate, q the intensity of the lateral load, and D_i , $i=1, \dots, 4$ bending rigidities. The notation of Okubo is retained for simplicity and to facilitate comparison with his work. In current notation,

$$D_{11} = D_1 \quad D_{12} = D_2 \quad D_{22} = D_3 \quad D_{66} = D_4/2 \quad (2)$$

A solution to Eq. (1) can be found by using the complex variable method [3]. A special case of the general form of the homogeneous part w_H of w , used by Okubo, to obtain the deflection function for a thin, circular plate of unit radius, unit thickness, and unit applied load intensity, is

$$w_H = \text{Re}[f_1(x + ik_1y) + f_2(x + ik_2y)] \quad (3)$$

where f_1, f_2 are the arbitrary functions and k_1, k_2 the roots of the characteristic equation

$$D_1 - 2(D_2 + D_4)k^2 + D_3k^4 = 0 \quad (4)$$

obtained through substitution of Eq. (3) in Eq. (1). The general solution to Eq. (1) is then

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$$\begin{aligned}
w = w_H + w_P = & \frac{c'^2}{4} \sum_{n=2}^{\infty} A_n \left\{ \frac{\cosh(2n+2)\alpha'}{(2n+2)(2n+1)} \cos(2n+2)\beta - \left[\frac{1}{(2n+1)2n} + \frac{1}{2n(2n-1)} \right] \right. \\
& \times \cosh 2n\alpha' \cos 2n\beta + \frac{\cosh(2n-2)\alpha'}{(2n-1)(2n-2)} \cos(2n-2)\beta \left. \right\} + \frac{c''^2}{4} \sum_{n=2}^{\infty} B_n \left\{ \frac{\cosh(2n+2)\alpha''}{(2n+2)(2n+1)} \cos(2n+2)\beta \right. \\
& - \left[\frac{1}{(2n+1)2n} + \frac{1}{2n(2n-1)} \right] \cosh 2n\alpha'' \cos 2n\beta + \frac{\cosh(2n-2)\alpha''}{(2n-1)(2n-2)} \cos(2n-2)\beta \left. \right\} + C_1 x^4 + C_2 x^2 y^2 + C_3 y^4 + C_4 x^2 + C_5 y^2 + C_6
\end{aligned} \quad (5)$$

where w_P is a particular solution, given on the last line in Eq. (5). The angle β is taken positive counter clockwise with $\beta=0$ on the positive x axis. The constants c', c'' in Eq. (5) and constants a', a'' to appear below are determined from k_1, k_2 and geometrical conditions in mapping the Cartesian coordinates onto two sets of curvilinear coordinates α', β and α'', β , while C_1-C_6, A_n, B_n are determined by the boundary conditions of the plate, see Ref. [2] for derivation details.

Okubo's solution is applicable to the special case in which material principal axes and geometrical axes coincide. This condition is trivial for a circular plate. Okubo also determined the plate deflection along the x and y axes, respectively, bending moments, stresses and related items, etc., except the support reaction, which is dealt with in the next section.

3 Plate Support Reaction

The equivalent shearing force at the boundary of a plate is [1]

$$V'_\alpha = Q_\alpha - \frac{\partial M_{\alpha\beta}}{a \partial \beta} \quad (6)$$

Here, α and β are the normal and tangential directions to the boundary of the plate, respectively, and a the plate radius, equal to unity for a plate of unit radius.

$$Q_\alpha = Q_x \cos \beta + Q_y \sin \beta \quad (7)$$

and

$$M_{\alpha\beta} = (M_x - M_y) \sin \beta \cos \beta + M_{xy} (\cos^2 \beta - \sin^2 \beta) \quad (8)$$

The shearing forces are

$$Q_x = -\frac{\partial}{\partial x} \left[D_1 \frac{\partial^2 w}{\partial x^2} + (D_2 + D_4) \frac{\partial^2 w}{\partial y^2} \right] \quad (9a)$$

$$Q_y = -\frac{\partial}{\partial y} \left[D_3 \frac{\partial^2 w}{\partial y^2} + (D_2 + D_4) \frac{\partial^2 w}{\partial x^2} \right] \quad (9b)$$

and the bending and twisting moments

$$M_x = -\left(D_1 \frac{\partial^2 w}{\partial x^2} + D_2 \frac{\partial^2 w}{\partial y^2} \right) \quad (10a)$$

$$M_y = -\left(D_3 \frac{\partial^2 w}{\partial y^2} + D_2 \frac{\partial^2 w}{\partial x^2} \right) \quad (10b)$$

$$M_{xy} = D_4 \frac{\partial^2 w}{\partial x \partial y} \quad (10c)$$

in which the curvatures are [2]

$$\begin{aligned}
\frac{\partial^2 w}{\partial x^2} = & \sum_{n=2}^{\infty} (A_n \cosh 2na' + B_n \cosh 2na'') \cos 2n\beta + 6C_1 + C_2 \\
& + (6C_1 - C_2) \cos 2\beta + 2C_4
\end{aligned} \quad (11a)$$

$$\begin{aligned}
\frac{\partial^2 w}{\partial y^2} = & -\sum_{n=2}^{\infty} (A_n k_1^2 \cosh 2na' + B_n k_2^2 \cosh 2na'') \cos 2n\beta + 6C_3 + C_2 \\
& - (6C_3 - C_2) \cos 2\beta + 2C_5
\end{aligned} \quad (11b)$$

$$\begin{aligned}
\frac{\partial^2 w}{\partial x \partial y} = & -\sum_{n=2}^{\infty} (A_n k_1 \sinh 2na' + B_n k_2 \sinh 2na'') \sin 2n\beta \\
& + 2C_2 \sin 2\beta
\end{aligned} \quad (11c)$$

The auxiliary complex variable $\zeta = \alpha + i\beta$ admits the identity

$$x +iky = c \cosh \zeta \quad (12)$$

cf. Eqs. (5) in Ref. [2]. From Eq. (12), the derivatives

$$\frac{\partial \zeta}{\partial x} = \frac{1}{c \sinh \zeta} \quad (13a)$$

and

$$\frac{\partial \zeta}{\partial y} = \frac{ik}{c \sinh \zeta} \quad (13b)$$

are obtained. The partial derivatives of the deflection function in Q_x and Q_y of the form $(\cosh 2n\alpha \cos 2n\beta)_{,x}$, etc., where $()_{,x}$ denotes differentiation with respect to x , are complex, and can conveniently be obtained through differentiation of an associated complex function. A suitable associated function is $\cosh 2n\zeta$, since its real part is $\cosh 2n\alpha \cos 2n\beta$. For example, for $n=2$, we have, on the one hand,

$$(\cosh 4\zeta)_{,x} = 4 \frac{\partial \zeta}{\partial x} \sinh 4\zeta = \frac{16}{c} \cosh 2\zeta \cosh \zeta \quad (14)$$

where

$$\begin{aligned}
\cosh 2\zeta \cosh \zeta = & 1[(\cosh 2\alpha \cos 2\beta)(\cosh \alpha \cos \beta) \\
& - (\sinh 2\alpha \sin 2\beta)(\sinh \alpha \sin \beta)] \\
& + i[(\cosh 2\alpha \cos 2\beta)(\sinh \alpha \sin \beta) \\
& + (\sinh 2\alpha \sin 2\beta)(\cosh \alpha \cos \beta)]
\end{aligned} \quad (15)$$

Note that the factors of 1 and i in this expression are complex and do not denote the real and imaginary parts, respectively, of the derivative $(\cosh 4\zeta)_{,x}$.

On the other hand, we have

$$\begin{aligned}
(\cosh 4\zeta)_{,x} = & (\cosh 4\alpha \cos 4\beta + i \sinh 4\alpha \sin 4\beta)_{,x} \\
= & (\cosh 4\alpha \cos 4\beta)_{,x} + i(\sinh 4\alpha \sin 4\beta)_{,x}
\end{aligned} \quad (16)$$

Identification of corresponding terms yields

$$\begin{aligned}
(\cosh 4\alpha \cos 4\beta)_{,x} = & \frac{16}{c} [(\cosh 2\alpha \cos 2\beta)(\cosh \alpha \cos \beta) \\
& - (\sinh 2\alpha \sin 2\beta)(\sinh \alpha \sin \beta)]
\end{aligned} \quad (17a)$$

and similarly for the y derivative

$$(\cosh 4\alpha \cos 4\beta)_{,y} = -k \frac{16}{c} [(\cosh 2\alpha \cos 2\beta)(\sinh \alpha \sin \beta) + (\sinh 2\alpha \sin 2\beta)(\cosh \alpha \cos \beta)] \quad (17b)$$

and so on for functions of increasing n .

In this procedure, we have exploited the fact that the rule of addition for the hyperbolic cosine function of a complex argument, on the one hand, and differentiation, on the other, are commutative. The manipulation of the function $\cosh 2n\zeta$ for increasing n is cumbersome to generalize and individual manipulations are often most convenient. A common feature, however, is that $\sinh \zeta$, the denominator in Eqs. (13a) and (13b), is always a factor of $\sinh 2n\zeta$.

Substitution of Eqs. (9) and (11) in Eq. (7) and using the procedure given by Eqs. (12)–(16), (17a), and (17b) yield

$$Q_\alpha = -4[3D_1C_1 + (D_2 + D_4)C_2 + 3D_3C_3 + 3(D_1C_1 - D_3C_3)\cos 2\beta] - \frac{2}{c'}[D_1 - k_1^2(D_2 + D_4)] \sum_{n=2}^{\infty} 2nA_n g_n(a', \beta) - \frac{2}{c''}[D_1 - k_2^2(D_2 + D_4)] \sum_{n=2}^{\infty} 2nB_n g_n(a'', \beta) - \frac{2k_1}{c'}[k_1^2D_3 - (D_2 + D_4)] \sum_{n=2}^{\infty} 2nA_n h_n(a', \beta) - \frac{2k_2}{c''}[k_2^2D_3 - (D_2 + D_4)] \sum_{n=2}^{\infty} 2nB_n h_n(a'', \beta) \quad (18)$$

$$+ D_4)] \sum_{n=2}^{\infty} 2nB_n h_n(a'', \beta) = -4[3D_1C_1 + (D_2 + D_4)C_2 + 3D_3C_3 + 3(D_1C_1 - D_3C_3)\cos 2\beta] - \sum (Q_\alpha)_n \quad (18)$$

where the two first terms of the functions g_n and h_n are

$$g_2(\alpha, \beta) = 2 \cosh 2\alpha \cosh \alpha \cos 2\beta \cos^2 \beta - \sinh 2\alpha \sinh \alpha \sin^2 2\beta \quad (19a)$$

$$g_3(\alpha, \beta) = (2 \cosh 4\alpha \cos 4\beta + 1) \cosh \alpha \cos^2 \beta - \sinh 4\alpha \sinh \alpha \sin 4\beta \sin 2\beta \quad (19b)$$

$$h_2(\alpha, \beta) = 2 \cosh 2\alpha \sinh \alpha \cos 2\beta \sin^2 \beta + \sinh 2\alpha \cosh \alpha \sin^2 2\beta \quad (19c)$$

$$h_3(\alpha, \beta) = (2 \cosh 4\alpha \cos 4\beta + 1) \sinh \alpha \sin^2 \beta + \sinh 4\alpha \cosh \alpha \sin 4\beta \sin 2\beta \quad (19d)$$

Further, Eqs. (10) and (11) inserted in Eq. (8) yield the twisting moment on the boundary

$$M_{\alpha\beta} = \frac{1}{2} \left[(D_2 - D_1) \frac{\partial^2 w}{\partial x^2} + (D_3 - D_2) \frac{\partial^2 w}{\partial y^2} \right] \sin 2\beta + D_4 \frac{\partial^2 w}{\partial x \partial y} \cos 2\beta \quad (20)$$

from which is obtained the derivative along the boundary

$$\begin{aligned} \frac{\partial M_{\alpha\beta}}{\partial \beta} = & [6(D_2 - D_1)C_1 + (D_3 - D_1)C_2 + 6(D_3 - D_2)C_3 + 2(D_2 - D_1)C_4 + 2(D_3 - D_2)C_5] \cos 2\beta + [6(D_2 - D_1)C_1 + (D_1 - 2D_2 + D_3 + 4D_4)C_2 + 6(D_2 - D_3)C_3] \cos 4\beta - \frac{1}{4} \left(\sum_{n=2}^{\infty} (2n-2)A_n \{ [D_2 - D_1 - k_1^2(D_3 - D_2)] \cosh 2na' + 2D_4k_1 \sinh 2na' \} + \sum_{n=2}^{\infty} (2n-2)B_n \{ [D_2 - D_1 - k_2^2(D_3 - D_2)] \cosh 2na'' + 2D_4k_2 \sinh 2na'' \} \right) \cos(2n-2)\beta + \frac{1}{4} \left(\sum_{n=2}^{\infty} (2n+2)A_n \{ [D_2 - D_1 - k_1^2(D_3 - D_2)] \cosh 2na' - 2D_4k_1 \sinh 2na' \} + \sum_{n=2}^{\infty} (2n+2)B_n \{ [D_2 - D_1 - k_2^2(D_3 - D_2)] \cosh 2na'' - 2D_4k_2 \sinh 2na'' \} \right) \cos(2n+2)\beta \end{aligned} \quad (21)$$

for a plate of unit radius.

Finally, the support reaction $V_\alpha = -V'_\alpha$ is

$$V_\alpha = -Q_\alpha + \frac{\partial M_{\alpha\beta}}{\partial \beta} = 4[3D_1C_1 + (D_2 + D_4)C_2 + 3D_3C_3] + [6(D_1 + D_2)C_1 + (D_3 - D_1)C_2 - 6(D_3 + D_2)C_3 + 2(D_2 - D_1)C_4 + 2(D_3 - D_2)C_5] \cos 2\beta + [6(D_2 - D_1)C_1 + (D_1 - 2D_2 + D_3 + 4D_4)C_2 + 6(D_2 - D_3)C_3] \cos 4\beta + \sum (Q_\alpha)_n + \sum (\partial M_{\alpha\beta} / \partial \beta)_n \quad (22)$$

where $\sum (Q_\alpha)_n$ and $\sum (\partial M_{\alpha\beta} / \partial \beta)_n$ denote the summation terms in Eqs. (18) and (21), respectively.

The support reaction V_α for a given material is calculated in two steps. In the first step, the constants k_1, a', c' , etc., are determined and then the equation system consisting of Eqs. (10)–(12) in Ref. [12], which represents the simply supported boundary con-

ditions of vanishing deflection and moment along the plate edge, is solved in order to obtain the constants $C_i, i = 1, \dots, 6$ and A_n, B_n , for $n \geq 2$ until some convergence criterion is fulfilled. The subsequent solutions of increasing order n , which ideally converge toward a hypothetical exact solution, are approximate in that the boundary condition of vanishing radial bending moment on the plate boundary is not fully satisfied in Okubo's equation system, leaving a residual moment

$$\begin{aligned} M_{\text{res}} = & -\frac{1}{2} \left[A_n \left(\frac{1}{2} [D_1 - D_2 - k_1^2(D_2 - D_3)] \cosh 2na' \right) + k_1 D_4 \sinh 2na' \right] + B_n \left(\frac{1}{2} [D_1 - D_2 - k_2^2(D_2 - D_3)] \cosh 2na'' + k_2 D_4 \sinh 2na'' \right) \cos 2n\beta \\ = & f_{\text{res}} \cos 2n\beta \end{aligned} \quad (23)$$

Table 1 Materials and material properties

(a) Nickel aluminide [7]				
$C_{11}=211.5$	$C_{12}=143.2$	$C_{44}=112.1$	(GPa)	
(b) PZT-5H [8]				
$C_{11}=126$	$C_{12}=55$	$C_{13}=53$	$C_{33}=117$	$C_{44}=35.3$ (GPa)
(c) PZT-4 [9]				
$C_{11}=139$	$C_{12}=77.8$	$C_{13}=74.3$	$C_{33}=113$	$C_{44}=25.6$ (GPa)
(d) Orthotropic [10]				
$E_{11}=48.3$	$E_{22}=17.3$	$G_{12}=6.9$	(GPa)	
$\nu_{12}=0.3$				
(e) Orthotropic, from Ref. [2], see Appendix				
$D_1=8.605$	$D_2=4.366$	$D_3=50.59$	$D_4=13.00$	(force $\times$ length)
for a plate of unit thickness, units unknown				
(f) Bor N5505 [12]				
$E_{11}=204$	$E_{22}=18.5$	$G_{12}=5.59$	(GPa)	
$\nu_{12}=0.23$				
(g) Graphite epoxy [11]				
$E_{11}=150$	$E_{22}=9$	$G_{12}=7.1$	(GPa)	
$\nu_{12}=0.3$				
(h) H-IM6 epoxy [12]				
$E_{11}=203$	$E_{22}=11.2$	$G_{12}=8.4$	(GPa)	
$\nu_{12}=0.32$				

where n is here the number of the last term in the series expansion of w . For known solutions, see, e.g., Refs. [2,5], f_{res} decreases rapidly as n increases, which indicates rapidly converging solutions. In the second step, the support reaction is calculated with Eq. (22) and data obtained in the first step.

In Eq. (21), it is easily seen that the mean value and thus the resultant force of the terms with origin in (the derivative of) the twisting moment along the plate boundary vanish. On the other hand, some terms in the functions g_n and h_n in Eq. (19) have a mean value along the boundary that is not zero. However, on account of the characteristic equation (4), these nonzero mean values cancel out and they yield, in fact, a vanishing resultant. The nonzero resultant part of the support reaction comes from the constant part along the plate perimeter. Equation (22), after substitution of Eq. (1) and integrated over the plate perimeter, results in $\int_0^{2\pi} V_\alpha d\beta = \pi$, which amounts to the total load on a plate of unit radius and unit load intensity. The mean value of V_α is then 0.5, as required for equilibrium. This argument holds independently of the anisotropic properties of a plate and therefore, in particular, for an isotropic plate. In the latter case, the value of the reaction around the plate perimeter is furthermore constant and equal to 0.5. This feature is not seen directly in Eq. (22) and pertaining expressions and is therefore further dealt with in the Appendix.

In engineering applications, solutions for an arbitrary plate geometry and load are required. For a plate with radius a , thickness h , and a uniform load q , the support reaction V , which is proportional to qa/h^3 , is obtained from the dimensionless support reaction

$$V/(qa) = V_\alpha/h^3 \quad (24)$$

where V_α is given by Eq. (22).

4 Applications

In order to illustrate the outcome of Eq. (22), the support reaction has been calculated for a number of materials, selected so as to represent a wide variety of anisotropic materials commonly used in practice. In particular, the ratio of the elastic moduli in perpendicular material directions spans a wide range. The materials are shown in Table 1 with properties as given in the referenced

Table 2 Bending rigidities D_i (N m)

Material	D_1	D_2	D_3	D_4
a	9.547	3.855	9.547	18.68
b	7.892	2.489	8.499	5.883
c	6.107	2.726	7.954	4.267
d	1.49	0.447	4.159	1.15
e ^a	8.605	4.366	50.59	13
f	1.556	0.498	17.16	0.932
g	0.7541	0.2262	12.57	1.183
h	0.9386	0.3003	17.01	1.4

^aSI units are assumed for this material.

works. The C_{ij} are the elastic constants, E_{11} and E_{22} the elastic moduli, G_{12} the shear modulus, and ν_{12} Poisson's ratio. The bending rigidities D_i were calculated from the elastic constants with the expressions

$$D_1 = \frac{h^3}{12} \left(C_{11} - \frac{C_{12}^2}{C_{22}} \right) \quad (25a)$$

$$D_2 = \frac{h^3}{12} \left(C_{13} - \frac{C_{12}C_{23}}{C_{22}} \right) \quad (25b)$$

$$D_3 = \frac{h^3}{12} \left(C_{33} - \frac{C_{23}^2}{C_{22}} \right) \quad (25c)$$

$$D_4 = \frac{h^3}{6} C_{44} \quad (25d)$$

respectively, for Materials a, b, and c and from the elastic moduli, etc., through

$$D_1 = \frac{h^3}{12} \frac{E_{11}}{1 - (E_{11}/E_{22})\nu_{12}^2} \quad (26a)$$

$$D_2 = \nu_{12} D_3 \quad (26b)$$

$$D_3 = \frac{h^3}{12} \frac{E_{22}}{1 - (E_{11}/E_{22})\nu_{12}^2} \quad (26c)$$

$$D_4 = \frac{h^3}{6} G_{12} \quad (26d)$$

respectively, for Materials f, g, and h. For Material e, the bending rigidities are calculated in the Appendix. The selection of material axes for Material a, which is cubic symmetric (for symmetry classification, see, e.g., Ref. [13]), means that the result applies to a plate containing any pair of two material principal axes in its plane. For Materials b and c, which are tetragonal, the material principal axis normal to the plane of isotropy lies in the plane of the plate. The elastic constant in this (normal) direction is in both cases smaller than the two others, which in turn are equal. In the calculations, the y axis is oriented along the stiffest direction of a material, which means in terms of bending rigidities that $D_3 \geq D_1$. Calculated bending rigidities are shown in Table 2.

In the first step of the solution procedure, the roots k_1 and k_2 of Eq. (4) are determined. The roots are either real or complex and form in the latter case a pair of complex conjugates. The materials are classified according to a two-dimensional anisotropy indicator

$$(\xi, \eta) = (D_1, D_2 + D_4)/D_3 \quad (27)$$

which can be taken as a measure of deviation from isotropy as $(\xi, \eta) = (1, 1)$ for an isotropic material. The dimensionless numbers ξ and η can here be seen as relative bending rigidities and are determined by normalizing Eq. (4) such that the factor of k^4 becomes unity. The numbers ξ and η are applicable to plate problems in general as their analogs also appear in the biharmonic

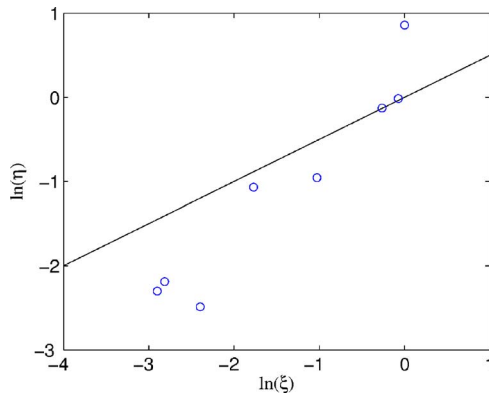


Fig. 1 Relative bending rigidity. Materials a–h form top right to down left.

Material	$\xi = D_1/D_3$	$\eta = (D_2 + D_4)/D_3$
a	1	2.361
b	0.929	0.985
c	0.768	0.879
d	0.358	0.384
e	0.17	0.343
f	0.091	0.083
g	0.06	0.112
h	0.055	0.1

differential equation for in-plane loading. The ξ and η values for all materials are shown in Fig. 1. The solid line in the figure is defined by the condition

$$\eta^2 - \xi = 0 \quad (28)$$

which separates regions of different types of root of Eq. (4). For a (ξ, η) point situated above, on, or under the line, there are two real roots, a real double root, or two complex conjugating roots, respectively. The ratio of the smallest to the largest ξ is of the order 1–60, thus spanning a wide range of anisotropy.

For the first three materials (a–c), the root pairs k_1 and k_2 are real for each material (the second and the third material fall just above the line (28)), while for the remainder (d–h), the root pairs are complex.

This behavior also appears in the solution of the equation system, Eqs. (10)–(12) in Ref. [2]. For real roots of Eq. (4), the constants A_n, B_n are real and vice versa. Thus, the type of constants, real or complex, in the solution is determined solely by material properties. In particular, real constants do not indicate incipient convergence of a solution and complex constants do not indicate a large residue. Note that the final solution is always real and for a solution with complex constants the imaginary parts are conjugates.

The ξ and η parameters have been found useful for classifying anisotropic materials in thin plate bending problems and they serve as an indicator of the order of a solution necessary to obtain a certain accuracy.

The constants C_i, A_n, B_n , and the residual moment f_{res} from the first solution step for all materials are shown in Table 3 for $n=4$. The constants have previously been calculated for the materials a and g and $n=3$ in Ref. [4]. For Material a, the constants A_n, B_n are virtually zero for $n \geq 3$. This means that the present result is identical to the previous and that a second order approximation is satisfactory. For Material g, the constants A_4, B_4 in the present work are small but still significant. This means that the accuracy of a third order approximation is less than that of a fourth order approximation. Note that, although of the same order, all constants are different in the two approximations, as they are obtained through solution of two different equation systems. A given constant in a third order approximation is in general different from the corresponding constant in a fourth order approximation. For example, $A_2(n=3)$ is different from $A_2(n=4)$.

The support reaction for all materials, obtained in the second solution step, is shown in Fig. 2. In order to check accuracy and convergence of approximations of increasing order, the support reaction is shown for the approximations of order $n=2-4$. All solutions have a jump between the first approximation (the particular solution only, not shown here) and subsequent approximations, which are much more close and meandering around one another. The relative difference calculated with the expression

$$\frac{|V_a(n=2 \text{ or } 3) - V_a(n=4)|}{(V_a)_m} \quad (29)$$

where $(V_a)_m$ is the mean value of the support reaction ($=0.5$), is assumed to be a reasonable estimation of the error in relation to the exact solution. It is also assumed that an error of the order a few percent of the support reaction, estimated in this way, is acceptable in most engineering applications.

Table 3 Residual moment (Eq. (23)) and constants C_i (in units of 10^{-3} GPa^{-1}), etc., for $n=4$

Material	f_{res}	C_1	C_2	C_3	C_4	C_5	C_6
(a) Nickel aluminide	0	1.392	2.009	1.392	−7.253	−7.253	5.958
(b) PZT-5H	10^{-7}	1.912	3.776	1.887	−9.655	−9.63	7.746
(c) PZT-4	10^{-7}	2.293	4.463	2.17	−10.93	−10.8	8.635
(d) Orthotropic (C and C)	−0.033	6.711	14.22	5.794	−32.31	−31.52	25.45
(e) Orthotropic (O)	−0.109	0.7161	1.395	0.5422	−3.177	−3.023	2.454
(f) Bor N5505	−0.660	2.872	6.619	1.984	−11.61	−11.05	8.683
(g) Graphite epoxy	−0.803	4.067	9.095	2.731	−16.33	−15.53	12.24
(h) H-IM6 epoxy	−0.873	3.123	6.990	2.044	−12.26	−11.62	9.126
Material	A_2	B_2	A_3	B_3	A_4	B_4	
(a)	0.474	−0.105	0	0	0	0	
(b)	$1.82 \cdot 10^{-3}$	$−7.26 \cdot 10^{-4}$	10^{-6}	10^{-7}	10^{-10}	0	
(c)	$1.81 \cdot 10^{-5}$	$−9.17 \cdot 10^{-7}$	10^{-8}	10^{-10}	10^{-11}	0	
(d)	$−0.395 \pm i0.50$		$(7.59 \pm i0.206) \cdot 10^{-3}$		$(7.12 \pm i1.80) \cdot 10^{-4}$		
(e)	$−(8.95 \pm i1.40) \cdot 10^{-2}$		$−(5.0 \pm i2.15) \cdot 10^{-3}$		$−(1.89 \pm i2.45) \cdot 10^{-4}$		
(f)	$−1.56 \pm i0.986$		$−(6.38 \pm i0.29) \cdot 10^{-2}$		$(9.51 \pm i0.85) \cdot 10^{-3}$		
(g)	$−2.81 \pm i0.413$		$−0.316 \pm i0.278$		$−(2.15 \pm i5.26) \cdot 10^{-2}$		
(h)	$−2.32 \pm i0.365$		$−0.274 \pm i0.243$		$−(1.98 \pm i4.79) \cdot 10^{-2}$		

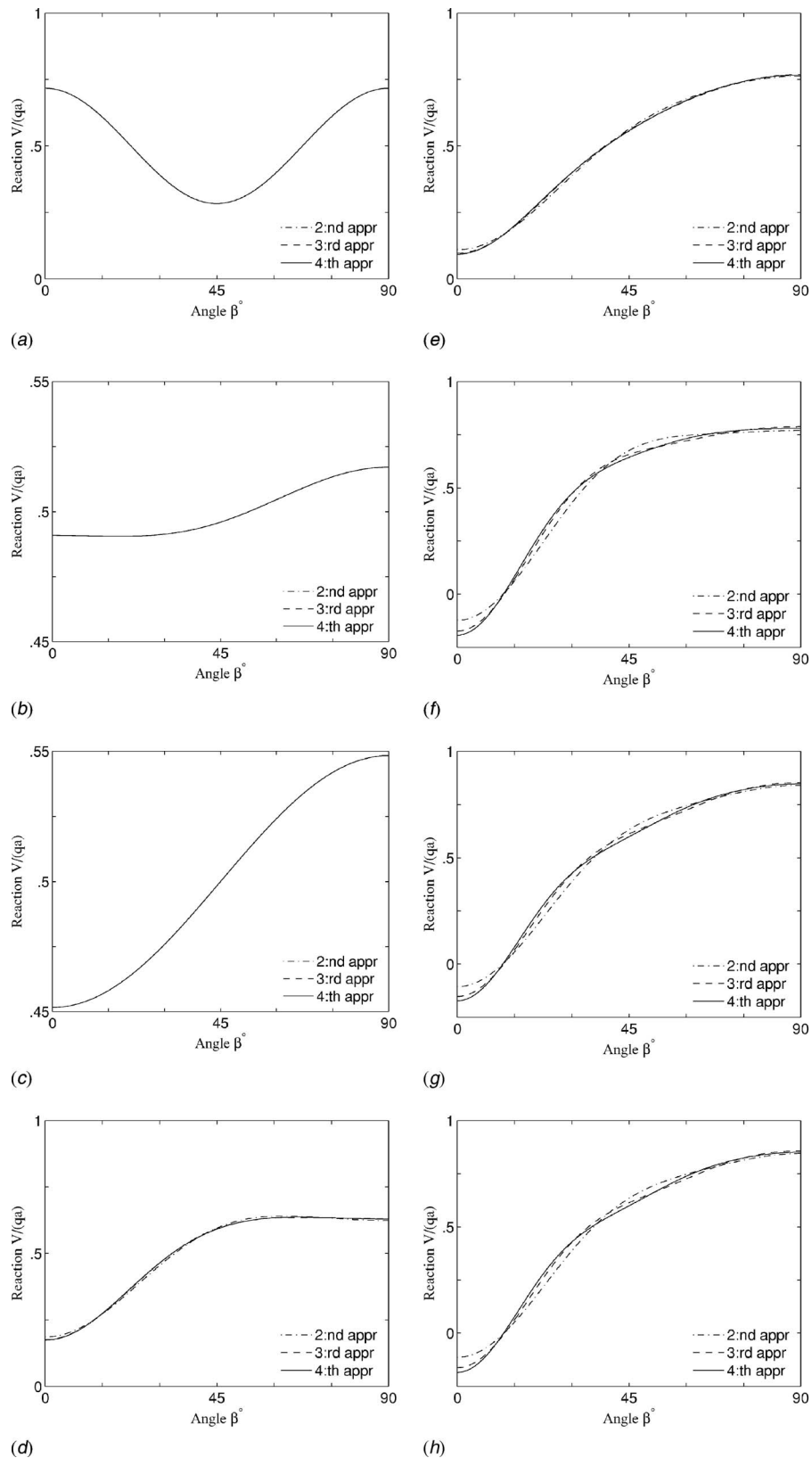


Fig. 2 Dimensionless support reaction versus position on plate perimeter for materials a–h

Due to symmetry, the support reaction for only one-fourth, or 90 deg, of the plate perimeter is given in general. In one case (Fig. 2(a)), this range is further halved to 45 deg, as $\xi=1$ for this material. As expected, the support reaction varies with position around the plate perimeter and its variation is related to the anisotropy of the material. Also, all curves are duly horizontal at points indicating material principal axes ($\beta=0$ deg and 90 deg), as expected from reasons of symmetry. Note the different load scaling in Figs. 2(b) and 2(c), where the variation of the support reaction is very small.

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All materials, except the first, are stiffest in the y direction and it is therefore intuitively reasonable to expect the maximum of the reaction to appear closer to $\beta=90$ deg than to $\beta=0$ deg and vice versa for the minimum reaction. A similar argument goes for the extreme values of the reaction of the first material, for which the corresponding values of β are 0 deg and 45 deg, respectively. In general, the minimum and the maximum of the reaction appear at $\beta=0$ deg and 90 deg or at $\beta=45$ deg and 0 deg, respectively, and the reaction is varying monotonously, or almost so, between the extreme values. There are two exceptions to this general behavior, the Materials b and d. In Material b, the reaction appears almost constant in a wide range of the order 30 deg, starting at $\beta=0$ deg and the minimum reaction appears to occur at an angle of the order 20 deg. In Material d, a corresponding behavior is seen in an equally wide range ending at $\beta=90$ deg and with maximum close to 70 deg. Due to the flatness of the curves, a small calculation error may displace the location of an extreme value greatly in the ranges concerned. In practice, this feature is of little or no concern.

The support reaction is partly negative for the last three materials, Figs. 2(f)–2(h). For a support not carrying negative loads, such materials will rise from the support in a neighborhood of $\beta=0$ deg when loaded and thus partly invalidate the present solution.

With respect to accuracy of the solutions, i.e., the order of approximation necessary to obtain a support reaction within the limits of engineering accuracy, the materials investigated are somewhat arbitrarily divided into three groups, which are denoted weakly, ordinary, and strongly anisotropic materials, respectively.

For weakly anisotropic materials, the second and higher approximations virtually coincide and an approximation of order 2 is therefore quite sufficient. Weakly anisotropic materials have ξ and η close to 1 and they satisfy the empirical relation

$$D_2 + D_4 > 3D_3 - 4.5D_1 \quad (30)$$

The Materials a–c belong to this group.

For ordinary anisotropic materials, a third order approximation is accurate for most materials. The error at the maximum of the support reaction along the plate perimeter and in the main part of the β range is of the order 1% or smaller but increases near minimum to typically 2–3%. Materials in this group satisfy the empirical relation

$$3D_3 - 4.5D_1 > D_2 + D_4 > D_3 - 4.5D_1 \quad (31)$$

This group includes the Materials d and e. Strongly anisotropic materials satisfy

$$D_2 + D_4 < D_3 - 4.5D_1 \quad (32)$$

and require a fourth order approximation in order to obtain engineering accuracy of the support reaction. The error at the maximum of the support reaction and in the main part of the β range is of the order 1% and increases near minimum to typically 3–4%. This group is exemplified by the Materials f–h, all three materials characterized by a partly negative support reaction. As seen in Table 3, for materials in this group, and at least with constants C_i of the order 15, say, and smaller, A_4 and B_4 constants of order 10^{-2} can be tolerated without exceeding an error of 3–4% in the support reaction. In comparison, for ordinary anisotropic materials, A_4 and B_4 are of the order 10^{-4} and the error of the support reaction is smaller than 2–3%.

The residual bending moment f_{res} , Table 3, is a further indicator of the anisotropic character of a material in bending, because f_{res} vanishes for an isotropic material. For weak anisotropy f_{res} is virtually zero and for strong anisotropy f_{res} is significantly greater than for ordinary anisotropy, although the figure has no absolute meaning. These relations apply for a given value of n . Further, the stronger the anisotropy of a given material the greater is also the value of n required to obtain an f_{res} smaller than a fixed limit.

Until further and sufficient experience is gained from investigations of a greater number of materials satisfying Eq. (32), an approximation of the fourth order of the support reaction seems to be a safe recommendation for engineering applications.

5 Discussion

The bending theory of a thin circular aeolotropic plate, with simply supported edge and uniform lateral load, first derived by Okubo [2], has been extended to include the plate support reaction, which was absent in the original theory. Approximate series expressions for the plate support reaction, which are directly applicable in practice, have been derived. The support reaction has been calculated for several materials, which were selected to represent a wide variety of anisotropic materials commonly used in practice. The reaction varies significantly along the plate perimeter and strongly anisotropic materials require in general a higher order series approximation to obtain engineering accuracy of the solution. With respect to anisotropy, the materials investigated have been divided into three different categories denoted weakly, ordinary, and strongly anisotropic materials and empirical relationships for classifying a given material are suggested.

The results in this work follow from up to eight order polynomial approximations of the deflection an anisotropic plate, which satisfy the plate equation. The plate reaction is subsequently calculated with standard thin plate theory. Neither any alternative analytical solution nor experimental results or numerical calculations in relation to the present problem are known to the author. Confirmation lacking, the results are presented tentatively and are recommended to be treated as such for the time being.

Acknowledgments

The author wishes to express his gratitude for kind guidance in material selection by Professor Lars Berglund at the Royal Institute of Technology in Stockholm.

Appendix

1 Limit Value of the Homogeneous Solution

In the isotropic case, the plate rigidities are related through the identities

$$D_1 = D_2 + D_4 = D_3 \quad (A1)$$

and the solutions of Eq. (4) are

$$k_1 = k_2 = 1 \quad (A2)$$

Further, there are no unique material axes and the coefficients of the particular solution are related through

$$C_1 = C_3 \quad C_2 = 2C_1 \quad C_4 = C_5 \quad (A3)$$

On account of the mapping conditions following Eqs. (5) in Ref. [2], we have

$$c' = \sqrt{1 - k_1^2} \quad (A4a)$$

and

$$c'' = \sqrt{1 - k_2^2} \quad (A4b)$$

From Eq. (A2), it follows that in this case

$$c' = c'' = 0 \quad (A5)$$

Then, from Eq. (5), it is seen directly that the homogeneous solution vanishes for an isotropic plate and only the particular solution $(V_a)_p$ in Eq. (22) remains

$$\begin{aligned} (V_a)_p = & 4[3D_1C_1 + (D_2 + D_4)C_2 + 3D_3C_3] + [6(3D_1 + D_2)C_1 + (D_3 \\ & - D_1)C_2 - 6(3D_3 + D_2)C_3]\cos 2\beta + [6(D_2 - D_1)C_1 + (D_1 \\ & - 2D_2 + D_3 + 4D_4)C_2 + 6(D_2 - D_3)C_3]\cos 4\beta \end{aligned} \quad (A6)$$

in which use has been made of Eq. (12:2) in Ref. [2] for simplification.

The value of the constant part of $(V_\alpha)_P$ is 0.5, as discussed in Sec. 3. Very short calculations show that the factors of $\cos 2\beta$ and of $\cos 4\beta$ are both zero, on account of Eqs. (A1) and (A3).

In summary, in the case of an isotropic plate, the reaction is constant and of value 0.5.

A somewhat more elaborated procedure is required to show that the homogeneous part of the solution (22), in fact, tends toward zero as a set of hypothetical anisotropic material properties tends toward isotropy. A formal proof is beyond the scope of this paper but a brief sketch is as follows.

For an isotropic material, it is seen by inspection that the coefficients of A_n and B_n in the equation system Eqs. (10)–(12) in Ref. [2] are equal for each equation and each n . This equation system is part of the first step in the two-step procedure to obtain V_α . In the coefficient matrix of this equation system, the columns of a pair A_n and B_n are thus identical. This means that the only possible solution of the equation system is $B_n = -A_n$. Further, in the summation terms of Eqs. (18) and (21), the factor of a given A_n is equal to the factor of the corresponding B_n . These two features taken together mean that all terms in the sums will cancel each other pairwise. The total sum value of the homogeneous solution is thus zero.

For a set of anisotropic material properties tending toward isotropy, all forms of “be,” etc., above can be thought of as “tend to” and the vanishing total of the homogeneous solution is thus the limit value at isotropy. Thus, in the limit for an anisotropic material tending toward isotropy, the value of the reaction is constant and equal to 0.5.

2 Bending Rigidities of Okubo and Recalculation of the Second Approximation

Okubo presents two approximations of the plate deflection, Eq. (9) in Ref. [2] (which is identical to Eq. (5) in this work); the first approximation comprises the particular solution only and the second includes also the first term of the homogeneous solution series. Okubo presents, however, just his results and he does not report any bending rigidities for the oak plate material, cut parallel to the grain, he is considering. On the other hand, the value of the roots $k_{1,2}^2$ of the characteristic equation (3) in Ref. [2] (equivalent to Eq. (4) here), are given. The four unknown bending rigidities can, however, be found by solving an equation system consisting of the four Eqs. (10) in Ref. [2], the first two of Eq. (12) in Ref. [2] and the expression for the residual bending moment pertaining to the particular solution in Ref. [2]. The constants C_i , $i = 1, \dots, 6$, ($C_1 = 0.000768, \dots$) are in this case considered as given and the bending rigidities as unknowns. The solution of the equation system is

$$D_1 = 8.605 \quad D_2 = 4.366 \quad D_3 = 50.587 \quad (A7)$$

$$D_4 = 13.00 \quad (\text{force} \times \text{length}) \quad (A7)$$

in unknown units. These bending rigidities yield in turn

$$k_{1,2}^2 = 0.3428 \mp 0.2279i \quad (A8)$$

in accordance with the values of $k_{1,2}^2$ given by Okubo. Therefore, we assume that the calculated bending rigidities are accurate. To allow comparison with other materials, SI units have been as-

sumed for the bending rigidities of this material, Table 2.

From the roots $k_{1,2}^2$, the constants a' and a'' can be calculated from Eq. (5) in Ref. [2] and the associated boundary conditions. By using Okubo's second approximation ($C_1 = 0.000738, \dots$), we can once again calculate bending rigidities with an equation system based on Eqs. (10) in Ref. [2] and in this case the first three of Eq. (12) in Ref. [2], in which also the given complex constants A_2 and B_2 are used in two of the equations. In this case, however, the values of the bending rigidities obtained above are not reproduced, the differences for corresponding rigidities being 3–15%.

Note that the correspondence between bending rigidities and solution constants is exact and one to one in both cases considered.

In view of the discrepancy between the two solutions, a new solution for the second approximation has been calculated using the bending rigidities obtained in the first case. The result is

$$C_1 = 0.000720 \quad C_3 = 0.000543 \quad C_5 = -0.003021$$

$$C_2 = 0.001386 \quad C_4 = -0.003180 \quad C_6 = 0.002454$$

$$A_2 = -0.0000839 + 0.0000181i$$

$$B_2 = -0.0000839 - 0.0000181i \quad (A9)$$

The greatest differences in comparison with Okubo's second approximation occur in the complex constants. Here, the real and the imaginary parts differ some 5% and 6%, respectively. With the exception of C_1 , for which the difference is some 2.5%, the difference of the remaining real-valued constants is less than 1%.

The residual bending moment at the edge is found to be $M_\alpha = -0.00211 \cos 6\beta$, which is 6% greater than in Ref. [2]. At the center of the plate, the bending moments are $M_\alpha = (0.2070 - 0.1247 \cos 2\beta)q$ and $M_{\alpha\beta} = -0.1247q \sin 2\beta$ which, oddly enough, coincide with Okubo's results.

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Angular Velocity Estimation From the Angular Acceleration Matrix

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Computing the angular velocity ω from the angular acceleration matrix is a nonlinear problem that arises when one wants to estimate the three-dimensional angular velocity of a rigid-body from point-acceleration measurements. In this paper, two new methods are proposed, which compute estimates of the angular velocity from the symmetric part $\mathbf{W}^S$ of the angular acceleration matrix. The first method uses a change of coordinate frame of $\mathbf{W}^S$ prior to performing the square-root operations. The new coordinate frame is an optimal representation of $\mathbf{W}^S$ with respect to the overall error amplification. In the second method, the eigenvector spanning the null space of $\mathbf{W}^S$ is estimated. As ω lies in this space, and because its magnitude is proportional to the absolute value of the trace of $\mathbf{W}^S$, it is a simple matter to obtain ω . A simulation shows that, for this example, the proposed methods are more accurate than those existing methods that use only centripetal acceleration measurements. Moreover, their errors are comparable to other existing methods that combine tangential and centripetal acceleration measurements. In addition, errors of 2.15% in the accelerometer measurements result in errors of approximately 3% in the angular-velocity estimates. This shows that accelerometers are competitive with angular-rate sensors for motions of the type of the simulated example, provided that position and orientation errors of the accelerometers are accounted for.

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Keywords: accelerometer, inertial measurement unit, kinematics, projectile guidance, angular velocity

1 Introduction

¹The angular acceleration matrix [1] $\mathbf{W} \in \mathbb{R}^{3 \times 3}$ is composed of a skew-symmetric and a symmetric part, which are the cross-product matrix (CPM) of the rigid-body angular acceleration and the square of the CPM of the angular velocity squared, respectively. Let A and B be two points of a rigid body, with position vectors $\mathbf{p}_A$ and $\mathbf{p}_B$, respectively. From the rigid-body equations, the acceleration $\ddot{\mathbf{p}}_B$ of point B can be computed from the acceleration $\ddot{\mathbf{p}}_A$ of point A as

$$\ddot{\mathbf{p}}_B = \ddot{\mathbf{p}}_A + \mathbf{W}(\mathbf{p}_B - \mathbf{p}_A) \quad (1)$$

where $\mathbf{W} = \dot{\boldsymbol{\Omega}} + \boldsymbol{\Omega}^2$, and $\boldsymbol{\Omega} \equiv \text{CPM}(\boldsymbol{\omega})$ is the CPM of the rigid-body angular velocity $\boldsymbol{\omega}$. Now, extracting $\boldsymbol{\omega}$ from matrix $\mathbf{W}$ is a problem that arises when trying to estimate the rigid-body angular motion from point-acceleration measurements. In short, this method consists in measuring the accelerations of several points on the rigid-body using accelerometers. Assuming that the positions of these points are known, one can then form a set of linear equations from Eq. (1). The ensuing linear system contains as many equations as there are uniaxial accelerometers attached to the rigid body. Hence, it is necessary that the number of accelerometers be greater than or equal to 9 in order to solve for the nine unknowns, namely, the entries of $\mathbf{W}$. One can then extract the angular acceleration and the angular velocity vectors from the angular acceleration matrix.

The foregoing approach for angular-velocity estimation is a crosscurrent when compared with the customary triad of mutually orthogonal angular-rate sensors (gyroscopes), which provide direct estimates of the angular velocity. The advantages that accelerometer-only inertial measurement units (IMUs) could bring

about come from the relative simplicity of accelerometers. In general, accelerometers are more reliable, less expensive, and require less power than angular-rate sensors.

Motivated by the above observations, several proofs of concept have been attempted. Peng and Golnaraghi [2] tested a cubic accelerometer-only IMU with an edge of one foot that was intended to measure the angular velocity of a car. Ang et al. [3], who used an all-accelerometer concept to control the vibrations of a hand-held microsurgical instrument, claimed that their solution was better than an angular sensor-based IMU. Algrain and Quinn [4] used accelerometers to stabilize the line of sight of an imaging device. Among the most interesting applications, we find the works by Chou and Sinha [5], Mital and King [6], Nusholtz [7], and Shea and Viano [8], who successfully used accelerometers to track the angular acceleration and angular velocity of the head of a dummy subjected to impact. Finally, the targeted application in this paper is the control of projectile trajectories. Indeed, long-range projectiles undergo an aerodynamic jump after launch, which contributes to their scattering. To cope with this problem, interesting all-accelerometer solutions were proposed in several papers from Pickel [9], Pamadi et al. [10], and Costello [11], to name a few. Apparently, from the literature, most of the successful implementations of accelerometer-only IMUs were done in high angular-rate applications. Therefore, we will demonstrate the validity of the progress done here by roughly simulating the trajectory of a brick spinning freely in space with an initial angular velocity of 10π rad/s.

Theoretical advances were reported in the field of accelerometer-only IMUs. Among them, we can cite the extensive works by Grammatikos [12], Schuler [13], Mostov [14], and Parsa [15]. From their work, we may draw three groups of methods for the estimation of the angular velocity from point-acceleration measurements.

1.1 Tangential Acceleration Methods. The first group of methods uses the time integration of the angular acceleration, which is obtained, in turn, from tangential acceleration (TA) measurements. Hence, we label it, for short, the group of TA methods.

¹CPM($\mathbf{v}$) is defined as $\partial(\mathbf{v} \times \mathbf{x})/\partial \mathbf{x}$, for any $\mathbf{x}, \mathbf{v} \in \mathbb{R}^3$.

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In this group, $\boldsymbol{\omega}$ is estimated from the skew-symmetric component of $\mathbf{W}$, which can be summarized symbolically as

$$\begin{aligned}\hat{\boldsymbol{\omega}} &= \text{vect}(\hat{\mathbf{W}}) \\ \hat{\boldsymbol{\omega}}(t) &= \int_0^t \hat{\boldsymbol{\omega}}(\tau) d\tau\end{aligned}\quad (2)$$

where $\hat{\boldsymbol{\omega}}$ is the axial vector² of $\hat{\mathbf{W}}$, and $(\hat{\cdot})$ is an estimate of $(\cdot)$. Several specific methods may be drawn from Eq. (2) by changing the numerical integration method, but, apart from that, the algorithm being linear, it is hard to imagine that there will be not more room for improvements. Hence, for the sake of conciseness, we choose to use only one integration method, namely, the trapezoidal rule. We will refer to this method by the acronym TA, indiscriminately from its group.

1.2 Centripetal Acceleration Methods. The second group includes all methods that consist in taking the square root of the centripetal acceleration (CA) that is represented by the symmetric component $\mathbf{W}^S$ of $\mathbf{W}$. This group will therefore be referred to as the CA methods. In this category, the customary approach is to combine linearly the diagonal terms of $\mathbf{W}$ in order to obtain the squares of the components of $\boldsymbol{\omega}$. The ensuing square-root operation yields the absolute values of the components of $\boldsymbol{\omega}$. One must thus cope with the sign ambiguity. Schuler [13] and Grammatikos [12] suggested that low-cost extra sensors be added to resolve the sign ambiguity. Another solution, proposed by Parsa [15], and which seems more reasonable to the authors, is to simply use the signs of the TA estimate. As a result, we obtain

$$\begin{aligned}\hat{\boldsymbol{\omega}}(t) &= \boldsymbol{\omega}(0) + \int_0^t \hat{\boldsymbol{\omega}}(\tau) d\tau \\ \hat{\xi}_i &= \hat{w}_{i,i} - (1/2)\text{tr}(\hat{\mathbf{W}}) \quad i = 1, 2, 3 \\ \tilde{\omega}_i &= \text{sgn}(\hat{\omega}_i)u(\hat{\xi}_i)\sqrt{\hat{\xi}_i} \quad i = 1, 2, 3\end{aligned}\quad (3)$$

where $\hat{w}_{i,j}$ is the (i,j) entry of $\hat{\mathbf{W}}^S$, $\text{sgn}(\cdot)$ is the signum function, $u(\cdot)$ is the Heaviside (step) function, and $(\hat{\cdot})$ stands for the estimate of $(\cdot)$ obtained from CA measurements. As it uses only the diagonal entries of $\mathbf{W}$, this estimation algorithm will be referred to as the CAD method.

An alternative method, which also pertains to the CA category, was reported by Peng and Golnaraghi [2], where the authors nonetheless point out that this algorithm is prone to singularity problems. This approach uses the off-diagonal entries of $\mathbf{W}^S$ to estimate the square of the components of the angular velocity. For that reason, we choose the label CAOD for this method. The sign ambiguity may be resolved by resorting, again, to the TA estimate of the angular velocity. This yields the algorithm

$$\begin{aligned}\hat{\boldsymbol{\omega}}(t) &= \boldsymbol{\omega}(0) + \int_0^t \hat{\boldsymbol{\omega}}(\tau) d\tau \\ \hat{\xi}_i &= \hat{w}_{i,j}\hat{w}_{k,i}/\hat{w}_{j,k}\end{aligned}\quad (4)$$

$$\tilde{\omega}_i = \text{sgn}(\hat{\omega}_i)u(\hat{\xi}_i)\sqrt{\hat{\xi}_i} \quad i, j, k = 1, 2, 3 \quad i \neq j \neq k$$

where one notes that ξ_i becomes undetermined whenever $w_{j,k}=0$, that is, when $\omega_j=0$ or $\omega_k=0$. In fact, these last equalities imply also that $w_{i,j}$ or $w_{i,k}$ be null, respectively. Hence, whenever any of the components of $\boldsymbol{\omega}$ goes to zero, the other two components experience indeterminacies of the type 0/0, thereby rendering the

algorithm unstable. This problem may be circumvented or attenuated by resorting to a different estimation method over certain ranges of angular velocity.

1.3 Tangential and Centripetal Acceleration Methods. A third group of estimation methods may be identified by observing that the TA and the CA methods can be combined in order to provide more robust angular-velocity estimates. Let us call it the group of tangential and centripetal acceleration (TCA) methods. Obviously, the existing CA methods detailed above make use of the TA method, but since it is only for the purpose of determining the signs of the angular-velocity components, we will not include them in the TCA category. Indeed, it is reasonable to think that only a small piece of the information contained in the TA estimate is used in the CA methods and that, therefore, their recombination with the TA estimate may yield more accurate results. Accordingly, all methods that use both the skew-symmetric and the symmetric component of $\mathbf{W}$, the former being used to change not only the signs of the square roots of the latter but also their magnitudes, will be considered as pertaining to the TCA group.

A first member of this group, which was proposed by Peng and Golnaraghi [2], will be called the TCAQ method, where the letter Q stands for quadratic. Indeed, these authors remarked that the quadratic equations

$$\tilde{\omega}_i^2 + \tilde{\omega}_i\hat{\omega}_j + (1/2)\text{tr}(\hat{\mathbf{W}}) - \hat{w}_{i,i} - \hat{w}_{i,j} = 0 \quad i, j = 1, 2, 3 \quad i \neq j \quad (5)$$

hold whenever the estimates are accurate. Angular velocity estimates may then be computed as

$$\begin{aligned}\hat{\boldsymbol{\omega}}(t) &= \boldsymbol{\omega}(0) + \int_0^t \hat{\boldsymbol{\omega}}(\tau) d\tau \\ \hat{\eta}_{i,j} &= \hat{\omega}_j^2 + 4\hat{w}_{i,i} + 4\hat{w}_{i,j} - 2\text{tr}(\hat{\mathbf{W}})\end{aligned}\quad (6)$$

$\tilde{\omega}_i = -(1/2)\hat{\omega}_j + (1/2)\text{sgn}(2\hat{\omega}_i + \hat{\omega}_j)u(\hat{\eta}_{i,j})\sqrt{\hat{\eta}_{i,j}}$ $i, j = 1, 2, 3 \quad i \neq j$ where it is apparent that two possible values of j exist for a given i and, therefore, two estimates of ω_i are available. In the subsequent simulations, we choose the estimate that corresponds to the maximum value of $\hat{\eta}_{i,j}$.

A second method, which was proposed by Parsa et al. [16], falls within the TCA category. The authors define a function $\mathbf{f}$ mapping the angular velocity onto the six-dimensional array $\mathbf{w}$ of the distinct entries of $\mathbf{W}^S$, that is,

$$\begin{aligned}\mathbf{f}(\boldsymbol{\omega}) &\equiv [-\omega_2^2 \quad -\omega_3^2 \quad -\omega_3^2 \quad -\omega_1^2 \quad -\omega_1^2 \\ &\quad -\omega_2^2 \quad \omega_1\omega_2 \quad \omega_2\omega_3 \quad \omega_3\omega_1]^T \\ &= [w_{1,1} \quad w_{2,2} \quad w_{3,3} \quad w_{1,2} \quad w_{2,3} \quad w_{3,1}]^T \\ &\equiv \mathbf{w}\end{aligned}\quad (7)$$

Note that $\mathbf{f}$ is neither injective nor surjective and, hence, it is not invertible. Nevertheless, one may still compute its gradient, which allows for its Taylor series expansion:

$$\mathbf{f}(\boldsymbol{\omega}) = \mathbf{f}(\hat{\boldsymbol{\omega}}) + \frac{\partial \mathbf{f}(\boldsymbol{\omega})}{\partial \boldsymbol{\omega}} \Delta \boldsymbol{\omega} + O(2) \quad (8)$$

where $\Delta \boldsymbol{\omega}$ will be the correction on the angular velocity estimate $\hat{\boldsymbol{\omega}}$ obtained from the TA method and where the gradient can be verified to be

$$\frac{\partial \mathbf{f}(\boldsymbol{\omega})}{\partial \boldsymbol{\omega}} = \begin{bmatrix} 0 & -2\omega_1 & -2\omega_1 & \omega_2 & 0 & \omega_3 \\ -2\omega_2 & 0 & -2\omega_2 & \omega_1 & \omega_3 & 0 \\ -2\omega_3 & -2\omega_3 & 0 & 0 & \omega_2 & \omega_1 \end{bmatrix}^T \quad (9)$$

One may also verify that $\partial \mathbf{f}(\boldsymbol{\omega})/\partial \boldsymbol{\omega}$ is of rank 3 if and only if $\|\boldsymbol{\omega}\|_2 \neq 0$, which allows the computation of its left Moore–Penrose

²The axial vector $\text{vect}(\mathbf{A})$ of matrix $\mathbf{A} \in \mathbb{R}^{3 \times 3}$ is defined such that $\text{vect}(\mathbf{A}) \times \mathbf{u} = (1/2)(\mathbf{A} - \mathbf{A}^T)\mathbf{u}$, for any vector $\mathbf{u} \in \mathbb{R}^3$.

inverse $[\partial \mathbf{f}(\boldsymbol{\omega})/\partial \boldsymbol{\omega}]^\dagger$. Upon estimating the left-hand side of Eq. (8) from $\hat{\mathbf{W}}^S$ and using $\hat{\boldsymbol{\omega}}$ to estimate the gradient of $\mathbf{f}$, we obtain

$$\Delta \boldsymbol{\omega} = [\partial \mathbf{f}(\boldsymbol{\omega})/\partial \boldsymbol{\omega}]_{\boldsymbol{\omega}=\hat{\boldsymbol{\omega}}}^\dagger [\hat{\mathbf{W}} - \mathbf{f}(\hat{\boldsymbol{\omega}})] \quad (10)$$

The resulting estimate $\tilde{\boldsymbol{\omega}}$ is computed as

$$\tilde{\boldsymbol{\omega}} = \hat{\boldsymbol{\omega}} + \Delta \boldsymbol{\omega} \quad (11)$$

We label this method TCAT, where T stands for Taylor series.

Another method was proposed by Krishnan [17] and by Schuler [13] to estimate the angular velocity of a rigid body from point-acceleration measurements. In this method, the accelerometers are mounted onto rotating disks and, hence, follow the exact same principle as most mechanical angular-rate sensors do, namely, by measuring the Coriolis acceleration. As this strategy is rather cumbersome and, strictly speaking, does not fall into the category of rigid-body motion, we will leave it aside in this paper. We also acknowledge the recent works of Genin et al. [18] and Parsa [15], who investigated the general problem of accelerometer placement for a rigid body moving in space, and the work of Williams and Fyfe [19], who studied the same problem for planar motion.

The aim of this paper is to provide new methods to estimate the angular velocity from the angular acceleration matrix that are implementable in real time. Two new methods are first devised and then compared with the existing TA, CA, and TCA methods through a simple simulation.

2 First Method: Expressing $\mathbf{W}$ in a Preferred Frame

One of the major drawbacks of the CA method is that, due to the square-root operation, the error amplification is unbounded when any component of the angular velocity approaches zero. This can be seen by computing the partial derivative of the three-dimensional array $\boldsymbol{\zeta}$ containing the squares of the components of $\boldsymbol{\omega}$ with respect to $\mathbf{w}' \equiv \text{diag}(\mathbf{W})$, where $\text{diag}(\cdot)$ returns the vector formed by the diagonal entries of $(\cdot)$. This gives

$$\frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{w}'} = 2\Delta \frac{\partial \boldsymbol{\omega}}{\partial \mathbf{w}'} = \left(\frac{1}{2}\right)\mathbf{B} \quad \text{where } \Delta \equiv \begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{bmatrix} \text{ and} \quad (12)$$

$$\mathbf{B} \equiv \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

From now on, we will refer to matrix $\partial \boldsymbol{\omega}/\partial \mathbf{w}'$ of Eq. (12) as the Jacobian matrix of the CA method. Indeed, this matrix gives, for a given estimation method, the sensitivity of the angular-velocity estimate to the diagonal entries of the angular acceleration matrix. From Eq. (12), we also see that when Δ becomes singular, the CA Jacobian matrix is undetermined. This situation is to be avoided at all costs, as it means that a small error in $\mathbf{w}'$ leads to unpredictably large errors in $\boldsymbol{\omega}$. On the other hand, from mere intuition, the best situation may be when Δ is "farthest" from being singular, that is, when it is isotropic. This happens when all the absolute values of the components of $\boldsymbol{\omega}$ are identical. In such a case, upon noting that $\mathbf{W}$ is a tensor, we may be able to express it in a frame $\mathcal{C}$ different from the body frame $\mathcal{B}$, in order to minimize the error amplification. Hence, we label this method CAC, the last C standing for the preferred computational frame $\mathcal{C}$.

2.1 Optimization Problem. To substantiate this claim, we first need to formulate the problem. Equation (12) is still valid in frame $\mathcal{C}$. We rewrite this equation as

$$[\Delta]_{\mathcal{C}} [\partial \boldsymbol{\omega}/\partial \mathbf{w}']_{\mathcal{C}} = (1/4)\mathbf{B} \quad (13)$$

where $[\cdot]_{\mathcal{C}}$ means that $(\cdot)$ is expressed in frame $\mathcal{C}$. Hence, we aim at minimizing the p -norm of the Jacobian matrix of the CA method, that is,

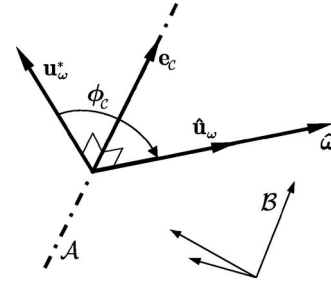


Fig. 1 Rotating $\mathbf{u}_{\omega}^*$ onto $\hat{\mathbf{u}}_{\omega}$

$$f_p \equiv \left\| \left[\frac{\partial \boldsymbol{\omega}}{\partial \mathbf{w}'} \right]_{\mathcal{C}} \right\|_p \rightarrow \min_{\mathcal{C}} \quad (14)$$

$\mathcal{C}$ being free of any constraint. The Euclidean norm, which corresponds to $p=2$, is chosen here because, due to its invariance, it is a common metric in vector spaces. Note that the Frobenius norm is also frame invariant. However, it does not bound any ratio of norms of the input and output vectors, and, therefore, it is meaningless for our application.

The optimum frame $\mathcal{C}$ under which f is minimum is found to be any frame for which we have

$$[\boldsymbol{\omega}]_{\mathcal{C}} = \|\boldsymbol{\omega}\|_2 \mathbf{u}_{\omega} \quad (15)$$

where $\mathbf{u}_{\omega} = (1/\sqrt{3})[\pm 1 \pm 1 \pm 1]^T$. Because the proof of this statement is lengthy, it is relegated to the Appendix. Also found in the Appendix are proofs that choosing the 1- or the ∞ -norms, which correspond to $p=1$ and $p=\infty$, respectively, leads to the same optimum frame $\mathcal{C}$.

2.2 Implementation. Let us first use $\hat{\boldsymbol{\omega}}$, obtained through the TA method, to compute an estimate $\hat{\mathbf{u}}_{\omega} \equiv \hat{\boldsymbol{\omega}}/\|\hat{\boldsymbol{\omega}}\|_2$ of the direction of the angular velocity. Thus, we need to find a frame $\mathcal{C}$ such that $[\hat{\mathbf{u}}_{\omega}]_{\mathcal{C}} = \mathbf{u}_{\omega}^* = [\mathbf{u}_{\omega}^*]_{\mathcal{B}}$, where $\mathbf{u}_{\omega}^* \equiv (1/\sqrt{3})[1 \ 1 \ 1]^T$. One solution to this problem can be found by performing a rotation around axis $\mathcal{A}$ orthogonal to $\hat{\mathbf{u}}_{\omega}$ and $\mathbf{u}_{\omega}^*$ by an angle ϕ_C , as shown in Fig. 1. Thus, we obtain $[\hat{\mathbf{u}}_{\omega}]_{\mathcal{B}} = \mathbf{Q}_{\mathcal{C}}[\mathbf{u}_{\omega}^*]_{\mathcal{B}} = \mathbf{Q}_{\mathcal{C}}[\hat{\mathbf{u}}_{\omega}]_{\mathcal{C}}$. Apparently, $\mathbf{Q}_{\mathcal{C}}$ is a matrix rotating frame $\mathcal{B}$ into frame $\mathcal{C}$. Hence [1],

$$\mathbf{Q}_{\mathcal{C}} = \mathbf{e}_C \mathbf{e}_C^T + \cos \phi_C (\mathbf{1}_{3 \times 3} - \mathbf{e}_C \mathbf{e}_C^T) + \sin \phi_C \mathbf{E}_C \quad (16)$$

where $\mathbf{e}_C$ is the unit vector giving the direction of $\mathcal{A}$, and $\mathbf{E}_C \equiv \text{CPM}(\mathbf{e}_C)$. Upon noting that $\mathbf{e}_C \sin \phi_C = \mathbf{u}_{\omega}^* \times \hat{\mathbf{u}}_{\omega}$ and $\cos \phi_C = \hat{\mathbf{u}}_{\omega}^T \mathbf{u}_{\omega}^*$, Eq. (16) is rewritten as

$$\mathbf{Q}_{\mathcal{C}} = \frac{(\mathbf{u}_{\omega}^* \times \hat{\mathbf{u}}_{\omega})(\mathbf{u}_{\omega}^* \times \hat{\mathbf{u}}_{\omega})^T}{1 + \hat{\mathbf{u}}_{\omega}^T \mathbf{u}_{\omega}^*} + \hat{\mathbf{u}}_{\omega}^T \mathbf{u}_{\omega}^* \mathbf{1}_{3 \times 3} + \text{CPM}(\mathbf{u}_{\omega}^* \times \hat{\mathbf{u}}_{\omega}) \quad (17)$$

where it is assumed that $\hat{\mathbf{u}}_{\omega}^T \mathbf{u}_{\omega}^* \neq -1$. In fact, $\hat{\mathbf{u}}_{\omega}^T \mathbf{u}_{\omega}^* = -1$ implies that $\hat{\mathbf{u}}_{\omega} = -\mathbf{u}_{\omega}^*$, which is also an optimum orientation. In this case, it is not necessary to change the coordinate frame, and hence, we choose $\mathbf{Q}_{\mathcal{C}} = \mathbf{1}_{3 \times 3}$. Hence, the algorithm is

if $\text{tr}(\mathbf{W}) \neq 0$ and $\|\hat{\boldsymbol{\omega}}\|_2 \neq 0$

$\hat{\mathbf{u}}_{\omega} = \hat{\boldsymbol{\omega}}/\|\hat{\boldsymbol{\omega}}\|_2$
if $\hat{\mathbf{u}}_{\omega}^T \mathbf{u}_{\omega}^* \neq -1$

$$\mathbf{Q}_{\mathcal{C}} = (\mathbf{u}_{\omega}^* \times \hat{\mathbf{u}}_{\omega})(\mathbf{u}_{\omega}^* \times \hat{\mathbf{u}}_{\omega})^T / (1 + \hat{\mathbf{u}}_{\omega}^T \mathbf{u}_{\omega}^*) + \hat{\mathbf{u}}_{\omega}^T \mathbf{u}_{\omega}^* \mathbf{1}_{3 \times 3} + \text{CPM}(\mathbf{u}_{\omega}^* \times \hat{\mathbf{u}}_{\omega})$$

else

$$\mathbf{Q}_{\mathcal{C}} = \mathbf{1}_{3 \times 3}$$

end

$$[\mathbf{W}]_{\mathcal{C}} = \mathbf{Q}_{\mathcal{C}}^T \mathbf{W} \mathbf{Q}_{\mathcal{C}}$$

$$[\mathbf{w}']_{\mathcal{C}} = \text{diag}([\mathbf{W}]_{\mathcal{C}})$$

$$[\boldsymbol{\zeta}]_{\mathcal{C}} = (1/2)\mathbf{B}[\mathbf{w}']_{\mathcal{C}}$$

for $i=1, 2, 3$

```

 $[\omega_i]_c = \sqrt{h([\xi_i]_c)[\xi_i]_c}$ 
end
 $\omega = Q_c[\omega]_c$ 
else
 $\omega = 0_3$ 
end

```

3 Second Method: Computing the Null Space of W^S

The idea behind this method is to use the invariants of W^S instead of expressing it in an optimum frame. When neglecting the measurement errors, we have $W^S = \Omega^2$, a symmetric negative-semidefinite matrix. The eigenvalues of Ω^2 are found to be $\{\lambda_1 = 0, \lambda_2 = -\|\omega\|_2^2, \lambda_3 = -\|\omega\|_2^2\}$, its eigenvectors being, respectively,

$$\begin{aligned} \mu_1 &= \mathbf{u}_\omega, \mu_2 = (1/\sqrt{\omega_1^2 + \omega_2^2})[-\omega_2 \ \omega_1 \ 0]^T, \\ \mu_3 &= (1/\sqrt{\omega_1^2 + \omega_3^2})[-\omega_3 \ 0 \ \omega_1]^T \end{aligned}$$

No information concerning the orientation of ω can be extracted from the eigenvalues: only $\|\omega\|_2$ is available. In fact, although it was never explicitly stated, this information was already present in the CA method, in the relation

$$-(1/2)\text{tr}(W) = -(1/2)\text{tr}(W^S) = -(1/2)(\lambda_1 + \lambda_2 + \lambda_3) = \|\omega\|_2^2 \quad (18)$$

The novelty of this approach lies in the information extracted from the eigenvectors of Ω^2 . As Ω^2 is symmetric, and provided that $\|\omega\|_2 \neq 0$, its eigenvectors are bound to be mutually orthogonal, and, since its rank is 2, two of these eigenvectors span its range, while the remaining one, linearly dependent on ω , spans the null space of Ω and, hence, of Ω^2 . Therefore, finding μ_1 for a given W^S gives us the direction of ω . Because the problem is essentially that of estimating the null space of Ω^2 , we refer to the ensuing algorithm as the CANS method.

3.1 Implementation. We will rely on the *QR* decomposition to compute the null space of interest in a predetermined number of steps.³ Let us use Householder reflections [20] to decompose W^S into QR , where R is an upper-triangular matrix and Q is an orthogonal matrix. Moreover, as W has one null eigenvalue, it has a rank of 2, which implies that the third diagonal entry of R will be null. This also indicates that the Householder method requires only two reflections; as a result, Q is bound to be proper orthogonal, thus representing a rotation. We thus have

$$0_3 = W^S \mu_1 = (W^S)^T \mu_1 = R^T Q^T \mu_1 = R^T \nu \quad (19)$$

where $\nu = Q^T \mu_1$. From Eq. (19), we see that ν must lie in the null space of R^T . Because the third column of R^T is null, we conclude directly that $\nu = e_3$ is a suitable solution. The eigenvector is obtained from the definition of μ , that is, $\mu_1 = Q\nu = q_3$, where q_3 is the vector formed from the third column of Q .

There is still one issue that needs to be resolved: we have obtained a vector that is parallel to ω , but we do not know whether it points in the correct direction. To fix this, we use the estimate $\hat{\omega}$ obtained from the TA method and take the direction that minimizes the magnitude of the error $\|\omega - \hat{\omega}\|_2$. As a result, ω is computed as $\omega = \text{sgn}(\mu_1^T \hat{\omega}) \|\omega\|_2 \mu_1$. Thus, the algorithm takes the form if $\text{tr}(W) < 0$

$$\begin{aligned} \|\omega\|_2 &= \sqrt{-(1/2)\text{tr}(W)} \\ W^S &= (1/2)(W + W^T) \\ \{Q, R\} &\leftarrow \text{Householder}(W^S) \\ \omega &= \text{sgn}(q_3^T \hat{\omega}) \|\omega\|_2 q_3 \end{aligned}$$

³It is common practice to find the null space of a matrix using the singular-value decomposition (SVD). However, algorithms that perform SVD are iterative, which is undesirable for angular-velocity estimation in real time, even for a relatively small matrix.

```

else
 $\omega = 0_3$ 
end

```

4 Simulation

In order to compare the robustness of the two foregoing methods with that of the previously proposed ones, the motion of a brick freely rotating in space was simulated. As the current objective is to compare different algorithms for the estimation of ω , the simulations were simplified to the minimum.

4.1 Accelerometer-Only Inertial Measurement Unit. The IMU used to estimate the angular velocity of the brick is assumed to be constituted of four triplets of mutually orthogonal accelerometers, attached to four of the vertices of the parallelepiped. The layout is depicted in Fig. 2, where unit vector a_i gives the sensitive direction of the i th accelerometer, whereas point P_i gives its location on the brick. Apparently, all direction vectors are parallel to four of the parallelepiped edges; moreover, $P_1 = P_2 = P_3$, $P_4 = P_5 = P_6$, $P_7 = P_8 = P_9$, and $P_{10} = P_{11} = P_{12}$. The brick dimensions are chosen to be $a = 0.15$ m, $b = 0.2$ m, and $c = 0.25$ m in the X -, Y -, and Z -axis directions, respectively. Let us also define frame B , which is attached to the brick and has its origin B located at the brick centroid. All vector quantities in this simulation are to be expressed in frame B .

4.2 Trajectory. We assume that neither force nor moment is applied to the brick throughout the simulation. Hence, the angular momentum is preserved, which gives

$$I_B \dot{\omega} + \omega \times I_B \omega = 0_3 \quad (20)$$

where I_B is the brick inertia matrix calculated with respect to point B and expressed in frame B . Upon assuming a constant density ρ of the brick material, we obtain

$$I_B = \frac{\rho abc}{12} \begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix} = \rho \begin{bmatrix} 64.1 & 0 & 0 \\ 0 & 53.1 & 0 \\ 0 & 0 & 39.1 \end{bmatrix} \times 10^{-6} \text{ m}^5 \quad (21)$$

Notice that, under these assumptions, the brick trajectory is independent of ρ . We choose the initial condition to be $\omega_0 = [13.33 \ 17.77 \ 22.21]^T$ rad/s, as represented in Fig. 2, where one can see that this vector is parallel to one of the great diagonals of the parallelepiped, its magnitude being 10π rad/s.

4.3 Accelerometer Readouts. The i th accelerometer output is decomposed into one deterministic part $a_{i,r}$, which is the actual point acceleration, and two stochastic parts $\delta a_{i,b}$ and $\delta a_{i,n}$, which are the bias error and the noise error, respectively. This gives, at an instant t ,

$$a_i(t) = a_{i,r}(t) + \delta a_{i,b} + \delta a_{i,n}(t) \quad (22)$$

In the above equation, the bias error is assumed to be time independent and normally distributed, while the noise is assumed to be white and normally distributed as well. The mean and variance of

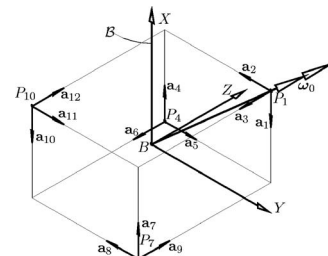


Fig. 2 A brick rotating freely in space

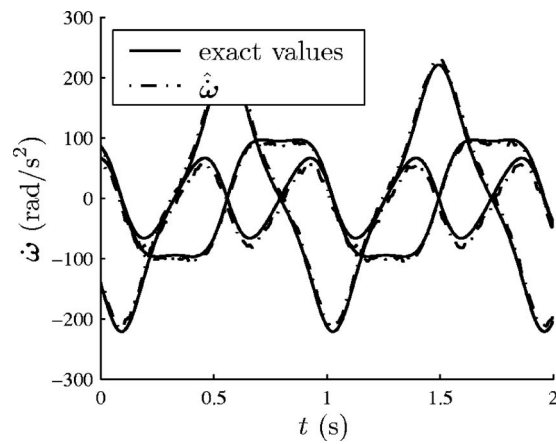


Fig. 3 Angular acceleration estimates

$\delta a_{i,b}$ are taken to be 0 and 300 mg, respectively, while the mean and variance of $\delta a_{i,n}$ are taken to be 0 and 75 mg, respectively. Note that a bias error of 300 mg corresponds to approximately 2% of the required range of the device. All random variables are assumed to be independent. A sampling rate of 100 Hz is assumed. In order to keep the simulation simple, no sensor orientation or position errors are included. Hence, the results obtained here serve as comparison among the different estimation methods.

4.4 Results. The estimated angular acceleration is shown in Fig. 3. Upon integrating the angular acceleration according to Eq. (2), we obtain the TA estimates of Fig. 4. As expected, these estimates drift constantly over time due to bias errors, which renders them useless for applications involving IMU autonomy of more than 1 s.

The estimates obtained from the existing and the proposed CA methods appear in Figs. 5 and 6, respectively. The CAD method exhibits discontinuities whenever one of the components approaches zero to the point that sign errors are introduced in the estimates. The same problem occurs with more amplitude in the CAOD method. Apparently, in this situation, the proposed CA methods are more robust than the existing ones, as can be seen from Fig. 6. Moreover, the CAC and the CANS behave well even when some of the components of the angular-velocity vector are close to zero.

On the other hand, from Fig. 7, it is hard to discriminate the accuracy of the TCA methods from that of the proposed CA methods. One may notice, however, slight discontinuities in the TCAQ estimates. In order to better compare the proposed CA methods

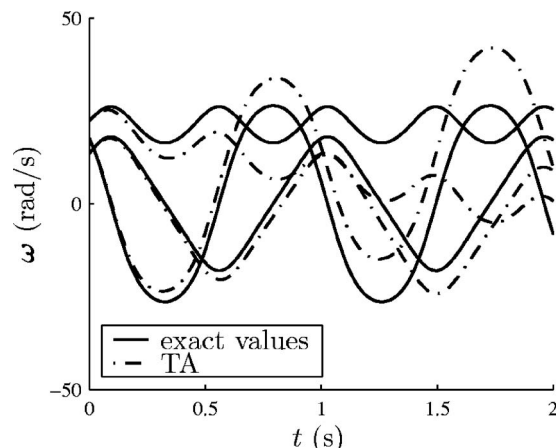


Fig. 4 Angular-velocity estimates from the existing TA method

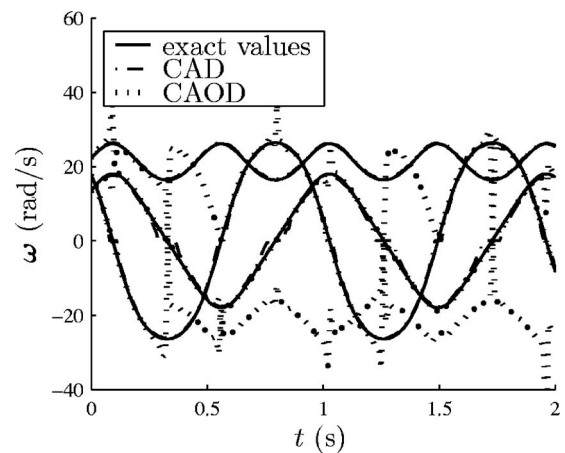


Fig. 5 Angular-velocity estimates from the existing CA methods

with the existing TCA methods, the errors on the angular-velocity estimates are traced in Figs. 8–10. Due to the instability problems when a component approaches zero, the CAD and CAOD methods turn out to be much less accurate than the other ones. Apparently, the CAC, CANS, and the TCAT methods are comparable, whereas the TCAQ method proves to be slightly less robust in this

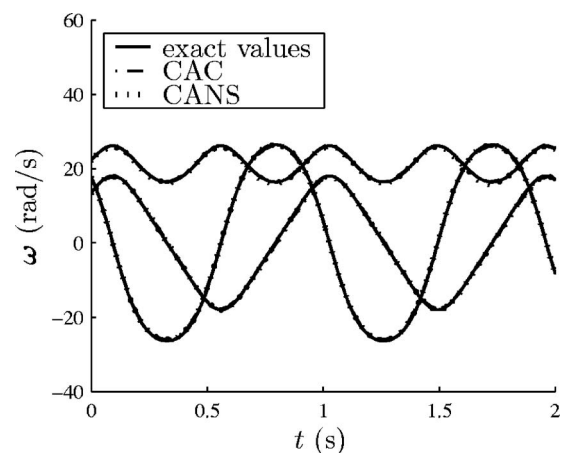


Fig. 6 Angular-velocity estimates from the proposed CA methods

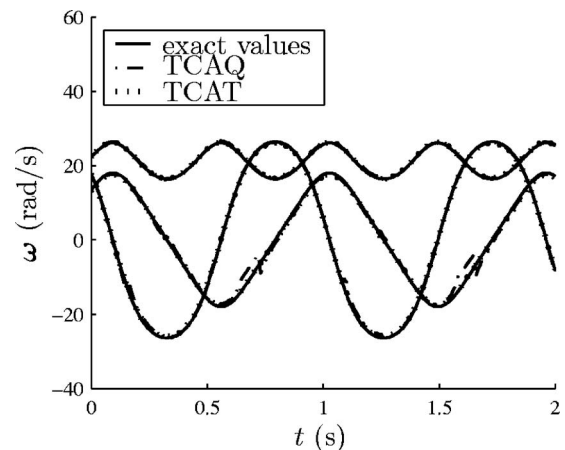


Fig. 7 Angular-velocity estimates from the existing TCA methods

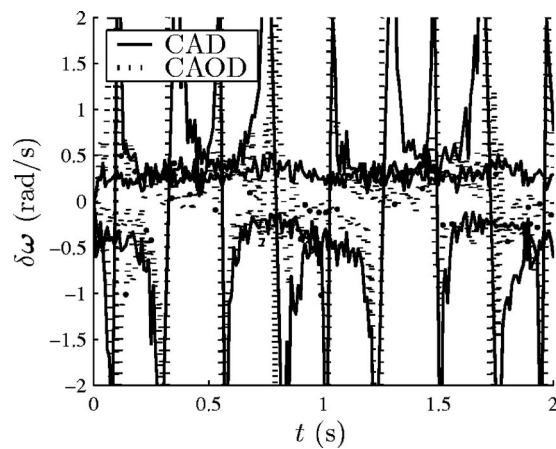


Fig. 8 Errors on the angular-velocity estimates

situation. In fact, from Eq. (6), one may notice that this last method is close to being singular whenever one of the pairs of equations

$$2\omega_i \approx -\omega_j \approx -\omega_k \quad i, j, k = 1, 2, 3 \quad i \neq j \neq k \quad (23)$$

is satisfied, as its associated discriminant approaches zero. Apparently, from Fig. 10, the recorded error peaks in the TCAQ estimates correspond to such situations.

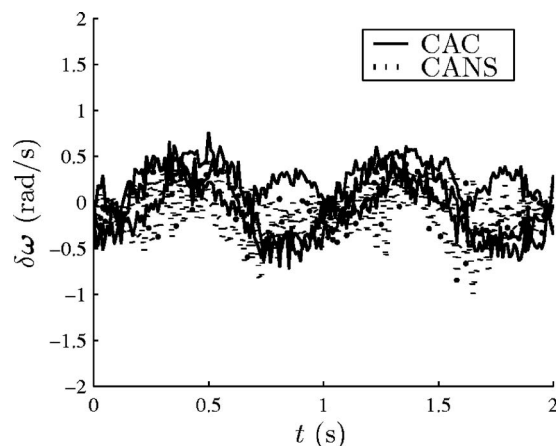


Fig. 9 Errors on the angular-velocity estimates

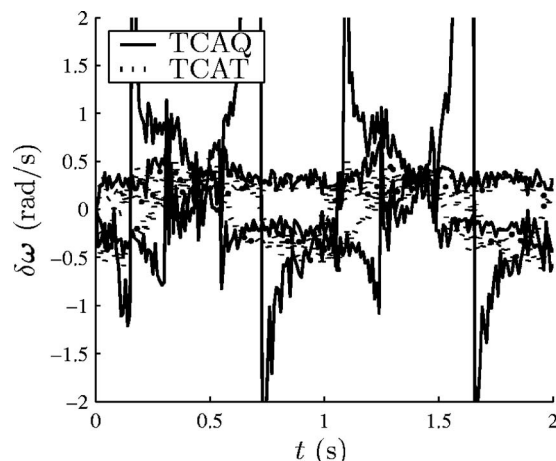


Fig. 10 Errors on the angular-velocity estimates

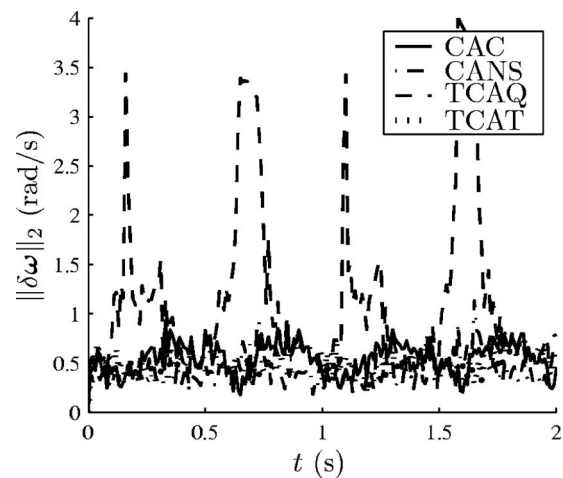


Fig. 11 Norms of the errors on the angular-velocity estimates

In summary, we see that the errors on the CA estimates of the angular velocity are stable over time, unlike the errors on the TA estimates. The Euclidean norms of the errors from the four most accurate methods in this simulation are displayed in Fig. 11. The performance of a method may be assessed by taking the rms value of the magnitude of its associated error, which gives, symbolically,

$$\delta\omega_{\text{rms}} = \sqrt{\frac{1}{n} \sum_{i=1}^n \|\hat{\omega}_i - \omega_i\|_2^2} \quad (24)$$

where ω_i is the true angular velocity at the i th measurement, $\hat{\omega}_i$ is its estimate, and n is the number of samples considered. The rms values resulting from the foregoing simulation are gathered in Table 1, along with their associated computational costs at each time-step. Because of the unstable nature of the TA method, its drift rate is given instead of its error norm rms value.

Sign errors render the CAD and CAOD methods unreliable whenever one of the angular-velocity components reaches zero, and, for that reason, their rms values are significantly higher than those of the other methods. It may be possible to reduce these errors by resorting to TA or TCA methods in the time intervals over which the estimates are prone to instabilities. However, this option is hard to justify, since it requires the evaluation of such an interval, and even harder when seeing that other CA methods circumvent that problem—except, of course, when all the angular-velocity components reach zero. We must state, however, that the CAD and CAOD methods are computationally less intensive than the CAC and the CANS.

On the other hand, it is hard to clearly discriminate the TCA methods and the proposed CA methods, as their performances are comparable in this simulation. To the advantage of the proposed CA methods, we may recall that they do not make use of all the information available in the angular acceleration matrix in that they use only its skew-symmetric component to estimate the signs

Table 1 Error and cost comparison

Method	$\delta\omega_{\text{rms}}$	Flops	Square roots
TA	30.2 rad/s ²	9	0
CAD	27.2 rad/s	18	3
CAOD	16.2 rad/s	21	3
CAC	0.547 rad/s	141	5
CANS	0.889 rad/s	114	3
TCAQ	1.4 rad/s	69	3
TCAT	0.625 rad/s	322	3

of the angular-velocity estimates. Hence, it is allowed to think that further recombination of the CAC and the CANS estimates with the TA estimate may improve the robustness of these methods.

5 Conclusions

Two new methods for estimating the angular velocity of a rigid body from its angular acceleration matrix were proposed. Simulation from noisy accelerometer signals showed that the new methods were more reliable, in the case studied, than the existing CA methods. Moreover, their accuracy was comparable to those of the TCA methods that combine TA and CA measurements.

Note also that the rms value of accelerometer bias errors that we introduced here account for 2.15% of the maximum point acceleration to be measured. On the other hand, the rms values of the errors on the most accurate angular-velocity estimates are approximately 3% of the maximum angular velocity. This shows that accelerometer-only strapdown IMUs have the potential of being competitive with their angular sensor-based counterparts for such a type of motion, provided that the sensors are accurately positioned.⁴

Moreover, the authors believe that there is still room for the improvement of these algorithms. Indeed, the TA and the CA methods have complementary advantages and disadvantages. Combining them correctly may result in the significant reduction of the rms error, without counting on possible advances in accelerometer technology. Note also that military applications are not the only beneficiaries of such improvements. In fact, any high angular speed guided rigid body could use such a system. Motion tracking of a dummy after impact is another application of these methods, but there are many more in biomechanics.

Appendix

In this section, we compute the global minima of the objective function f as defined in Eq. (14). For starters, let us exclude the situation where $\boldsymbol{\omega}=\mathbf{0}_3$, in which case changing frame does not improve anything. Furthermore, as we are to find the frames that minimize the overall error sensitivity, we can readily discard the worst ones, that is, the frames for which the error sensitivity is unbounded. As stated previously, in these frames, at least one of the angular-velocity components is null. From this last assumption, Δ is nonsingular over the selected optimization domain, and we can write

$$[\partial\boldsymbol{\omega}/\partial\mathbf{w}']_C = (1/4)[\Delta^{-1}]_C \mathbf{B} \quad (\text{A1})$$

1 Minimizing f_2

As the Euclidean norm of $\boldsymbol{\omega}$ is invariant, we can further simplify the problem by defining $\mathbf{D} \equiv \Delta/\|\boldsymbol{\omega}\|_2$, where

$$\mathbf{D} \equiv \begin{bmatrix} u_{\omega,1} & 0 & 0 \\ 0 & u_{\omega,2} & 0 \\ 0 & 0 & u_{\omega,3} \end{bmatrix} \text{ and } \mathbf{u}_{\omega} \equiv \begin{bmatrix} u_{\omega,1} \\ u_{\omega,2} \\ u_{\omega,3} \end{bmatrix} \equiv \boldsymbol{\omega}/\|\boldsymbol{\omega}\|_2 \quad (\text{A2})$$

is the unit vector pointing in the same direction as $\boldsymbol{\omega}$. Thus, Eq. (13) can be rewritten as

$$4\|\boldsymbol{\omega}\|_2 [\mathbf{D}]_C [\partial\boldsymbol{\omega}/\partial\mathbf{w}']_C = \mathbf{B} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T \quad (\text{A3})$$

where

$$\mathbf{U} \equiv \begin{bmatrix} 0 & -\sqrt{2/3} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \end{bmatrix} \quad \boldsymbol{\Sigma} \equiv \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

⁴Note that gyroscopes are also subjected to misalignments, but not to position errors.

and

$$\mathbf{V} \equiv \begin{bmatrix} 0 & -\sqrt{2/3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$$

is the singular-value decomposition of $\mathbf{B}$. Note that $\boldsymbol{\Sigma}$ can be further decomposed into $\boldsymbol{\Sigma} = (2)\mathbf{1}_{3 \times 3} - \mathbf{e}_3\mathbf{e}_3^T$, where $\mathbf{e}_i \in \mathbb{R}^3$ has all its components null except for the i th component, which is 1. We perform the foregoing substitution, solve for the Jacobian matrix, and take its Euclidean norm, which yield

$$4\|\boldsymbol{\omega}\|_2 \|\partial\boldsymbol{\omega}/\partial\mathbf{w}'\|_C = \|[\mathbf{D}]_C^{-1} \mathbf{U} (2\mathbf{1}_{3 \times 3} - \mathbf{e}_3\mathbf{e}_3^T) \mathbf{V}^T\|_2 \quad (\text{A4})$$

As the matrix Euclidean norm is invariant under rotations, we can write

$$4\|\boldsymbol{\omega}\|_2 f_2 = \|[\mathbf{D}]_C^{-1} \mathbf{U} (2\mathbf{1}_{3 \times 3} - \mathbf{e}_3\mathbf{e}_3^T) \mathbf{U}^T\|_2 = \|[\mathbf{D}]_C^{-1} (2\mathbf{1}_{3 \times 3} - \mathbf{u}_3\mathbf{u}_3^T)\|_2 \quad (\text{A5})$$

where $\mathbf{u}_3 \equiv -(1/\sqrt{3})[1 \ 1 \ 1]^T$ is the third column vector of $\mathbf{U}$. Upon defining the orthogonal projector $\mathbf{P}_{u,3} \equiv \mathbf{1}_{3 \times 3} - \mathbf{u}_3\mathbf{u}_3^T$ onto the plane normal to $\mathbf{u}_3$, we obtain

$$4\|\boldsymbol{\omega}\|_2 f_2 = \|[\mathbf{D}]_C^{-1} (\mathbf{1}_{3 \times 3} + \mathbf{P}_{u,3})\|_2 \quad (\text{A6})$$

The strategy is now to define a lower bound for the objective function which, if it is inclusive, will prove to be a global minimum of f_2 . Hence, by comparing with Eq. (A6), one can readily verify that

$$4\|\boldsymbol{\omega}\|_2 f_2 \geq \|[\mathbf{D}]_C^{-1} (2\mathbf{P}_{u,3})\|_2 = 2 \max_{\|\mathbf{x}\|_2=1} \|[\mathbf{D}]_C^{-1} \mathbf{P}_{u,3} \mathbf{x}\|_2 \quad (\text{A7})$$

The product $\mathbf{P}_{u,3} \mathbf{x}$ subject to the constraint $\|\mathbf{x}\|_2=1$ can be viewed as a mapping taking the unit sphere onto the unit disk $\mathcal{D}$ centered at the origin and lying in the plane $\mathcal{P}$ normal to $\mathbf{u}_3$. Furthermore, the action of the diagonal matrix $[\mathbf{D}]_C$ on $\mathcal{D}$ can be viewed as a scaling along the three orthogonal directions $\{\mathbf{e}_i\}_{i=1}^3$ corresponding to the three diagonal entries of $[\mathbf{D}]_C$, respectively. It is now apparent that a matrix $[\mathbf{D}]_C$ that minimizes the lower bound is one of the isotropic solutions, that is, $[\mathbf{D}]_C^{-1} = \sqrt{3}\mathbf{1}_{3 \times 3}$. Indeed, any attempt to reduce the i th component of $[\mathbf{D}]_C$ results in an augmentation of at least one of the other components through the relation $\|\mathbf{u}_{\omega}\|_2^2 = 1$, thereby increasing the radius of the circle in a direction orthogonal to $\mathbf{e}_i$. Hence, the lower bound minimum is attained when the circle is scaled uniformly in all directions, which corresponds to a lower bound value of $2\sqrt{3}$. On the other hand, upon substituting the foregoing value of $[\mathbf{D}]_C$ into Eq. (A6), we obtain

$$4\|\boldsymbol{\omega}\|_2 f_2 = \sqrt{3}\|\mathbf{1}_{3 \times 3} + \mathbf{P}_{u,3}\|_2 = 2\sqrt{3} \quad (\text{A8})$$

which is the same result as for the lower bound. Hence, $[\mathbf{D}]_C^{-1} = \sqrt{3}\mathbf{1}_{3 \times 3}$, or, equivalently, $\mathbf{u}_{\omega} = \mathbf{u}_{\omega}^* \equiv (1/\sqrt{3})[1 \ 1 \ 1]^T$, corresponds to a global minimum of the objective function f_2 . In fact, one can readily verify that any of the vectors

$$\mathbf{u}_{\omega} = (1/\sqrt{3})[\pm 1 \ \pm 1 \ \pm 1]^T \quad (\text{A9})$$

also corresponds to a global minimum of f_2 .

2 Minimizing f_1

Upon choosing $p=1$ in Eq. (14), we obtain

$$f_1 = \left\| \begin{bmatrix} \partial\boldsymbol{\omega} \\ \partial\mathbf{w}' \end{bmatrix} \right\|_C = \frac{\|[\mathbf{D}^{-1}]_C \mathbf{B}\|_1}{4\|\boldsymbol{\omega}\|_2} \rightarrow \min_C f_1 \quad (\text{A10})$$

Let us rewrite this minimization problem in terms of $\mathbf{u}_{\omega}$ as defined in Eq. (15), which requires the addition of a constraint equation. After simplification, this yields

$$f_1 = \frac{1}{4\|\omega\|_2} \sum_{i=1,2,3} \frac{1}{|u_{\omega,i}|} \rightarrow \min_{\mathbf{u}_\omega} f_1 \quad (\text{A11})$$

$$g(\mathbf{u}_\omega) = \|\mathbf{u}_\omega\|_2^2 - 1 = 0$$

Let us consider the objective function over the first octant of the Cartesian space only, over which it is continuous. This yields the optimization problem

$$f_1 = \frac{1}{4\|\omega\|_2} \sum_{i=1,2,3} \frac{1}{u_{\omega,i}} \rightarrow \min_{\mathbf{u}_\omega > \mathbf{0}_3} f_1$$

$$g(\mathbf{u}_\omega) = \|\mathbf{u}_\omega\|_2^2 - 1 = 0 \quad (\text{A12})$$

which allows a solution by the Lagrange multipliers method. Accordingly, we write

$$\nabla f_1 + \lambda \nabla g = \frac{-1}{4\|\omega\|_2^2} [1/u_{\omega,1}^2 \quad 1/u_{\omega,2}^2 \quad 1/u_{\omega,3}^2]^T + 2\lambda \mathbf{u}_\omega = \mathbf{0}_3 \quad (\text{A13})$$

where λ is the Lagrange multiplier. We obtain

$$-1/(4\|\omega\|_2^2 u_{\omega,i}^2) + 2\lambda u_{\omega,i} = 0 \text{ or } u_{\omega,i} = 1/(2\sqrt[3]{\lambda}) \quad i = 1, 2, 3 \quad (\text{A14})$$

which yields $\lambda = (\sqrt{3}/2)^3$, after substitution of Eq. (A14) in the constraint equation. The resulting optimum unit vector is $\mathbf{u}_\omega = (1/\sqrt{3})[1 \ 1 \ 1]^T$. Performing the same analysis over the seven other octants of the Cartesian space yields the optima

$$\mathbf{u}_\omega = (1/\sqrt{3})[\pm 1 \quad \pm 1 \quad \pm 1]^T \quad (\text{A15})$$

which are, apparently, the global minima of f_1 .

3 Minimizing f_∞

The objective function obtained from the selection $p = \infty$ is

$$f_\infty = \left\| \left[\frac{\partial \omega}{\partial \mathbf{w}'} \right]_C \right\|_\infty = \frac{\|[\mathbf{D}^{-1}]_C \mathbf{B}\|_\infty}{4\|\omega\|_2} \rightarrow \min_C f_\infty \quad (\text{A16})$$

and is rewritten in terms of $\mathbf{u}_\omega$ as

$$f_\infty = \frac{3}{4\|\omega\|_2} \max_{i=1,2,3} \frac{1}{|u_i|} \rightarrow \min_{\mathbf{u}_\omega} f_\infty$$

$$g(\mathbf{u}_\omega) = \|\mathbf{u}_\omega\|_2^2 - 1 = 0 \quad (\text{A17})$$

The solution of this problem by the method of Lagrange multipliers is cumbersome. Indeed, the presence of the maximum function which experiences discontinuities over the planes $u_{\omega,i} = \pm u_{\omega,j}$, $i, j = 1, 2, 3$, $i \neq j$, forces the partition of the space into 24 pseudopyramids having their apex at the origin and their lateral faces within the planes $u_{\omega,i} = 0$, $u_{\omega,j} = 0$, $u_{\omega,i} = u_{\omega,k}$, and $u_{\omega,j} = u_{\omega,k}$, $i = 1, 2, 3$, $i \neq j \neq k$.

Consider instead the relation

$$f_\infty \geq f_1 \quad (\text{A18})$$

which may be obtained by inspection of Eqs. (A11) and (A17). Upon evaluating the two functions at the global minima of f_1 given in Eq. (A20), we obtain

$$f_1(\mathbf{u}_\omega) = f_\infty(\mathbf{u}_\omega) = (3\sqrt{3})/(4\|\omega\|_2^2) \quad (\text{A19})$$

The inequality (Eq. (A18)), Eq. (A19), and the fact that both objective functions are subjected to the same constraint g imply that the values of the global minima of f_∞ are the same as those of f_1 , namely,

$$\mathbf{u}_\omega = (1/\sqrt{3})[\pm 1 \quad \pm 1 \quad \pm 1]^T \quad (\text{A20})$$

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A Strain Gradient Model for Fracture Prediction in Brittle Materials

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In this paper, we present a new model to predict the fracture in brittle materials from a geometrical weakness presenting an arbitrary stress concentration. The main idea is to combine the strain gradient elasticity with a cohesive model that includes both the displacement and the rotation jumps between the cohesive surfaces in the separation law. Three material parameters were used in the establishment of the fracture criterion. The first two parameters are the commonly used σ_c , the ultimate stress, and G_c , the critical energy release rate. The third parameter is the characteristic length l as in most of the strain gradient models. The proposed three-parameter model enables to take the different stress concentration levels into account, thus providing a criterion to predict fractures for any stress concentration, whether it is singular or not. Experimental results were selected to verify the accuracy and efficiency of the criterion. It was shown that the proposed model is physically reasonable, highly accurate, and easy to apply. It can be used in crack initiation prediction of engineering structures made of brittle materials.
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Keywords: Fracture criterion, strain gradient, cohesive model, PMMA, brittle or quasi-brittle materials, size effect, stress concentration

1 Introduction

Fracture prediction of structures made of brittle materials is an important issue in engineering designs. In real structures, the failures are often initiated from a few geometrical weaknesses near which stress concentrations are formed. The most dangerous geometrical weaknesses, such as the macrocracks, for example, can usually be avoided with appropriate structure designs. However, it is difficult to keep away from other geometrical weaknesses such as holes, notches, etc. The stress concentrations produced near these geometrical weaknesses, even at a weak level, may constitute a source of crack initiation. In brittle materials, the crack initiation is often followed by instable crack propagation, thus leading to the final failure of the structure. In this point of view, the formation of a main crack in brittle materials is often considered as the structure failure.

The stress concentrations are of many types and different levels. The failure prediction for all these stress concentrations is an essential research topic for engineers and scientists since Galileo. However, it seems that so far, fracture can be accurately predicted only for few types of stress states. For brittle materials, failure criteria for two simple situations are commonly accepted.

- (1) When a structure is under uniform uniaxial tension, the maximum tensile stress criterion is used for fracture prediction: fracture occurs when

$$\sigma \geq \sigma_c \quad (1)$$

i.e., when the maximum tensile stress attains the ultimate stress of the material.

- (2) When cracks exist in a structure, the criteria based on the linear fracture mechanics are appropriate to predict the fracture. The mostly used criterion is that proposed by Griffith

on the basis of the energy release rate when new surfaces are created during the cracking. The Griffith criterion predicts the crack growth when

$$G \geq G_c \quad (2)$$

where G and G_c are, respectively, the energy release rate and its critical fracture value.

When the stress distribution is not uniform but does not present a singularity, the maximum stress criterion (Eq. (1)) is still used in many engineering practices. However, it has been shown that in this case, a single material parameter such as the ultimate tensile stress is not sufficient to describe the fracture conditions; many factors such as the stress gradients, the multiaxial stress state, or the structure size may influence the material strength (Bazant [1]).

When the stress concentration presents a weaker singularity than a crack one, such a singularity can be found in the cases of sharp notches, interface cracking, or cracks in ductile materials; criteria based on the finite fracture mechanics (FFM) were developed and reported in the literature. In simple words, these criteria are kinds of combinations of criteria (1) and (2), using both the two material parameters σ_c and G_c in the fracture prediction [2–6]. Another class of two-parameter fracture criteria are issued from the so-called cohesive models [7,8]. Li and Zhang [9] have adapted a large number of these criteria to predict fractures caused by a nonsingular stress concentration. However, it was shown that all of them predict too conservative critical loads compared to the experimental results. In fact, each of these criteria is appropriate only for a particular stress concentration case for which it was developed. Its extension to a more general case often induces important errors. They believed that the two parameters used in these criteria are not sufficient to describe the fracture process under arbitrary stress concentrations and a third parameter should be added in order to better predict brittle fractures. This consideration led them to construct a three-parameter criterion giving more accurate fracture predictions. Nevertheless, we think that it will be beneficial to develop physically more reasonable and technically more practical criteria for this purpose.

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In this paper, we present a fracture model combining a linear strain gradient formulation and a cohesive zone model. The strain gradient theory, associated with the couple-stress elasticity, was initially developed by Toupin [10] and Mindlin and Tiersten [11]. This theory was successfully applied to simulate the size dependency of the deformation behavior in micron scale in metals [12–15] and polymers [16,17]. This theory provides a powerful tool for modeling the material constitutive laws in a microscopic scale. However, it is still rare in the literature to study the fracture process by means of the strain gradient theory.

In this work, the establishment of the cohesive model was based on a conventional separation law by adding to its potential a variable representing the relative rotation between two cohesive surfaces. Consequently, the cohesive forces issued from this potential include not only the tensile forces but also the couple stress between the separated surfaces. This requires the use of the strain gradient elasticity theory in describing the constitutive law of the bulk material. The numerical implementation of this fracture model needs to develop a cohesive finite element including the relative rotation effect. In practice, the cohesive elements can be placed at the fracture path between bulk elements. The fracture model thus built was used to reproduce the experimentation results obtained by Li and Zhang [9] who carried out uniaxial tension tests on dogbone shaped polymethyl metacrylate (PMMA) plates with a central hole. It has been shown that the proposed model is capable to provide highly accurate fracture predictions for all the specimens considered, covering a large range of stress gradient levels.

The organization of this paper is as follows. First, we present briefly the strain gradient theory used in this work and the construction of the cohesive model. Next, we formulate the finite elements to model the bulk material and the cohesive zone. After that, we present the confrontation between the predictions issued from the proposed model and the test results. Finally, we advance some discussions on the efficiency of the proposed fracture model and tried to reveal its physical significances.

2 Theoretical Considerations

In this section, we recall some notions of the strain gradient elasticity and the couple-stress theory. Then, we establish a cohesive potential that takes both the displacement and the rotation jumps between the separated surfaces into account. These theoretical considerations form the frame of the fracture model of brittle materials developed in this work.

2.1 Field Equations for Strain Gradient Elasticity. The theory of strain gradient elasticity was first developed by Toupin [10] and Mindlin and Tiersten [11]. Numerous schemes were developed on the basis of their theory. In one of these schemes, the solid is taken to be homogeneous and isotropic with an energy density function W that depends on scalar invariants of the strain tensor $\boldsymbol{\varepsilon}$ and the deformation curvature tensor $\boldsymbol{\chi}$ with

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{u} \nabla + \nabla \mathbf{u}) \quad (3)$$

$$\boldsymbol{\chi} = \boldsymbol{\theta} \nabla \quad (4)$$

where ∇ is the gradient operator, $\mathbf{u}$ is the displacement vector, and $\boldsymbol{\theta}$ is the rotation vector. Since

$$\boldsymbol{\theta} = -\frac{1}{2} \nabla \times \mathbf{u} \quad (5)$$

the curvature tensor $\boldsymbol{\chi}$ is issued from the second derivatives of the displacement vector and can therefore be considered as a strain gradient tensor. In order to describe the stress state of such a deformable body, couple-stress theory is invoked to complete the characterization. The stress quantities that are work conjugate to these two strain quantities are the symmetric part of the Cauchy stress $\boldsymbol{\sigma}$ and the asymmetric couple stress $\mathbf{m}$. In fact, let $\mathbf{t}$ be the

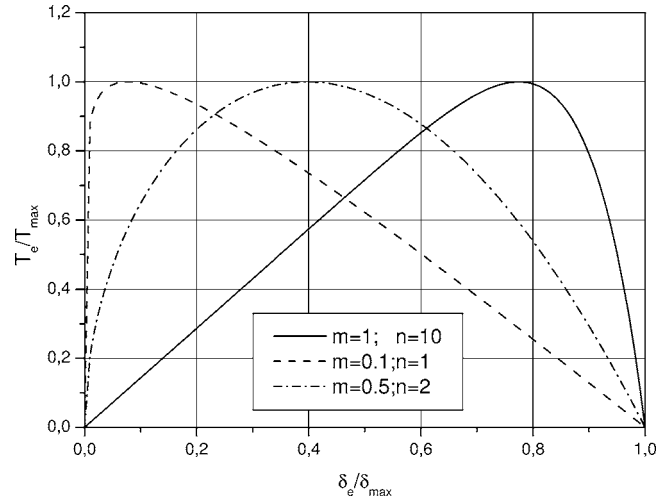


Fig. 1 Separation laws between cohesive surfaces

Cauchy stress tensor, $\mathbf{t} = \boldsymbol{\sigma} + \boldsymbol{\tau}$, where $\boldsymbol{\tau}$ is the antisymmetric part of $\mathbf{t}$ and the couple-stress tensor $\mathbf{m}$ is related to $\boldsymbol{\tau}$ by the formula $(1/2)\mathbf{e} \cdot \nabla \cdot \mathbf{m} = \boldsymbol{\tau}$ with $\mathbf{e}$ being the permutation tensor. These stress quantities are related to the strain quantities $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$ through the energy density W as follows:

$$\boldsymbol{\sigma} = \frac{\partial W(\boldsymbol{\varepsilon}, \boldsymbol{\chi})}{\partial \boldsymbol{\varepsilon}} \quad (6)$$

$$\mathbf{m}^T = \frac{\partial W(\boldsymbol{\varepsilon}, \boldsymbol{\chi})}{\partial \boldsymbol{\chi}} \quad (7)$$

For elastic bodies, one of the most used strain energy functions is written as follows (Yang et al. [18]):

$$W(\boldsymbol{\varepsilon}, \boldsymbol{\chi}) = \frac{1}{2} \lambda (\text{tr } \boldsymbol{\varepsilon})^2 + \mu (\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} + l^2 \boldsymbol{\chi} : \boldsymbol{\chi}) \quad (8)$$

where λ and μ are the Lamé constants and l is a length scale parameter. According to Eqs. (6) and (7), we obtain the constitutive law of the considered elastic body,

$$\boldsymbol{\sigma} = \lambda (\text{tr } \boldsymbol{\varepsilon}) \mathbf{I} + 2\mu \boldsymbol{\varepsilon} \quad (9)$$

$$\mathbf{m}^T = 2\mu l^2 \boldsymbol{\chi} \quad (10)$$

It is clear that when $l=0$, Eqs. (9) and (10) are degenerated to the pure Hooke law.

A principle of virtual work was postulated by Fleck et al. [14]. Let $\mathbf{T}_j = (\sigma_{ij} + \tau_{ij})n_i$ be the traction vector acting on a surface element with unit normal $\mathbf{n}$ and $\mathbf{q}_j = m_{ij}n_i$ be the couple stress traction vector. The principle of virtual work statement for all admissible variations $\delta \mathbf{u}$ is

$$\int_S (T_i \delta u_i + q_i \delta m_i) dS = \int_V (\sigma_{ij} \delta \varepsilon_{ij} + m_{ij} \delta \chi_{ij}) dV \quad (11)$$

The associated equilibrium equations in the absence of body forces or couples are

$$\sigma_{ji,j} + \tau_{ji,j} = 0 \quad (12)$$

$$\tau_{jk} + \frac{1}{2} e_{ijk} m_{pi,p} = 0 \quad (13)$$

In plane problems, the only nonzero component in the rotation vector $\boldsymbol{\theta}$ is θ_3 . Therefore, the nonzero components in the deformation curvature tensor $\boldsymbol{\chi}$ are $\chi_{31} = \partial \theta_3 / \partial x$ and $\chi_{32} = \partial \theta_3 / \partial x$ and the nonzero components in the couple-stress tensor $\mathbf{m}$ are $m_{13} = 2\mu l^2 \chi_{31}$ and $m_{23} = 2\mu l^2 \chi_{32}$.

2.2 Cohesive Model With Rotation Jump. Barenblatt [7] assumed that a cohesive zone exists immediately ahead of the crack tip. The cohesive forces depend on the separation distance between the two crack lips. This zone develops progressively as the remote loading increases before the separation energy provided by the remote loads overcomes the cohesive energy due to the cohesive forces. Once this critical state is achieved, the crack propagation is supposed to occur. This model was successfully used to predict fractures in different materials due to different mechanisms [19–22].

In traditional cohesive models, the cohesive force vector $\mathbf{T}$ essentially depends on the separation vector between the two surfaces $[\delta]$. This will change when the strain gradient is taken into

consideration. Within the framework of the above-mentioned couple-stress theory, the cohesive forces will include the couple-stress traction vector $\mathbf{q}$ due to the relative rotation $[\theta]$ between the two separated surfaces. In general, we can define a cohesive potential $\varphi([\delta], [\theta])$ such that

$$\mathbf{T} = \frac{\partial \varphi}{\partial [\delta]} \quad \mathbf{q} = \frac{\partial \varphi}{\partial [\theta]} \quad (14)$$

In the present work, only the plane problem is considered. If we assume that the displacement jump in the cohesive zone only occurs in the normal direction of the separated surfaces, the nonzero quantities in $[\delta]$ and $[\theta]$ are $[\delta_n]$ and $[\theta_3]$ that we notice as δ and θ for brevity. Thus, we can define a cohesive potential as follows:

$$\varphi = \begin{cases} 0 & \text{for } \delta_e < 0 \\ a\delta_{\max} \left[\frac{1}{m+1} \left(\frac{\delta_e}{\delta_{\max}} \right)^{m+1} - \frac{1}{n+1} \left(\frac{\delta_e}{\delta_{\max}} \right)^{n+1} \right] & \text{for } 0 \leq \delta_e \leq \delta_{\max} \\ a\delta_{\max} \left(\frac{1}{m+1} - \frac{1}{n+1} \right) & \text{for } \delta_{\max} < \delta_e \end{cases} \quad (15)$$

with δ_e defined as the effective displacement jump as follows:

$$\delta_e^2 = \delta^2 + l_c^2 \theta^2 \quad (16)$$

where a , m , n , l_c , and $\delta_{\max}$ are material constants. In the case when $l_c=0$, this potential is degenerated to a conventional potential similar to that of Lennard-Jones [23].

The dual quantities related to the normal separation and the relative rotation are the cohesive force T and the couple-stress traction q . According to Eq. (14), we obtain, for $0 \leq \delta_e \leq \delta_{\max}$,

$$\begin{aligned} T &= \frac{\partial \varphi}{\partial \delta} = a \left[\left(\frac{\delta_e}{\delta_{\max}} \right)^m - \left(\frac{\delta_e}{\delta_{\max}} \right)^n \right] \frac{\delta}{\delta_e} \\ q &= \frac{\partial \varphi}{\partial \theta} = a \left[\left(\frac{\delta_e}{\delta_{\max}} \right)^m - \left(\frac{\delta_e}{\delta_{\max}} \right)^n \right] \frac{l_c^2 \theta}{\delta_e} \end{aligned} \quad (17)$$

If we define a quantity T_e called the effective cohesive force such that

$$T_e^2 = T^2 + \left(\frac{q}{l_c} \right)^2 = a^2 \left[\left(\frac{\delta_e}{\delta_{\max}} \right)^m - \left(\frac{\delta_e}{\delta_{\max}} \right)^n \right]^2 \quad (18)$$

it is clear that

$$T_e = \frac{\partial \varphi}{\partial \delta_e} \quad (19)$$

2.3 Properties and Identification of the Proposed Cohesive Model. Even for conventional cohesive models, the identification of material parameters is not an easy task. Amongst different cohesive models, the main difference lies in the shape of the traction-separation response and the constants used to describe that shape. The shape of the traction-separation response of the above-mentioned cohesive model can be adjusted by choosing different parameters m and n . Figure 1 shows the δ_e - T_e relations by using different values of m and n . For brittle materials, one often supposes that the δ_e - T_e relations are essentially linear for small δ_e . Once δ_e reaches a critical value, the effective cohesive force T_e drops dramatically and thus the fracture process exhibits a brittle aspect. This leads us to choose $m=1$ with a large n , $n=10$, for example. Thus, we rewrite the cohesive potential (Eq. (15))

$$\varphi = a\delta_{\max} \left[\frac{1}{2} \left(\frac{\delta_e}{\delta_{\max}} \right)^2 - \frac{1}{n+1} \left(\frac{\delta_e}{\delta_{\max}} \right)^{n+1} \right] \quad (20)$$

The δ_e - T_e relation is, according to Eq. (19),

$$T_e = \frac{\partial \varphi}{\partial \delta_e} = a \left[\frac{\delta_e}{\delta_{\max}} - \left(\frac{\delta_e}{\delta_{\max}} \right)^n \right] \quad (21)$$

This relation is also illustrated in Fig. 1. Therefore, we have still three material parameters a , l_c , and $\delta_{\max}$ to determine in this cohesive model.

For conventional cohesive models, there is common belief that these models can be described by two independent parameters [24,25]. These parameters may be two of three parameters, namely, the maximum cohesive strength σ_c , the maximum separation distance $\delta_{\max}$, or the cohesive energy γ_c , that can be related to the two precedent parameters. (It is believed that γ_c is equivalent to the critical energy release rate G_c .) If the gradient of the separation distance, i.e., the relative rotation, is introduced into a cohesive model, we have to consider at least three material parameters. For the potential used in the present work, these material parameters are a , l_c , and $\delta_{\max}$.

Let us first give an estimation of the value of l_c . To this end, we adopt the hypothesis that the constitutive behavior is essentially identical for both the bulk material and the material in cohesive zones, especially in their elastic regime. This hypothesis is justified by the fact that the nonlinear part of the cohesive potential is nearly negligible compared to that of the linear part. Therefore, we can write, for a large range of δ ,

$$\varphi \approx a\delta_{\max} \frac{1}{2} \left(\frac{\delta_e}{\delta_{\max}} \right)^2 = a \frac{\delta^2 + l_c^2 \theta^2}{2\delta_{\max}} \quad (22)$$

Consider now a cohesive zone of length dx and of height $2h$. The deformation of this cohesive zone is characterized by an average separation distance δ and an average relative rotation $\theta = \theta(h)y/h$, as shown in Fig. 2. The nonzero strain components and their gradients corresponding to this deformation are $\varepsilon_{22} = \delta/2h$ and $\chi_{32} = \partial\theta/\partial y = \theta(h)/h$. The other strain components and their gradients are neglected. Since the cohesive material has the same constitutive behavior as the bulk material, the total strain energy stocked in the cohesive zone can be calculated according to Eq. (8),

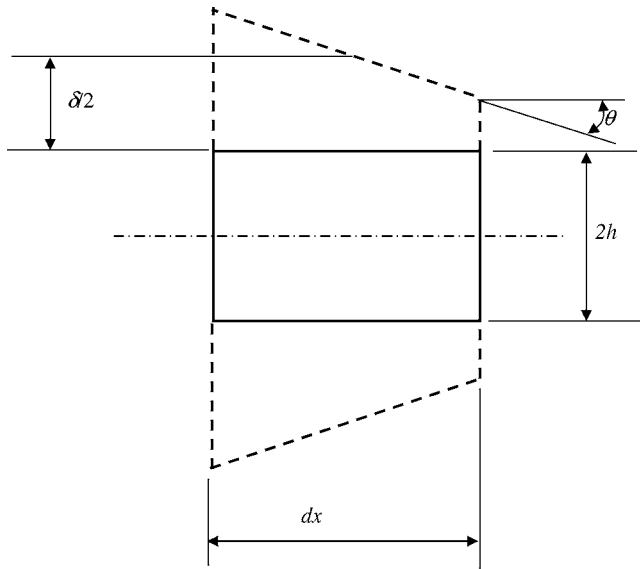


Fig. 2 A deformed infinitesimal cohesive element

$$dW = \left[\int_{-h}^h \left(\frac{E}{2} \varepsilon_{22}^2 + \mu l^2 \chi_{32}^2 \right) dy \right] dx = \left(\frac{E \delta^2}{4h} + 2\mu l^2 \frac{\theta^2}{h} \right) dx \quad (23)$$

Comparing to Eq. (22), we have $\varphi = dW/dx$ and therefore

$$\frac{a}{2\delta_{\max}} = \frac{E}{4h} \frac{a l_c^2}{2\delta_{\max}} = \frac{2\mu l^2}{h} \quad (24)$$

These equations lead to

$$l_c = \frac{2}{\sqrt{1+\nu}} l \quad (25)$$

This result shows that the values of l_c and l are of the same order. In the case of $\nu=0.3$, we have $l_c=1.754l$.

Now, let us determine the values of the parameters a and $\delta_{\max}$. First, according to Eq. (17), we compute the cohesive tractions

$$T = a \left[\frac{\delta_e}{\delta_{\max}} - \left(\frac{\delta_e}{\delta_{\max}} \right)^n \right] \frac{\delta}{\delta_e} \quad (26)$$

$$q = a \left[\frac{\delta_e}{\delta_{\max}} - \left(\frac{\delta_e}{\delta_{\max}} \right)^n \right] \frac{l_c^2 \theta}{\delta_e}$$

The critical energy release rate G_c is defined as the work done by the cohesive forces on a unit cohesive length and during whole the separation procedure, namely,

$$G_c = \lim_{d \rightarrow 0} \frac{1}{d} \int_0^d \left\{ \int_0^{\delta_c} T_n d\delta + \int_0^{\theta_c} q d\theta \right\} dx$$

$$= \lim_{d \rightarrow 0} \frac{1}{d} \int_0^d \left\{ \int_0^{\delta_{\max}} T_e d\delta_e \right\} dx \quad (27)$$

This definition leads to

$$G_c = \varphi(\delta_e = \delta_{\max}) = a \delta_{\max} \left(\frac{1}{2} - \frac{1}{n+1} \right) \quad (28)$$

It is clear that G_c just represents the area under the δ_e - T_e curve. On the other hand, we suppose that the effective cohesive traction T_e is limited by the ultimate stress of the material σ_c , which is measured by uniform tension tests. Therefore, from Eq. (21), we look for the maximal value of T_e ,

$$\frac{\partial T_e}{\partial \delta_e} = \frac{a}{\delta_{\max}} \left[1 - n \left(\frac{\delta_e}{\delta_{\max}} \right)^{n-1} \right] = 0 \Rightarrow \delta_e = \delta_{\max} n^{1/(1-n)} \quad (29)$$

and therefore

$$T_{e \max} = a [n^{1/(1-n)} - n^{n/(1-n)}] = \sigma_c \quad (30)$$

From Eqs. (28) and (30), we obtain

$$a = \frac{\sigma_c}{[n^{1/(1-n)} - n^{n/(1-n)}]} \quad \delta_{\max} = \frac{G_c [n^{1/(1-n)} - n^{n/(1-n)}]}{\sigma_c (1/2 - 1/(n+1))} \quad (31)$$

Thus, the constants a and $\delta_{\max}$ are related to the material parameters σ_c and G_c . In the case when $n=10$, we have $a=1.435\sigma_c$ and $\delta_{\max}=1.703G_c/\sigma_c$.

3 Finite Element Modeling

In order to construct a finite element model with strain gradient, we need to establish two types of elements: the first one is the bidimensional linear elements with strain gradient and the second one is a cohesive element with relative rotation between the separated surfaces.

3.1 Hybrid Elements for Plane Strain Deformation. The establishment of the strain gradient element in continuous elastic solid can be performed by introducing the following variational principle [26,27]:

$$\Pi(\mathbf{u}, \boldsymbol{\theta}, \boldsymbol{\tau}) = \int_V W(\boldsymbol{\varepsilon}, \boldsymbol{\chi}) dV + \int_V \boldsymbol{\tau}_{ij} \alpha_{ij} dV - \int_{S_T} (T_i^0 u_i + q_i^0 \theta_i) dS$$

$$\delta \Pi(\mathbf{u}, \boldsymbol{\theta}, \boldsymbol{\tau}) = 0 \quad (32)$$

with

$$\alpha_{ij} = e_{ijk} \theta_k - \frac{1}{2} (u_{j,i} - u_{i,j}) \quad (33)$$

In Eq. (32), both the displacements u_i and the rotations θ_i are considered as independent variables. For a couple-stress solid, $\alpha_{ij}=0$ enforces the relationship between rotations θ_i and displacements u_i . The variational statement (Eq. (32)) is derived through enforcing $\alpha_{ij}=0$ in an averaged sense in Eq. (5) by the use of the Lagrange multipliers τ_{ij} , which are the antisymmetric stress components. This principle was first used by Herrmann [26] as the basis for a finite element formulation of couple-stress solids and later by other authors.

Plane strain elements based on Eq. (32) have as primary dependent variables the displacement components (u_1, u_2) and the rotation θ_3 and the antisymmetric component of the stress $\tau = \tau_{12} = -\tau_{21}$. Within this framework, we have formulated 6- and 8-noded elements, see Fig. 3. For these elements, there are three nodal values at each node, namely, (u_1, u_2, θ_3), and one constant τ within each element. Therefore, both displacements and rotations are continuous across elements, but not τ . The total degrees of freedom for an N -node element are thus $(3N+1)$. These are standard isoparametric elements, with the same shape functions for u_1, u_2 , and θ_3 . The formulation details are omitted here for brevity.

3.2 Cohesive Elements. The cohesive elements can be constructed to represent the cohesive model described in Sec. 2.2. Consider the two solid elements Ω_1 and Ω_2 separated by an infinitesimally thin element with surfaces S_1 and S_2 being the part of Ω_1 and Ω_2 , respectively. Mathematically, we consider surfaces $S_1=S_2=S$ in the initial configuration and their corresponding normal vectors such that $\mathbf{n}_1=\mathbf{n}_2=\mathbf{n}$. Thus, we define a 6-node cohesive element with the Nodes 1, 2, and 3 belonging to S_1 and the Nodes 4, 5, and 6 belonging to S_2 , as illustrated in Fig. 4. The displacement jump between the two surfaces described in the global system is therefore

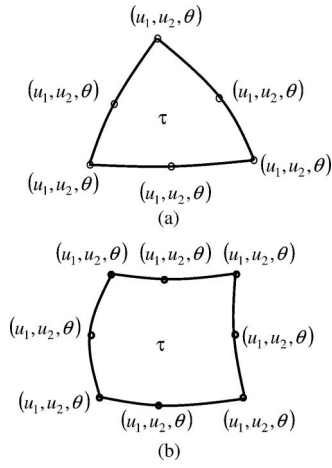


Fig. 3 Hybrid strain gradient bidimensional elements: (a) a 6-noded element and (b) a height-noded element

$$[\mathbf{u}]_{xy} = \mathbf{u}^+ - \mathbf{u}^- \quad (34)$$

where $[\mathbf{u}]_{xy} = \{[u_1] [u_2] [\theta_3]\}^T$ is the displacement jump vector between the displacement vectors $\mathbf{u}^+$ and $\mathbf{u}^-$ on S_1 and S_2 , respectively. In the local coordinate system, we have $[\mathbf{u}]_{nt} = \{[u_n] [u_t] [\theta_3]\}^T$; notice that the tangent displacement jump is not considered in this work. For brevity and in accordance with the notation in Sec. 2.2, we notice $[\mathbf{u}]_{nt} = \{\delta \ 0 \ \theta\}^T$. Between these two representations of the displacement jump vector, we have the following relationship:

$$[\mathbf{u}]_{xy} = \mathbf{R}[\mathbf{u}]_{nt} \quad (35)$$

with

$$\mathbf{R} = \begin{bmatrix} n_x & -n_y & 0 \\ n_y & n_x & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (36)$$

According to the separation law defined in Eq. (26), the cohesive traction vector in the local coordinate system can be related to the displacement jump vector by

$$\mathbf{T}_{nt} = \Phi[\mathbf{u}]_{nt} \quad (37)$$

with

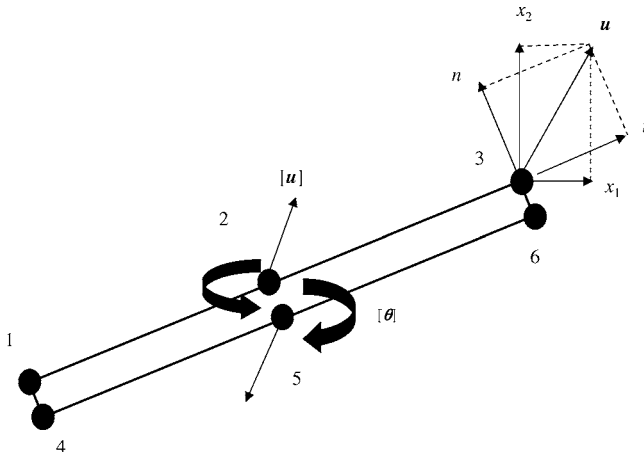


Fig. 4 Cohesive element

$$\Phi = a \left[\frac{1}{\delta_{\max}} - \left(\frac{\delta_e}{\delta_{\max}} \right)^{1/n} \frac{1}{\delta_e} \right] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \phi_t & 0 \\ 0 & 0 & l_c^2 \end{bmatrix} \quad (38)$$

Since $[u_t] = 0$, we can prescribe a large value to the term ϕ_t . The cohesive force vector in the global coordinate system can be obtained by applying the coordinate transformation, namely,

$$\mathbf{T}_{xy} = \mathbf{R} \mathbf{T}_{nt} \quad (39)$$

Let us now represent the displacements within the cohesive element by means of the nodal displacement vector $\mathbf{q} = \{u_1 \ v_1 \ \theta_1 \ \dots \ u_6 \ v_6 \ \theta_6\}^T$, namely,

$$[\mathbf{u}]_{xy} = \mathbf{N} \mathbf{q} \quad (40)$$

where $\mathbf{N}$ is the shape function matrix. For any admissible virtual displacement field, the virtual work principle states

$$\delta W_{\text{int}} = \delta W_{\text{ext}} \quad (41)$$

where W_{int} is the work of the internal forces within the cohesive element, according to Eqs. (35) and (39),

$$\begin{aligned} W_{\text{int}} &= \int_L \mathbf{T}_{xy}^T [\mathbf{u}]_{xy} dl = \int_L \mathbf{T}_{nt}^T \mathbf{R}^T [\mathbf{u}]_{xy} dl = \int_L [\mathbf{u}]_{nt}^T \Phi^T \mathbf{R}^T [\mathbf{u}]_{xy} dl \\ &= \int_L [\mathbf{u}]_{xy}^T \mathbf{R}^{-T} \Phi^T \mathbf{R}^T [\mathbf{u}]_{xy} dl = \mathbf{q}^T \left[\int_L \mathbf{N}^T \mathbf{R} \Phi^T \mathbf{R}^T \mathbf{N} dl \right] \mathbf{q} \end{aligned} \quad (42)$$

W_{ext} is the work of the external forces. Let $\mathbf{F}$ be the external force vector applied on the nodes of the element; we can write

$$W_{\text{ext}} = \mathbf{q}^T \mathbf{F} \quad (43)$$

Therefore, the virtual work principle leads to

$$\mathbf{K} \mathbf{q} = \mathbf{F} \quad (44)$$

where

$$\mathbf{K} = \int_L \mathbf{N}^T \mathbf{R}^{-T} \Phi^T \mathbf{R}^T \mathbf{N} dl \quad (45)$$

is the rigidity matrix of the cohesive element. It is clear that this rigidity matrix is not a constant matrix; thus, a nonlinear system has to be resolved by using iterative procedures.

4 Verification of the Model by Experimental Results

In this section, we will verify the proposed strain gradient cohesive model by using the experimental results obtained by Li and Zhang [9] who carried out uniaxial tension tests on dogbone shaped PMMA plates with a central hole. The mechanical characteristics of the used material are as follows: the elastic modulus $E = 3000$ MPa, the Poisson ratio $\nu = 0.36$, the ultimate tensile stress $\sigma_c = 72$ MPa, and the critical release energy rate $G_c = 290$ J/m. The section of the specimens is 10×30 mm². Central holes were drilled with different diameters, namely, $d = 0.6$ mm, 1.2 mm, 2 mm, and 3 mm. Specimens without holes were also prepared and tested for comparison. The test procedures and the results were detailed in Ref. [9].

Important size effect can be observed from the test results within the chosen hole size range. It is seen that the strength of the central-holed specimens depends strongly on the hole diameter. First, the critical fracture load increases as the hole size decreases. Second, it tends to the critical material strength σ_c as the hole size tends to zero and to $\sigma_c/3$ as the hole size is large enough.

The stress distribution near a circular hole in an infinite plate under uniaxial tensile stress is well known [28]. When the hole sizes are much smaller with respect to the rest specimen dimensions, the maximum tensile stress occurs at the hole boundary with $\sigma_{\theta\theta}(r=a, \theta=0) = 3\sigma_{\infty}$ whatever the hole size. According to

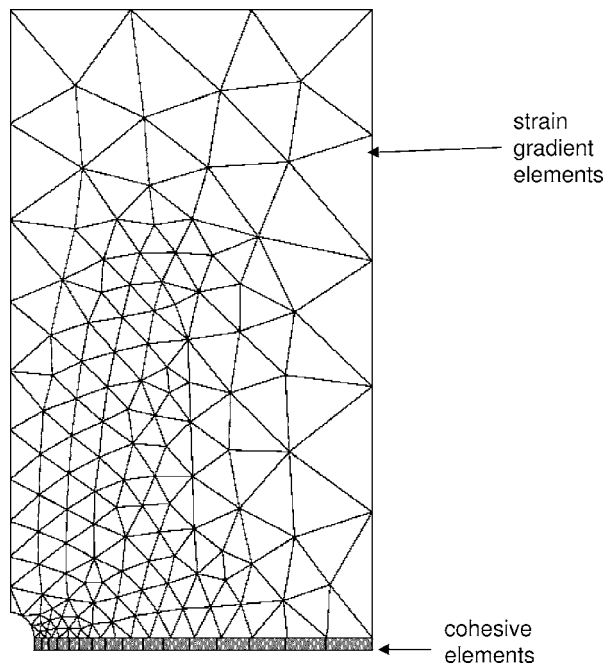


Fig. 5 A typical mesh of the finite element models

criterion (1), the specimens would be broken at $\sigma_{\infty} = \sigma_0/3$ whatever the hole size. It is clear that this prediction is not correct compared with the experimental results.

We will apply the established model to predict the fracture of the tested specimens. To this end, we construct a finite element model representing the experimentation. The plates with a central hole were meshed with 6-node triangle strain elements. Only a quarter of the plate was meshed for reasons of symmetry. The cohesive elements were placed on the ligament in order to assess the fracture process. The remote load was applied on the top of the plate by prescribing displacements in the y direction. A typical finite element mesh is represented in Fig. 5.

The fracture loads were calculated by using an incremental procedure. The prescribed displacement was applied gradually. At each step, iterations were carried out until convergence. When a particular load level is reached, iterations will lead to the complete separation of the ligament. This load will be considered as the critical fracture load level of the specimen.

The only unknown material parameter in this model is the characteristic length l in the strain gradient elements (the characteristic length l_c in the cohesive elements is related to l through Eq. (25)). Figure 6 illustrates the critical fracture loads predicted by the present model for different hole sizes and different values of the parameter l . The experimental results are also plotted for comparison. First, we can observe that the present model can describe the size effect observed in the experimentation whatever the value of l , i.e., the critical fracture load increases as the hole size decreases. However, we can also remark the very important influence of l on the prediction accuracy. Roughly speaking, neglecting the strain gradient effect ($l=0$) will lead to a too conservative fracture prediction. On the contrary, a too large l will overestimate the critical fracture loads. For the PMMA used in the present study, a value of $l=0.5$ mm gives a suitable fit to the experimental data.

5 Discussions

In this paper, we have established a fracture model to predict the failure of a structure initiated from a stress concentration source. This model is established on the strain gradient finite elements associated with the basis of the cohesive elements. In this model, three material parameters are necessary in order to de-

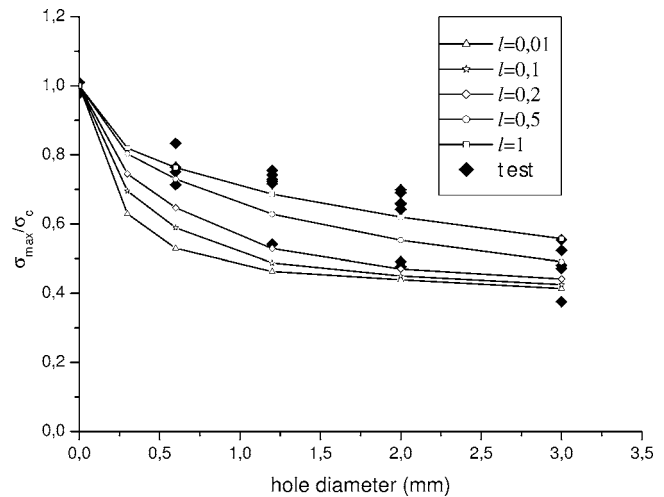


Fig. 6 Predicted critical loads at fracture with different length parameters l

scribe the cohesive behaviors, namely, the ultimate tensile stress σ_c , the critical release energy rate G_c , and a length parameter l . Comparing with conventional cohesive models, we have introduced a supplementary length parameter l . In the following, we will examine the role of this parameter and try to reveal its physical significance.

5.1 Other Cohesive Models. First, one can suppose that other cohesive models may fit better the experimental data; thus, the introduction of the strain gradient theory may be not absolutely necessary. In fact, different cohesive models are available in the literature. In the following, we will use two typical cohesive models, namely, the Dugdale model [8] and the linear softening cohesive model [20], to predict the critical fracture loads of the considered specimens.

5.1.1 Dugdale Model. The application of the Dugdale model in the present problem can be described as follows. An assumed crack, corresponding to the damaged zone along the ligament, is opened by remote loading σ_{∞} with an undetermined length d_c . The uniformly distributed cohesive forces are applied on the crack lips. The cohesive potential $\varphi([u])$ can be written as follows:

$$\varphi = [u]\sigma_c \quad (46)$$

This assumed crack is stable while the remote load is smaller than its critical value. The two unknown critical parameters σ_{∞} and d_c can be determined by considering the following conditions:

- (1) The stresses are regular at the ends of the cohesive zone, i.e., the resultant stress intensity factor induced by the remote loading and the cohesive forces is nil, namely,

$$K(\sigma_{\infty}, d_c) = 0 \quad (47)$$

- (2) The cohesive surfaces separate when the cohesive potential reaches the critical energy release rate of the material,

$$\varphi(\sigma_{\infty}, d_c) = G_c \quad (48)$$

The detailed description of this application to the present holed plates was given by Li and Zhang [9].

5.1.2 Linear Softening Cohesive Model. Another simple relation is the linear softening cohesive fracture model [20]. In this model, the interface begins to separate when the traction reaches a maximum stress σ_c , then the traction decreases linearly as the surface separation increases, and vanishes for a critical separation value $[u]_{\max}$. Thus, the relationship between cohesive force and surface separation is written as follows:

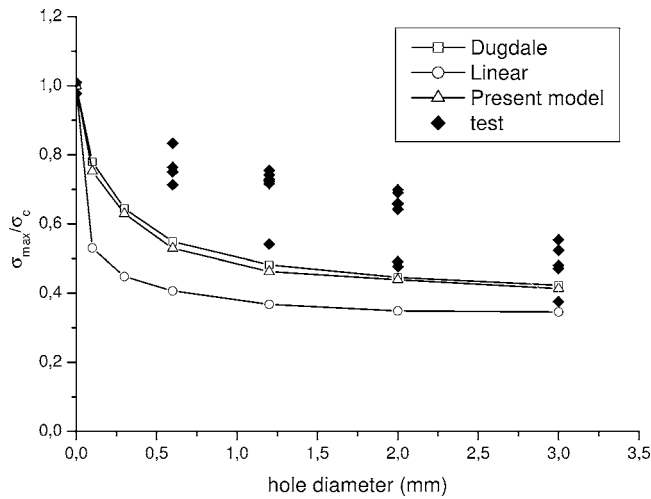


Fig. 7 Predicted critical loads at fracture with different cohesive models

$$T = \sigma_c \left(1 - \frac{[u]}{[u]_{\max}} \right) \quad (49)$$

It is generally accepted that the area under the traction-displacement curve represents the cohesive energy $\varphi_{\max}$ or the fracture toughness G_c [29]. Hence,

$$G_c = \frac{1}{2} [u]_{\max} \sigma_c \quad (50)$$

The application of this model to the present problem is similar to that of the Dugdale model (Li and Zhang [9]).

Figure 7 illustrates the simulation results of the critical remote stresses for different hole sizes by using the above-mentioned cohesive models. We can remark that the Dugdale model predicts the best fit to the experimental data. Similar results were obtained with the proposed cohesive model. The prediction of the linear softening model is much more conservative. In general, all these models give lower critical loads at fracture compared with the experimental data. Since the shapes of the traction-separation curves of other cohesive models are often enveloped between the Dugdale model and the linear model, we can expect that the predictions given by using other cohesive models are also enveloped by those obtained by using these two cohesive models. These results suggest that conventional cohesive models are not capable to accurately describe the size effect observed in the tests.

5.2 Influence of the Values of G_c . The mediocre performance of the conventional cohesive model may also be attributed to the inaccuracy of the material parameters, namely, σ_c and G_c . In general, σ_c is measured from uniaxial uniform tensile tests and quite a good accuracy can be guaranteed. On the other hand, the critical release energy rate G_c is measured from a crack propagation test, in which the measurement accuracy may be worse than that of σ_c . Hereafter, we will only examine the influence of the accuracy of G_c on the prediction of the critical fracture loads. To this end, we calculated the critical fracture loads by using different values of G_c and by fixing a small value of l ($l=0.01$ mm) such that the couple stress induced by the relative rotation can be neglected. Different values of the critical strain energy release rate, namely, gG_c with the factor $g=1, 2, 4, 8$, and 16 , were used in these calculations. Figure 8 shows the results of this calculation. From this figure, we remark that only large values of G_c can fit well the experimental data. If the effect of l is neglected, we should need a G_c about ten times larger than the measured value in order to fit well the present experimental data. Consequently, the measure-

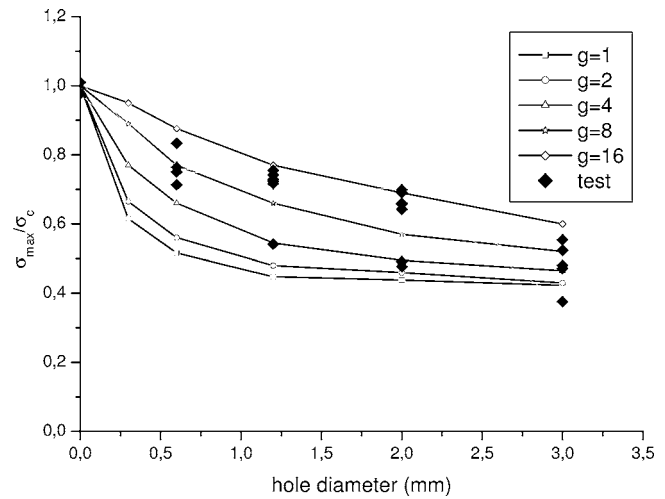


Fig. 8 Predicted critical loads at fracture with different critical strain energy release rates gG_c

ment inaccuracy of G_c that we estimated to be about 30% cannot explain the underestimation of the critical fracture loads by the used cohesive model.

5.3 Validity of the Present Model. It is well known that the strain gradient theory is suitable to describe the size effects observed in a microscopic scale in some materials. The main parameters in this theory are the length parameters introduced in the material law, for example, the parameter l in Eq. (8). Physically speaking, these length parameters represent a microscopic scale of the material heterogeneities such as the distance between atoms in crystals, the average chain length in polymers, the average size of grains in alloys, and so on. In the literature, the measurement of this parameter is often based on the materials response at a microscopic scale. In the context of elasticity, a few works have employed atomistic methods and/or phonon dispersion curves to obtain this order parameter [30–34]. These works indicated that the characteristic length scale for nonlocal effects is within the order of several nanometers for pure metals; it may be larger for polymers and composites. Lam and Chong [16] and Lam et al. [17] estimated that the nonlocal length in epoxy polymers can reach 10–50 μm .

We have not found available measurements of this parameter for PMMA materials in the literature so far. For the PMMA used in the present work, we found that $l \approx 0.5$ mm provides a satisfactory agreement between test results and model predictions. Comparing to the values usually measured for polymers, this value seems too large. Hereafter, we attempt to give a reasonable physical explication of this apparent abnormality.

As we have noticed, the parameter l in strain gradient theories represents a microscopic length scale of the material. This length is proportional to the average dimension of the heterogeneities in the material. That is why l is larger in polymers or in composites than in crystals. It is also to notice that this parameter is often measured in the intact state of the materials. However, when we would like to use this parameter to describe the fracture process, we have to consider the fact that the material may be highly damaged at fracture. As a consequence, the heterogeneities may be largely developed compared to those in the intact state of the material. Therefore, l must be larger for damaged materials than for intact materials.

We can find some experimental works in the literature to support this argument. For instance, Ravi-chadar and Yang [35] have studied the dynamic fracture in several polymers including the PMMA. The examination of the fracture surfaces reveals clearly that the operative micromechanisms that govern dynamic fracture

in brittle materials are the nucleation, growth, and coalescence of secondary microcracks. Others studies [36] also showed the existence of a large number of microcracks before the final failure in the PMMA under monotone or cyclic loading. Under certain conditions, the length of these microcracks may attain 0.4–0.6 mm before the final failure. Therefore, the heterogeneity increases in the damaged PMMA due to the nucleation and the growth of the secondary microcracks; thus, the characteristic length l in the damaged PMMA can become much larger. From this point of view, a large value of l found in this work is not really surprising.

According to this analysis, a logical approach is to adopt an evolution law for l as function of damage. Taking a fixed large value of l as in the present work may modify the macroscopic behavior of the material in its undamaged state and introduce errors in the calculations. In order to ensure the validity of the present model, a necessary condition is that the global stiffness of the specimens is not significantly modified by the use of large value of l . For this, we can perform the calculations with a conventional model, i.e., $l=0$, and with a strain gradient model, i.e., $l \neq 0$. This verification was carried out with the holed plates considered in this work. Our calculations show that the supplementary stiffness introduced by the strain gradients can be neglected in all the cases treated. The maximum stiffness differences were found for the plate with the hole diameter $d=3$ mm. The apparent Young modulus of the holed plate (average stress/apparent strain) is 2940 MPa by taking $l=0$ and 2946 MPa by taking $l=0.5$ mm. The supplementary global stiffness is only about 0.2%. Therefore, we can confirm that the present fracture model does not modify significantly the macroscopic material behavior even with a large length parameter $l=0.5$ mm.

5.4 Mechanical Implications. The present model shows an excellent capacity in predicting the fracture process induced by nonsingular stress concentrations of brittle materials. In particular, the size effect due to macroscopic defaults (holes in the present study) is better described by using the present model than other conventional cohesive models. Therefore, it will be interesting to be able to reveal the physical significance of the present model.

First, it is necessary to notice that the conventional cohesive models present capacities in describing the size effects of fracture process. In the present study, we can observe that the cohesive models without considering the strain gradient effect can provide correct critical loads in the limit cases, i.e., when the hole is sufficiently large and when the hole diameter tends to zero. However, between these limits, the prediction accuracy is rather mediocre.

When we introduce the strain gradients in the cohesive model, we can remark two complementary effects. First, the strain gradients modify the cohesive energy constitution. The cohesive energy needed to separate the two surfaces includes not only the part due to the displacement jump but also the part due to the relative rotation. Consequently, if we admit the hypothesis of Griffith, i.e., the surface energy rate needed at fracture is a constant quantity for all fracture cases, the fracture under uniform stresses would be more difficult than that under stress concentration because in the first case, there exists only displacement jump between the separated surfaces in the cohesive energy.

The second effect is that the introduction of the strain gradients makes the material stiffer when the rotational deformation is important. As a consequence, the material near a stress concentration source is stiffer than that under a uniform traction. This leads to smaller displacement and rotation jumps between the cohesive surfaces. This may explain why the present model provides higher fracture loads for specimens with holes than the conventional models.

Mathematically speaking, in the present model, the second effect is predominant because it is the cause of the first one. In fact, the couple stresses in the cohesive elements must be in equilibrium with those in the bulk elements. Therefore, the cohesive energy related to the rotation jump in the cohesive elements van-

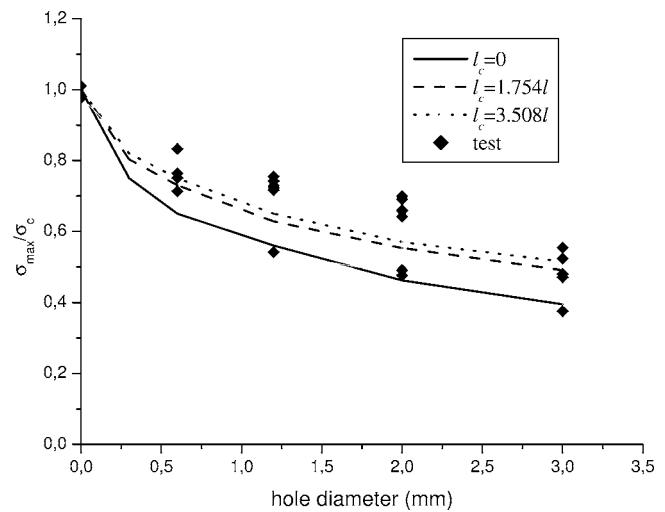


Fig. 9 Predicted critical loads at fracture with different values of l_c ($l=0.5$ mm)

ishes if there are no node rotations in the bulk elements. On the contrary, if the rotation effect is neglected in the cohesive elements, the couple stresses vanish in the bulk elements only at the boundaries connected to the cohesive element. Thus, its influence to the global behavior of the model should be secondary.

In summary, the introduction of the gradient effect in the bulk elements is essential to simulate the size effects induced by stress concentrations. In order to evaluate the influence of the gradient effect in the cohesive elements, we carried out fracture simulations by using fixed value of l ($l=0.5$ mm) and different values of l_c ($l_c=0, 1.754l, 3.508l$). Figure 9 illustrates the result of these simulations. From Fig. 9, we can remark that the strain gradient in cohesive elements plays an important role. The effect of the parameter l_c is not negligible. However, increasing l_c beyond a certain value will not significantly increase the critical fracture loads. In the present work, l_c is related to l through Eq. (25) according to a simple consistency analysis. We believe that this relationship is physically reasonable.

6 Concluding Remarks

In this paper, we have constructed a fracture model that associates the linear constitutive behavior with strain gradient to a cohesive model with relative rotation effects. Special finite elements were established for the implementation of this model. This model was then used to predict the fracture loads of a family of specimens with centered hole. Confrontations between the predicted fracture loads and the experimental results show that the present fracture model possesses excellent capacities to provide accurate critical loads at fracture for specimens with different hole sizes.

In conventional cohesive models or other fracture criteria issued from fracture mechanics, two material parameters, the ultimate stress σ_c and the critical energy release rate G_c , are mostly used to describe the fracture process. In the present fracture model, a third material parameter, the length parameter l in the constitutive law with strain gradients, is utilized in addition. Introduction of this additional parameter enables the present model to count more precisely for stress concentrations of different levels, which may be singular or not.

In this work, we have attempted to assess the influence of the default (holes, inclusions) size on the fracture behavior of a brittle material. Experimentations showed that this influence is obvious not only in a microscopic scale but also in a macroscopic scale. This size dependence cannot be well described by conventional fracture theories. The introduction of the strain gradient theory seems to be an appropriate approach to repair this weakness.

It is clear that the present work is just one of the first attempts in this topic. Other theoretical and experimental works will be necessary in order to provide more pertinent models.

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LES of Wall-Bounded Flows Using a New Subgrid Scale Model Based on Energy Spectrum Dissipation

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*A new one-equation subgrid scale (SGS) model that makes use of the transport equation for the SGS kinetic energy (k_{SGS}) to calculate a representative velocity scale for the SGS fluid motion is proposed. In the k_{SGS} transport equation used, a novel approach is developed for the calculation of the rate of dissipation of the SGS kinetic energy (ϵ). This new approach leads to an analytical computation of ϵ via the assumption of a form for the energy spectrum. This introduces a more accurate representation of the dissipation term, which is then also used for the calculation of a representative length scale for the SGS based on their energy content. Therefore, the SG length scale is not associated simply with the grid resolution or the largest of the SGS but with a length scale representative of the overall SGS energy content. The formulation of the model is presented in detail, and the new approach is tested on a series of channel flow test cases with Reynolds number based on friction velocity varying from 180 to 1800. The model is compared with the Smagorinsky model (1963, "General Circulation Experiments With the Primitive Equations: 1. The Basic Experiment," *Mon. Weather Rev.*, **91**, pp. 90–164) and the one-equation model of Yoshizawa and Horiuti (1985, "A Statistically-Derived Subgrid Scale Kinetic Energy Model for the Large Eddy Simulation of Turbulent Flows," *J. Phys. Soc. Jpn.*, **54**(8), pp. 2834–2839). The results indicate that the proposed model can provide, on a given mesh, a more accurate representation of the SG scale effects.*

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1 Introduction

From the early stages of large eddy simulation (LES) development, subgrid scale (SGS) models based on the transport equation for the SGS kinetic energy k_{SGS} have been proposed, motivated by a wish to increase the physical accuracy of SGS modeling beyond the original Smagorinsky algebraic eddy viscosity level and reflecting a similar development of improved Reynolds averaged Navier–Stokes (RANS) turbulence models. Early attempts indicated no significant advantages over algebraic models, at least as far as mean velocity field prediction was concerned. However, in terms of turbulence characteristics, transport equation models showed a more accurate description [1]. These early models of course relied on a priori specification of a number of model coefficients required in the transport equation and eddy viscosity formulation. In the last decade, considerable work has been published, which combines a transport equation SGS model with a dynamic procedure in order to obtain these model coefficients dynamically as part of the flow simulation. Ghosal et al. [2] used a constraint localized form of the dynamic procedure to obtain the coefficients in a k_{SGS} -equation model but presented only calculations for decaying turbulence behind a grid. Menon and Kim [3] used a similar formulation for high Reynolds number decaying and forced isotropic turbulence and a temporally evolving turbulent mixing layer to obtain results in good agreement with experimental and direct numerical simulation (DNS) data. More recently, Krajnovic et al. [4] used both the Menon and Kim [3] model and a model proposed by Davidson [5] for a recirculating flow around a surface mounted cube, producing good results.

However, during this study, it was noted that for this complex recirculating flow, grid resolution influenced the effectiveness of the transport equation models and improved the accuracy of the flow prediction. Evidence from the Krajnovic et al. [4] work implies that the modeling improvements introduced so far by the combination of a one-equation SGS model and the dynamic procedure are still not sufficient to cope with situations when an insufficiently resolved coarse grid is used and when in some flow regions, particularly near solid walls, a significant fraction of the turbulence energy is contained in the SGS of the flow.

In addition, particular implementations of the dynamic procedure have been reported to generate numerical instabilities [6], leading to the adoption of a "clipping" approach, which, depending on numerical details, may be ad hoc or based on physical realizability considerations [3,6]. This practice has a direct effect on the SGS stresses, potentially compromising the range of flow problems where the model can deliver good performance. Evidence to support this contention is that a homogeneous flow direction has normally been required for this clipping practice to achieve stability [7,8].

Considering the above, further improvement in the area of SGS modeling could either be achieved by the introduction of a second transport equation (as in RANS formulations) or by the improved formulation of one-equation models. Following the second school of thought, Kajishima and Nomachi [9] pointed out the importance of an improved estimation of energy transfer between the resolved and SGS and introduced a new one-equation model based on the idea that the dynamic procedure is also suitable and applicable to the calculation of energy transfer between grid and SGS [9]. Hence, their model expressed the production term in the k_{SGS} equation using the Smagorinsky formulation but employed the dynamic procedure to calculate the value of the Smagorinsky constant while retaining the usual formulation of the SGS eddy viscosity based on k_{SGS} . The model was applied to plane channel

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flows and a rotating channel flow case giving good results in both cases when the grid spacing was less than 30 wall units in the spanwise direction. However, clipping of k_{SGS} was still required in areas where this quantity would have assumed negative values.

The net level of k_{SGS} produced is, of course, a balance between energy transfer from the resolved scales and dissipation into internal energy. It would therefore seem wise that any attempt to improve modeling of the k_{SGS} equation should consider both of these processes. This paper therefore presents the development of a new SGS model that introduces a new approach regarding the definition of various characteristic subgrid length scales used in the model via a careful consideration of the relevant flow physics. The increased use of flow physics in the model formulation leads to a one-equation model that does not suffer from some of the limitations discussed above. The new model retains the use of a transport equation for k_{SGS} , in the form proposed by Yoshizawa and Horiuti [10] and Horiuti [1], but with the dissipation term ε calculated explicitly from the integration of an assumed energy spectrum shape over the frequency range (length scale range) not resolved by the grid and hence corresponding to the entire SGS region in wave-number space. In this way, the resulting behavior of ε is more physically correct; the SGS model proposed here is hence termed an energy spectrum dissipation (ESD) model. Furthermore, the improved estimation of ε allows the model to produce a prediction of the local Kolmogorov length scale η and hence the maximum eddy wave number κ_d . This enables an energy-weighted average wave number κ_{SGS} to be defined for the SGS (nonresolved) motions, and it is argued here that this constitutes a more representative wave number from which the SG length scale can be extracted. This is necessary because in many cases, particularly at high Re numbers and for near-wall turbulence where insufficient resolution is often typical in complex geometry LES calculations, the nonresolved wave numbers still contain a considerable amount of energy, and this should play an important role in the determination of both turbulence velocity and length scales used to evaluate the SGS eddy viscosity ν_t . Such a role is not fulfilled by the classical, grid based SG length scale used in the majority of SGS models but is reflected in the transport equation based SGS model proposed here via a length scale evaluated from the energy-weighted wave number κ_{SGS} . This introduces a more accurate representation of the whole range of nonresolved scales, in a manner more likely to be independent of the computational grid employed, and therefore successful for a wider range of levels of nonresolved SGS energy.

In the following section, a detailed presentation of the mathematical formulation of the model and its numerical implementation is given. Section 3 gives a brief description of the LES code employed, the SGS models used for the comparative study, and the test cases considered. The description of the results and the discussion are given in Sec. 4 followed by conclusions in Sec. 5.

2 Development and Implementation of an Energy Spectrum Dissipation Model

The driving motivation for this new model was the desire to formulate a more accurate description of the SGS motions while retaining the Boussinesq eddy viscosity assumption based on a turbulence velocity scale extracted from a transport equation for k_{SGS} . Yoshizawa and Horiuti [10] proposed a one-equation SGS model based on the transport equation for k_{SGS} , which has been used for wall-bounded flows giving good results (Horiuti [1]). This model was therefore used as the base line for the development of the new model presented here. According to the Yoshizawa and Horiuti [10] model, the SGS kinematic eddy viscosity ν_t is given by

$$\nu_t = C_\mu \bar{\Delta}_G \sqrt{k_{SGS}} \quad (1)$$

In Eq. (1), $C_\mu = 0.05$, $\bar{\Delta}_G = (\Delta x \Delta y \Delta z)^{1/3}$ (the subscript G indicating a grid-defined SG length scale), and k_{SGS} is derived from the solution of the following transport equation:

$$\begin{aligned} \frac{\partial k_{SGS}}{\partial t} + \frac{\partial \bar{u}_j k_{SGS}}{\partial x_j} = & C_1 \frac{\partial}{\partial x_j} \left(\bar{\Delta}_G \sqrt{k_{SGS}} \frac{\partial k_{SGS}}{\partial x_j} \right) + \nu \frac{\partial^2 k_{SGS}}{\partial x_j \partial x_j} - \tau_{ij} \bar{S}_{ij} \\ & - C_2 \frac{k_{SGS}^{3/2}}{\bar{\Delta}_G} \end{aligned} \quad (2)$$

where $C_1 = 0.1$, $C_2 = 1.0$, $\tau_{ij} = -2\nu_t \bar{S}_{ij}$, and $\bar{S}_{ij} = \frac{1}{2}(\partial \bar{u}_i / \partial x_j + \partial \bar{u}_j / \partial x_i)$.

The model proposed here uses the formulation presented above but introduces two new features: (i) an alternative approach to the calculation of the local SG length scale (used in calculating ν_t , Eq. (1), and in the k_{SGS} turbulent transport term) based on the energy content of all SGSs rather than a grid geometrical characteristic and (ii) an improved representation of ε in Eq. (2), replacing the $-C_2(k_{SGS}^{3/2}/\bar{\Delta}_G)$ term.

In the majority of SGS models, the length scale in ν_t is specified purely via the local grid size, therefore representing essentially the SGSs closest to the grid cutoff wave number $\kappa_c = 2\pi/2\bar{\Delta}_G$, where due to the Nyquist criterion, the smallest resolved length scale on a grid of cell size $\bar{\Delta}_G$ is $2\bar{\Delta}_G$. Although these are the most energetic of the SGSs, this does not necessarily mean that a length scale so derived represents the optimum approach to capture the effects of all SGSs on the resolved scale motions. This aspect will, of course, be particularly true when grid resolution is such that an appreciable level of energy is still present in the SGSs. This scenario almost certainly occurs in the near-wall region of most LES simulations at high Re numbers.

An improved description of the SG length scale should be related to a representative SGS wave number. Figure 1 gives a graphical representation of this idea. In Fig. 1, a typical energy spectrum $E(\kappa)$ is shown, with the grid cutoff wave number κ_c lying in the inertial subregion and the dissipation wave number κ_d being the largest energy-containing eddy wave number given by

$$\kappa_d = \frac{2\pi}{\eta} \quad (3)$$

where η is the Kolmogorov length scale (the smallest physically realizable length scale),

$$\eta = \left(\frac{\nu^3}{\varepsilon} \right)^{1/4} \quad (4)$$

Since the Kolmogorov scale is the smallest scale observed in the turbulent motion, its magnitude at high Re is dictated only by the energy dissipation rate ε and the fluid viscosity [11].

A representative wave number for the SGSs κ_{SGS} may be defined via the convenient physical interpretation of an energy-weighted average of all wave numbers between κ_c and κ_d . Taking into account that

$$k_{SGS} = \int_{\kappa_c}^{\kappa_d} E(\kappa) d\kappa \quad (5)$$

κ_{SGS} can be defined via

$$\kappa_{SGS} = \frac{1}{k_{SGS}} \int_{\kappa_c}^{\kappa_d} \kappa E(\kappa) d\kappa \quad (6)$$

The corresponding length scale can then be computed as

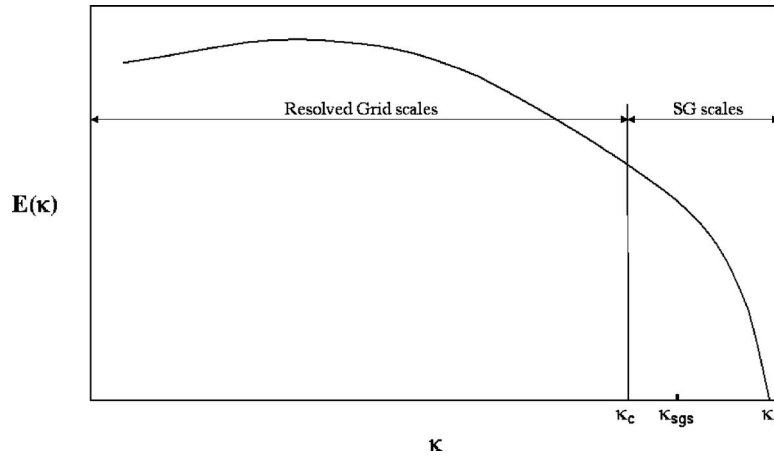


Fig. 1 Schematic of energy distribution among wave numbers

$$\bar{\Delta}_{ES} = \frac{2\pi}{\kappa_{SGS}} \quad (7)$$

where the subscript ES indicates that the length scale is evaluated via the energy spectrum.

It is clear that the calculation of κ_{SGS} requires k_{SGS} , which may be obtained from the solution of its modeled transport equation. Furthermore, it requires a function that describes accurately the shape of the energy spectrum $E(\kappa)$ in as wide a range of flows as possible, also ensuring, of course, that the selected spectrum shape is consistent with the expected $\kappa^{-5/3}$ behavior for the inertial sub-range at high Re since it is expected that κ_c will lie in this range. In addition, the energy spectrum function is constrained by its relation to the dissipation rate at high Re [11,12],

$$\varepsilon = 2\nu \int_0^\infty \kappa^2 E(\kappa) d\kappa \quad (8)$$

The Kolmogorov spectrum

$$E(\kappa) = K_o \varepsilon^{2/3} \kappa^{-5/3} \quad (K_o = 1.4) \quad (9)$$

commonly quoted in many turbulence textbooks is not a suitable candidate since, although it fits inertial sub-range conditions well, it does not describe accurately the rate of decrease of the energy content as κ_d is approached [12], and another form has to be employed.

Over the years, a number of researchers [13–16] have proposed a variety of energy spectra. Pope [11] proposed a general form for $E(\kappa)$, given by Eq. (10) below, which demonstrates a good correlation with experimental data obtained in a number of flows ranging from fully developed pipe and channel flows to grid turbulence and homogeneous shear flows, with a wide Re number variation. This indicates that one real advantage of extracting information from the energy spectrum is that its form is, to a certain degree, flow-type independent, giving a universal character to the information obtained,

$$E(\kappa) = C \varepsilon^{2/3} \kappa^{-5/3} f_L(\kappa L) f_\eta(\kappa \eta) \quad (10)$$

where $f_L(\kappa L)$ and $f_\eta(\kappa \eta)$ are specified nondimensional functions given by

$$f_L(\kappa L) = \left\{ \frac{\kappa L}{[(\kappa L)^2 - c_L]^{1/2}} \right\}^{11/3} \quad (11)$$

$$f_\eta(\kappa \eta) = \exp(-\beta[(\kappa \eta)^4 + c_\eta^{1/4} - c_\eta]) \quad (12)$$

where L is the length scale characterizing the large eddies ($L \equiv k^{3/2}/\varepsilon$) and C , c_L , c_η and β are positive constants [11].

While Pope's spectrum is arguably a good candidate, its mathematical complexity could lead to numerical complexities and expense when used in full LES calculations. Therefore, at this stage of development and exploration of the model, an alternative approach was selected, namely, the model spectrum proposed by Kovaszny [14], which was preferred due to its mathematical simplicity. The Kovaszny spectrum was developed based on locally isotropic turbulence,

$$E(\kappa) = K_o \varepsilon^{2/3} \kappa^{-5/3} \left[1 - \frac{K_o}{2} \left(\frac{\kappa}{\kappa_d} \right)^{4/3} \right]^2 \quad (13)$$

Equation (13) satisfies Eq. (8) and reduces to the Kolmogorov spectrum for low values of κ . The spectrum vanishes at $\kappa/\kappa_d = (2/K_o)^{3/4}$, which fixes the value of K_o at 2 to match the Kolmogorov scale. However, it may be advantageous to retain K_o as a model parameter to allow better description of the spectrum shape at wave numbers containing most of the nonresolved energy while departing strictly from the Kolmogorov constraint.

If the form of $E(\kappa)$ is assumed to be given by Eq. (13), this equation can be substituted in Eq. (5) and integrated to give

$$k_{SGS} = \kappa_d^{-2/3} \left(-\frac{3}{2}A - 3AB + \frac{AB^2}{2} \right) + 3AB\kappa_c^{2/3}\kappa_d - \frac{AB^2}{2}\kappa_c^2\kappa_d^{-8/3} + \frac{3}{2}A\kappa_c^{-2/3} \quad (14)$$

where $A = K_o \varepsilon^{2/3}$, $B = K_o/2$, and $\kappa_d = 2\pi \varepsilon^{1/4}/\nu^{3/4}$.

The left hand side of Eq. (14) can be assumed known from the solution of the k_{SGS} transport equation. Hence, the only unknown in Eq. (14) is ε . Rearranging Eq. (14) in terms of ε gives

$$k_{SGS} = \varepsilon^{1/2} (B^3 D^{2/3} - 6B^2 D^{2/3} - 3BD^{2/3}) + \varepsilon^{1/3} (6B^2 \kappa_c^{2/3} D^{4/3} + \varepsilon^{2/3} (3B\kappa_c^{-2/3}) - B^3 \kappa_c^2 D^{8/3}) \quad (15)$$

where $D = \nu^{3/4}/2\pi$. Moving k_{SGS} to the right hand side, Eq. (15) can be rewritten as

$$f(\varepsilon) = 0 = \varepsilon^{1/2} (B^3 D^{2/3} - 6B^2 D^{2/3} - 3BD^{2/3}) + \varepsilon^{1/3} (6B^2 \kappa_c^{2/3} D^{4/3} + \varepsilon^{2/3} (3B\kappa_c^{-2/3}) - (B^3 \kappa_c^2 D^{8/3} + k_{SGS})) \quad (16)$$

Equation (16) can be written as a fourth order polynomial in $\psi = \varepsilon^{1/6}$ in the following form:

$$g(\psi) = 0 = \psi^4 (3B\kappa_c^{-2/3}) + \psi^3 (B^3 D^{2/3} - 6B^2 D^{2/3} - 3BD^{2/3}) + \psi^2 (6B^2 \kappa_c^{2/3} D^{4/3} - (B^3 \kappa_c^2 D^{8/3} + k_{SGS})) \quad (17)$$

For the range of physically realizable values of all parameters involved in Eq. (17) (i.e., $K_o \approx 2.0$ and κ_c , k_{SGS} , $\nu \geq 0.0$), the terms

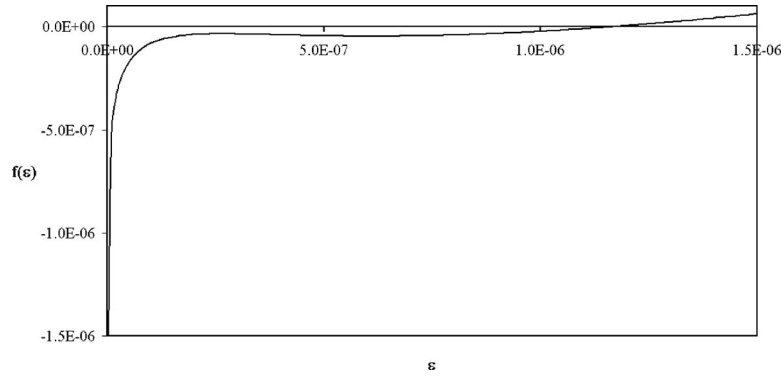


Fig. 2 Variation of $f(\epsilon)$ with ϵ for $\kappa_{c \max}$

$3B\kappa_c^{-2/3}(=C_{g1})$ and $6B^2\kappa_c^{2/3}D^{4/3}(=C_{g3})$ are positive definite, while the terms $B^3D^{2/3}-6B^2D^{2/3}-3BD^{2/3}(=C_{g2})$ and $-(B^3\kappa_c^2D^{8/3}+k_{SGS})(=C_{g4})$ are negative definite. Using the Routh–Hurwitz criterion [17], it can be shown that Eq. (17) has three roots with a positive real part. Factorization of Eq. (17) by ψ^2 leads to

$$g(\psi) = 0 = \psi^2 \left(\psi^2 + \frac{C_{g2}}{C_{g1}}\psi + \frac{C_{g3}}{C_{g1}} \right) + \frac{C_{g4}}{C_{g1}} \quad (18)$$

The quadratic expression in parentheses in Eq. (18) has a pair of complex conjugate roots with a positive real part due to the nature of the physically realizable values of the coefficients, leading to the conclusion that Eq. (17) has only one positive real root. Due to the positive definite nature of ϵ , the only physically realizable root is that corresponding to this positive real root. The above argument may be confirmed by plotting $f(\epsilon)$ against ϵ for two limiting cases of $\kappa_{c \min}$ and $\kappa_{c \max}$ (defined below) with $k_{SGS}=0.0$. Note that for any isothermal incompressible flow, K_o and ν are constants, and therefore the only variable parameters to be expected from the flow simulation will be κ_c (due to the use of a nonuniform grid) and k_{SGS} . κ_c will typically vary between $\kappa_{c \max}$ (corresponding to the near-wall cells that are typically the smallest in the grid) and $\kappa_{c \min}$ (corresponding to the largest cells in the grid, typically well away from any walls). Regarding k_{SGS} , the limiting case to consider is when it becomes zero since then, for the same value of ϵ , $f(\epsilon)$ is closest to the $f(\epsilon)=0$ axis, as can be seen from Eq. (16).

Extracting $\kappa_{c \min}$ and $\kappa_{c \max}$ from a grid used below for a fully developed channel flow simulation at $Re_\tau=180$, two plots were generated, presented in Fig. 2 for $\kappa_{c \max}$ and in Fig. 3 for $\kappa_{c \min}$, both with $k_{SGS}=0.0$. The plots show that in this extreme case, $f(\epsilon)$ does indeed cross zero at only one point, corresponding to the single positive real root of $g(\psi)$. The second observation is that

this root is larger in magnitude when closer to the wall, as shown in Fig. 2, indicating a correct representation of the physical behavior of ϵ . Finally, Figs. 2 and 3 verify that the nature and the number of roots of $f(\epsilon)$ does not vary with κ_c , as indicated previously by the mathematical analysis of $g(\psi)$.

From the above, it was concluded that the solution of Eq. (15) can return an accurate representation of ϵ when provided with the appropriate input of κ_c and k_{SGS} from the numerical LES solution. As soon as this value of ϵ is found, κ_d can be calculated and used in Eq. (6) to calculate κ_{SGS} and hence $\bar{\Delta}_{ES}$ from Eq. (7).

In general, any root identification algorithm may be used to find the root of Eq. (16). For the present study, a modified Newton–Raphson method [18] was employed; this uses the classic Newton–Raphson algorithm but with an extra bisection step taken every time the algorithm begins to search outside a prespecified range of ϵ values or does not narrow down the field of possible roots in a steady manner [18]. The search range of ϵ for the identification of the Eq. (16) root is user defined and is based on a number of considerations. Since ϵ cannot be negative, the search range lower limit can be obviously set to zero. The upper limit is set to a value ten times the value of ϵ from the previous time step. This definition of the upper limit can be changed, but it was found that this approach was adequate for all simulations performed and reported below. This aspect of the method should, however, be investigated further.

The transport equation used for k_{SGS} is that given by Eqs. (1) and (2) above, with the differences being that (i) the dissipation term in Eq. (2) is not the modeled form $-C_2k_{SGS}^{3/2}/\bar{\Delta}_G$ but rather ϵ calculated directly using the procedure described above and (ii) the SG length scale $\bar{\Delta}_{ES}$ is used to replace $\bar{\Delta}_G$ in both the ν_t definition (Eq. (1)) and the turbulent diffusion term in Eq. (2).

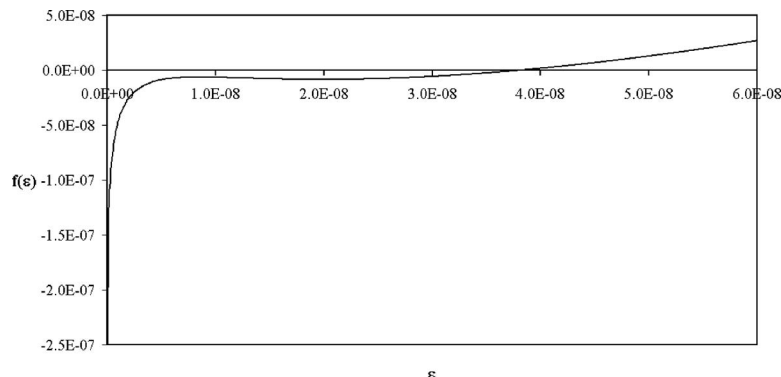


Fig. 3 Variation of $f(\epsilon)$ with ϵ for $\kappa_{c \min}$

The model proposed here is termed as the ESD model and involves the following steps.

1. Use the value of k_{SGS} from the previous time step and the value of κ_c at each grid node to calculate ε from the root of Eq. (15), using the enhanced Newton–Raphson method.
2. Use this value of ε to compute κ_d .
3. Use κ_d in Eq. (6) to calculate κ_{SGS} and hence $\bar{\Delta}_{ES}$.
4. Use $\bar{\Delta}_{ES}$ to compute ν_t .
5. Use ε for the solution of the transport equation for k_{SGS} .

Note that the new length scale $\bar{\Delta}_{ES}$ is also introduced into the production term of the k_{SGS} transport equation by using the new formulation of $\nu_t = C_\mu \bar{\Delta}_{ES} \sqrt{k_{SGS}}$ in the calculation of τ_{ij} . At the beginning of the simulation, initial values of k_{SGS} and ε are calculated as follows:

$$k_{SGS} = 2\bar{S}_{ij}\bar{S}_{ij} \frac{C_\mu \bar{\Delta}_G^2}{C_1} \quad (19)$$

$$\varepsilon = C_2 \frac{k_{SGS}^{3/2}}{\bar{\Delta}_G} \quad (20)$$

Equation (19) was proposed by Ghosal et al. [2] and Dejoan and Schiestel [19] and is based on an equilibrium assumption between k_{SGS} production and dissipation; $\bar{S}_{ij}$ is the resolved scale strain rate tensor and C_μ , C_1 , and C_2 are the coefficients used in the model of Yoshizawa and Horiuti [10].

3 Description of the Large Eddy Simulation Code, Subgrid Scale Models Used, and Simulations Performed

An in-house code, DELTA [20,21], was used for the present study. DELTA is based on a finite-volume approach for the solution of the governing equations. It uses a collocated variable arrangement on a nonorthogonal curvilinear block-structured mesh, in combination with Rhie and Chow smoothing to avoid pressure-velocity decoupling. DELTA adopts a derivative of the SIMPLE pressure correction method, designed to handle both incompressible and compressible flows [22]. Originally written as a RANS solver, DELTA was extended to allow LES calculations by introducing four new features: an explicit time-stepping formulation employing a third order accurate low storage Runge–Kutta method [23], a second order central differencing for convection term discretization, an appropriate scaling of the Rhie and Chow smoothing terms to take account of the very small time steps needed in LES calculations (due to the significant variations in cell volume size across the solution domain), and the Smagorinsky SGS model [24] with the near-wall damping formulation of Piomelli et al. [25]. Further development of the LES version of the code involved the introduction of a number of SGS models [26].

For the present study, two SGS models have been compared against the new model proposed here, namely, the algebraic Smagorinsky model [24] and the original k_{SGS} -equation model proposed by Yoshizawa and Horiuti [10]. Both models are based on a high Re formulation, and therefore they require a near-wall damping function to account for the low Re effects present in the near-wall region.

The Smagorinsky model expresses the eddy viscosity ν_t as

$$\nu_t = l^2 \bar{S} = (C_s \bar{\Delta}_G)^2 \bar{S} \quad (21)$$

where l is the Smagorinsky mixing length, an idea based on Prandtl's mixing length hypothesis. l is assumed to be given by the product of the Smagorinsky constant C_s and the filter width $\bar{\Delta}_G$. $\bar{S}$ is the filtered rate of strain given by

$$\bar{S} \equiv (2\bar{S}_{ij}\bar{S}_{ij})^{1/2} \quad (22)$$

The Smagorinsky constant was calculated from the near-wall damping function of Piomelli et al. [25], given by

$$C_s = C_{so} [1 - \exp(-y^+/A^+)^3]^{1/2} \quad (23)$$

where $C_{so}=0.1$ and $A^+=25$.

The Yoshizawa and Horiuti [10] k_{SGS} -equation model is a one-equation model that solves Eq. (2) and uses a form of Eq. (1) to calculate the SGS viscosity ν_t . In the present study, the version of this model incorporating the near-wall damping function proposed by Yoshizawa et al. [27] was used; this introduces low Re damping into the eddy viscosity (Eq. (1)) via

$$\nu_t = F_{wY} C_\mu \bar{\Delta}_G \sqrt{k_{SGS}} \quad (24)$$

where

$$F_{wY} = 1 - \exp \left[- \left(C_w \frac{\sqrt{2k_{SGS}}}{\bar{S}\bar{\Delta}_G} \right)^2 \right] \quad (25)$$

and $C_w=21.0$. The combination of the Yoshizawa and Horiuti [10] k_{SGS} -equation model and the near-wall damping function proposed by Yoshizawa et al. [27] is hereafter referred to as the “standard” k_{SGS} -equation approach.

For the present ESD model, the modifications introduced do not address the need for low Re damping at the length scale, so a damping function is still needed. In order to examine how sensitive LES predictions are to the particular near-wall low Re damping formulation introduced into the ESD SGS model approach, a second alternative suggested by Inagi et al. [28] was explored along with the near-wall damping function of Yoshizawa et al. [27]. The Inagi et al. [28] formulation proposes a different damping function,

$$F_{wI} = \frac{1}{1 + \bar{\Delta}_G \bar{S} / C_T \sqrt{k_{SGS}}} \quad (26)$$

where $C_T=10.0$.

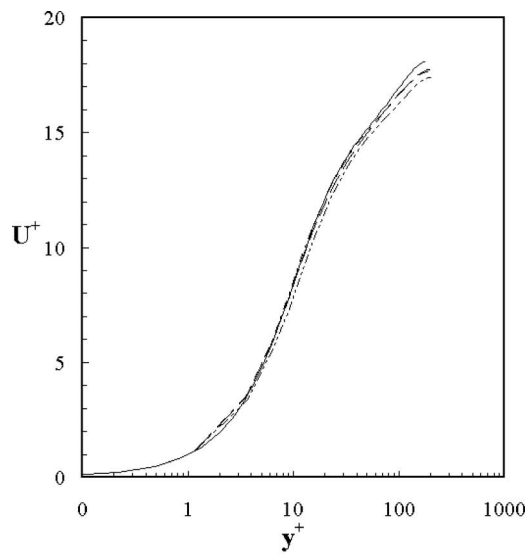
This function was used by Inagi et al. [28] in combination with a one-equation k_{SGS} SGS model for channel and rotating channel flows, flow over a backward facing step, and flow around a bluff body and provided superior results to the function (Eq. (23)) of Piomelli et al. [25].

Preliminary calculations were performed using both Eqs. (25) and (26) as low Re damping functions, with superior results being obtained with the Inagi et al. [28] approach (Eq. (26)). Therefore, the Inagi et al. [28] approach was adopted to be used in conjunction with the ESD SGS model. This was only modified to allow the physically more representative length scale proposed by the ESD approach to be used, so that Eq. (26) was changed to

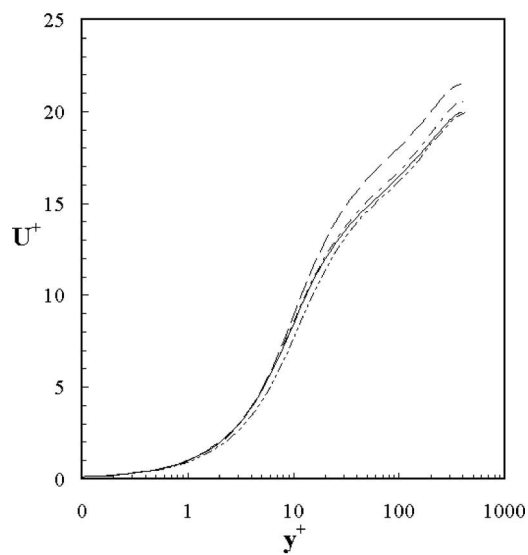
$$F_{wI} = \frac{1}{1 + \bar{\Delta}_{ES} \bar{S} / C_T \sqrt{k_{SGS}}} \quad (27)$$

The three SGS models were tested on four fully developed channel flow cases with Reynolds numbers (based on the channel half height δ and the friction velocity u_τ) Re_τ of 180, 395, 640, and 1800. The extent of the flow domain for each case, as well as the grid characteristics, can be seen in Table 1.

Reference data used below for the first and second cases were provided by DNS calculations performed by Kim et al. [29] and Moser et al. [30]. For the $Re_\tau=180$ simulation, Kim et al. used a $192 \times 129 \times 160$ grid to resolve the turbulent flow through a channel of dimensions $4\pi\delta \times 2\delta \times 2\pi\delta$. The resulting cell dimensions in wall units were $\Delta x^+ \approx 12$, $0.05 < \Delta y^+ < 4.4$, and $\Delta z^+ \approx 7$. For the $Re_\tau=395$ simulation, Moser et al. used a $256 \times 193 \times 192$ grid to resolve the turbulent flow through a channel of dimensions $2\pi\delta \times 2\delta \times 2\pi\delta$. Their resulting resolution in terms of wall units was $\Delta x^+ \approx 10$, $0.05 < \Delta y^+ < 6.5$, and $\Delta z^+ \approx 6.5$. For the third case, where $Re_\tau=640$, reference DNS data were provided by the calculation performed by Iwamoto [31]. The extent of the flow domain in that calculation was $2.5\pi\delta \times 2\delta \times \pi\delta$, and the number of points



(a)

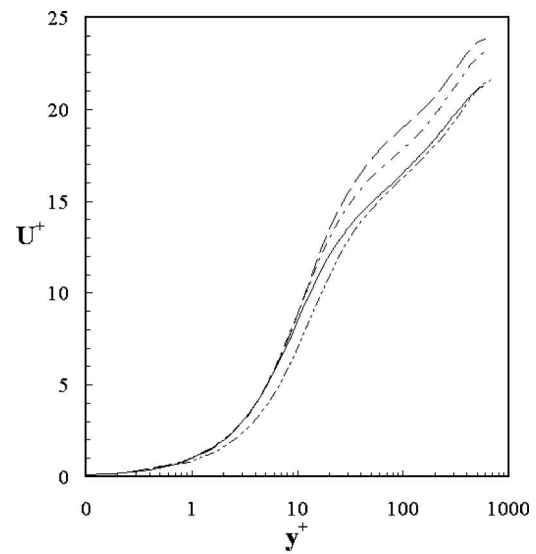


(b)

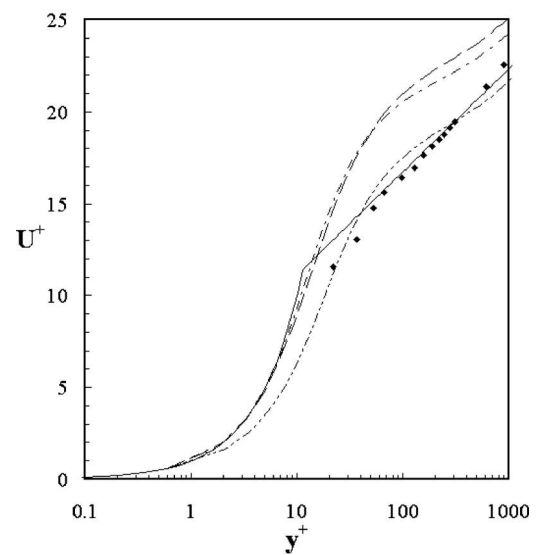
— DNS — Smag. - - k-eqn - . - ESD

(c)

Fig. 4 U^+ versus y^+ for (a) $Re_\tau=180$ and (b) $Re_\tau=395$



(a)



(b)

— Log-Law + W-W — Smag. - - k-eqn - . - ESD

(c)

Fig. 5 U^+ versus y^+ for (a) $Re_\tau=640$ and (b) $Re_\tau=1800$

used was $288 \times 384 \times 257$ in the x , y , and z directions respectively. This resulted in $\Delta x^+ = 17.7$, $0.049 < \Delta y^+ < 7.98$, and $\Delta z^+ = 5.32$. In the fourth case, the bulk Reynolds number was significantly higher at $Re_b = 38,000$. This test case was originally presented by Piomelli [7], who used a dynamic SGS model. Shah and Ferziger [32] also predicted this flow, employing a noneddy viscosity based SGS model. Experimental results for this flow are

available from the study of Wei and Willmarth [33] using a water-tunnel and local density approximation (LDA) instrumentation. In the computations performed for the present study, the flow domain size and the grid resolution used for this case, as given in Table 1, matched those employed by Piomelli [7] and Shah and Ferziger [32].

Table 1 Description of test cases

Case name	Re_b	Re_τ	$L_x \times L_y \times L_z$	$N_x \times N_y \times N_z$	Δx^+	Δy^+	Δz^+
1	3000	180	$12\delta \times 2\delta \times 2\delta$	$70 \times 56 \times 38$	34	1.16–15	10
2	6960	395	$2.5\pi\delta \times 2\delta \times \pi\delta$	$64 \times 64 \times 64$	50	0.18–39	20
3	12,155	640	$2.5\pi\delta \times 2\delta \times \pi\delta$	$64 \times 64 \times 64$	81	0.30–63	32
4	38,000	1800	$2.5\pi\delta \times 2\delta \times (\pi\delta/2)$	$64 \times 80 \times 80$	218	0.66–136	35

Table 2 Predictions of C_f

Case name	Model	C_f	C_f Dean	Error (%)	U_c/U_b	$(U_c/U_b)_{\text{Dean}}$	Error (%)
1	Smag	0.008351	0.008296	0.66	1.145	1.157	-1.04
2	Smag	0.005665	0.006720	-15.71	1.140	1.146	-0.52
3	Smag	0.004428	0.005846	-24.27	1.135	1.139	-0.35
4	Smag	0.003478	0.004397	-20.90	1.079	1.124	-4.00
1	k_{SGS} -eqn	0.008427	0.008296	1.58	1.147	1.157	-0.86
2	k_{SGS} -eqn	0.006282	0.006720	-6.51	1.150	1.146	0.35
3	k_{SGS} -eqn	0.004874	0.005846	-16.63	1.140	1.139	0.09
4	k_{SGS} -eqn	0.003707	0.004397	-15.67	1.074	1.124	-4.45
1	ESD	0.008731	0.008296	5.25	1.150	1.157	-0.61
2	ESD	0.006701	0.006720	-0.30	1.157	1.146	0.96
3	ESD	0.005694	0.005846	-2.60	1.147	1.139	0.70
4	ESD	0.004611	0.004397	4.89	1.096	1.124	-2.49

4 Results and Discussion

The time-averaged statistics presented in the following were collected over ten flowthrough times in order to ensure that they were statistically stationary. During these calculations, it was found that the ESD model was approximately 50% more expensive than the original k_{SGS} model due to the root search algorithm required, resulting in an increase of the computational time required per time step by approximately 18%. This was an indication that future implementations should either involve a quicker algorithm or adopt an analytical solution of the polynomial equation for ε whenever this is possible. However, even this increase in computational cost was less than the increase that would result from the introduction of a further transport equation for ε .

Figures 4 and 5 show the results of the mean flow field, in terms of the U^+/y^+ distribution, for $\text{Re}_\tau=180, 395, 640$, and 1800. In all figures presented, the Smagorinsky model prediction is labeled as Smag, the new model prediction as ESD, the Yoshizawa and Horiuti model [10] as k -eqn, and the DNS data [29,31] as DNS. In case 4, the reference data used was from the experiments of Wei and Willmarth [33], labeled as WW.

In case 1, the influence of the SGS model was insignificant due to the low Re_τ and the overall good grid resolution. All the models behaved similarly with ESD being slightly more dissipative than the others. The new model gave the largest error in the prediction of the skin friction coefficient C_f (but still only 5%) but the smallest error in the prediction of the centerline to bulk velocity ratio U_c/U_b (see Table 2). The fact that the grid was not particularly fine in the region next to the wall (note that $y^+=1.16$) is believed to be the main source of error in the prediction of C_f . In Table 2,

the predicted values of U_c/U_b and C_f are compared to expected values calculated from the empirical formulas proposed by Dean [34] with corresponding percentage errors.

As Re increases, without a corresponding grid refinement, the role of the SGS model in the calculation of the flow becomes increasingly important and the physical realism of the model is more closely tested. Hence, it can be seen that in the rest of the flow cases considered, the various SGS models give clearly distinguishable predictions. In case 2, the Smagorinsky model over-predicted the log-law region due to a 16% underprediction of C_f , which feeds through the skin friction velocity and hence to the nondimensionalization of U . This, however, did not affect the slope of the log-law region, which was captured accurately. This feature of the U^+ distribution was captured by all models tested. The underprediction of C_f and the overprediction of the velocity profile in the log-law region were reproduced consistently by the basic Smagorinsky SGS model in all cases, with the overshoot worsening as Re_τ increased. The k_{SGS} -equation model and the ESD model produced a similar level of accuracy for the overall distribution of U^+ for case 2, although the k_{SGS} -equation model returned a much larger error in C_f (6.51% compared to -0.3%). The accuracy of the capture of the overall streamwise velocity profile was also similar, as indicated by the small error in U_c/U_b with both one-equation SGS models. A factor that contributed to this was the near-wall damping function used, which had been "calibrated" for this specific case.

It is very noticeable that the ESD model captured the log-law region very well for higher Re cases 3 and 4 and managed to

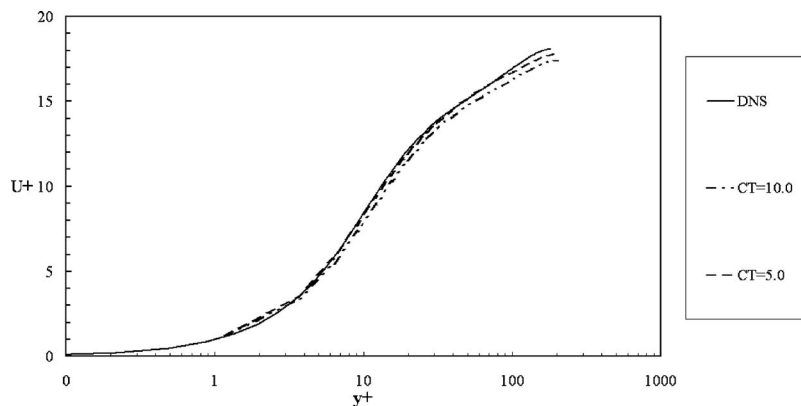
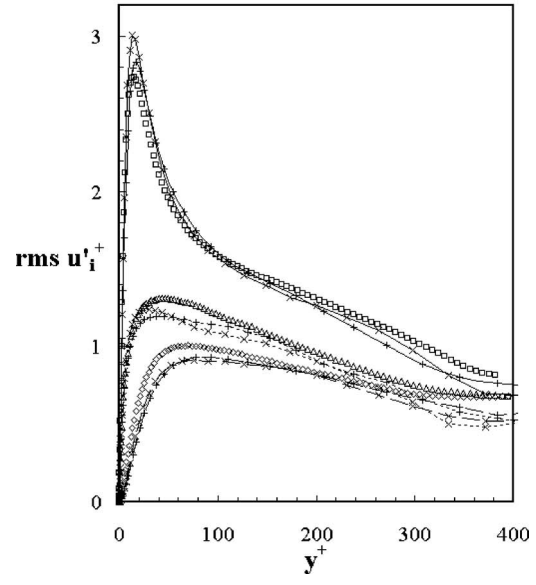
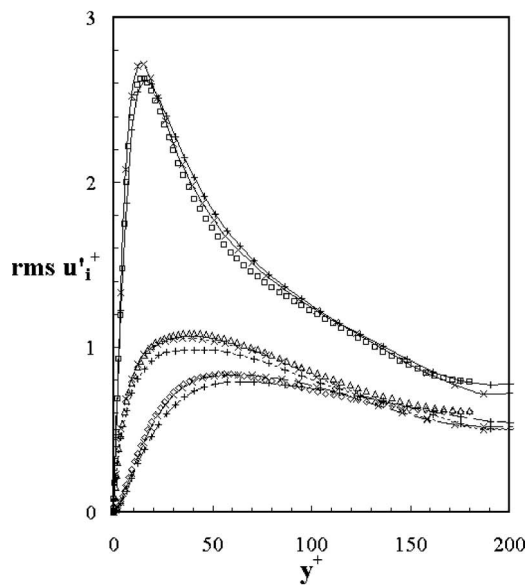
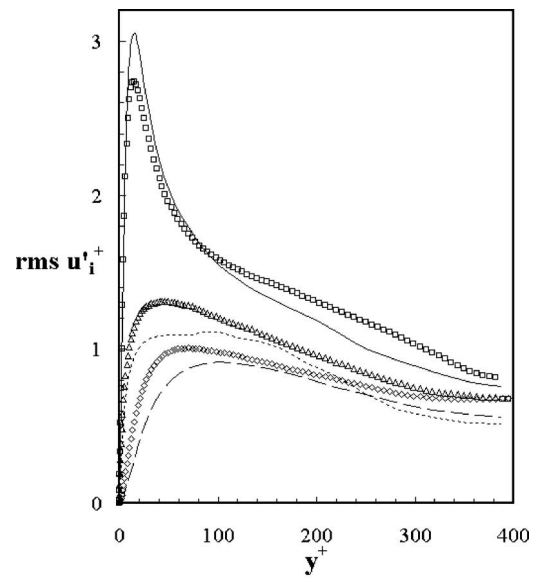
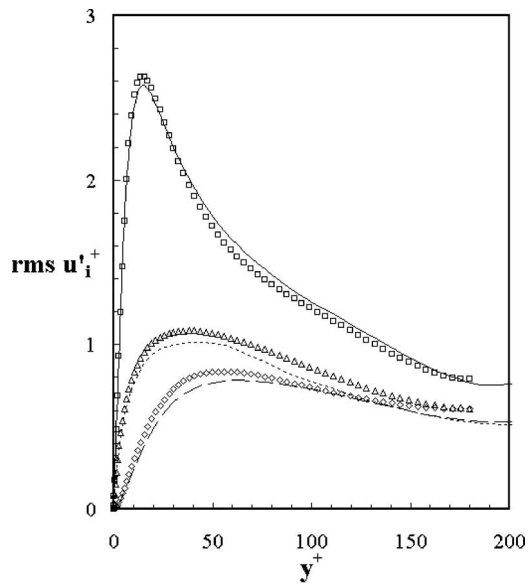


Fig. 6 U^+ versus y^+ for $\text{Re}_\tau=180$ using the ESD model and Inagi et al. [28] function with $C_T=5.0$ and 10.0



$\square$ u' DNS $\circ$ v' DNS $\triangle$ w' DNS $—$ u' Smag $- -$ v' Smag $\cdots$ w' Smag
 $—$ u' k-eqn $- -$ v' k-eqn $\cdots$ w' k-eqn $—$ u' ESD $- -$ v' ESD $\cdots$ w' ESD

Fig. 7 Nondimensional rms values of $\bar{u}^+$, $\bar{v}^+$, and $\bar{w}^+$ versus y^+ for $Re_\tau=180$

return the most accurate prediction for C_f overall. However, the model did not reproduce the upper part of the linear sublayer and the buffer layer very well (i.e., $y^+ < 30$).

There are two factors that have contributed to the ESD under-prediction of U^+ in the linear sublayer and buffer layer regions, namely, the chosen energy spectrum shape and the near-wall damping function employed. The Kovaszny spectrum was developed on the assumption of equilibrium between turbulent energy production and dissipation. However, in the regions of interest, this assumption is not valid, and as a result, the model spectrum is not truly representative of the energy distribution. As a result, the computed length scale may not reflect the characteristic scales of the SGS structures, producing excessive dissipation. The use of a more accurate model spectrum shape, such as Pope's [11] could be beneficial. This is because Pope's spectrum applies to a much wider range of Re compared to Kovaszny's spectrum. As a result, even in cases of locally low Re, where turbulence production and

Fig. 8 Nondimensional rms values of $\bar{u}^+$, $\bar{v}^+$, and $\bar{w}^+$ versus y^+ for $Re_\tau=395$

dissipation wave numbers are close, but without a clear inertial region, Pope's spectrum would be able to return a more accurate spectrum shape, resulting in a more accurate $\bar{\Delta}_{ES}$, while Kovaszny's spectrum is more likely to be in error.

As far as the near-wall damping function is concerned, Inagi et al. [28] calibrated this function for $Re_\tau=395$. This may have had an adverse effect on the use of the function for other Re numbers. This was confirmed by a test simulation performed using $C_T=5.0$ (rather than 10.0) for $Re_\tau=180$. The results in terms of wall-normal U^+ distribution show a considerable improvement over the whole y^+ range, as can be seen in Fig. 6. The improvement extends to the C_f prediction error, which was reduced from 5.25% to 1.11%.

The marginally more accurate prediction of U_c/U_b by the other models did not counteract their larger error in C_f , resulting in the overprediction of the log-law region, with the worst case being that of the Smagorinsky model. This same pattern can also be seen in the final flow case. In case 4, the Smagorinsky and k_{SGS} -equation models produced similar flow behaviors in the linear sublayer and buffer regions with some differentiation at the

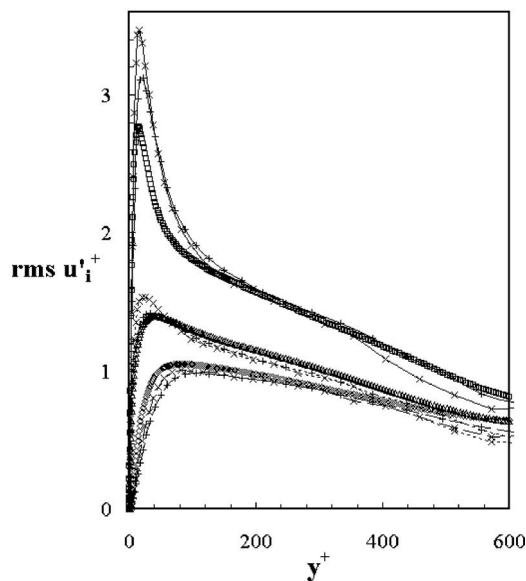
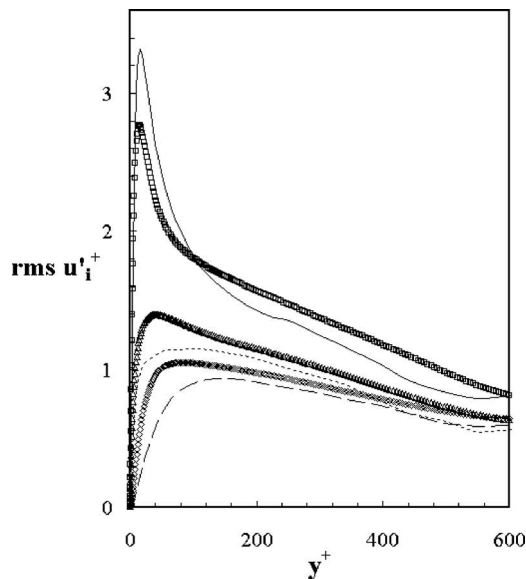


Fig. 9 Nondimensional rms values of $\bar{u}'^+$, $\bar{v}'^+$, and $\bar{w}'^+$ versus y^+ for $Re_\tau=640$

end of the latter, mainly due to the better prediction of the skin friction by the transport equation model. The ESD model, however, gave a clearly improved overall prediction although with some deviations, as noted above, up to $y^+ \approx 30$. The slight overshoot at the end of the buffer layer is typical for linear eddy viscosity based models and has been reported by a number of researchers [32,35].

From the observations made, it is clear that as Re_τ was increased (at an approximately constant grid resolution) and the energy in the SGS was therefore larger due to the lack of grid refinement, the error in the prediction of C_f increased for both Smagorinsky and standard one-equation SGS models. However, the behavior of the ESD model as Re_τ was increased indicates a much more robust maintenance of accuracy as Re increases, with predictive accuracy for parameters such as C_f and Reynolds averaged velocity distributions remaining essentially similar over all test cases, in contrast to the other models. This confirms the idea that information extracted from the energy spectrum, even if this adopts an approximate modeled form, can be used for a variety of flow conditions giving satisfactory results.

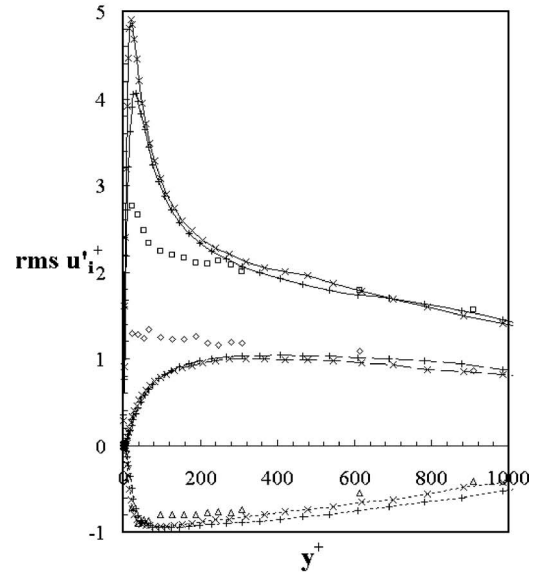
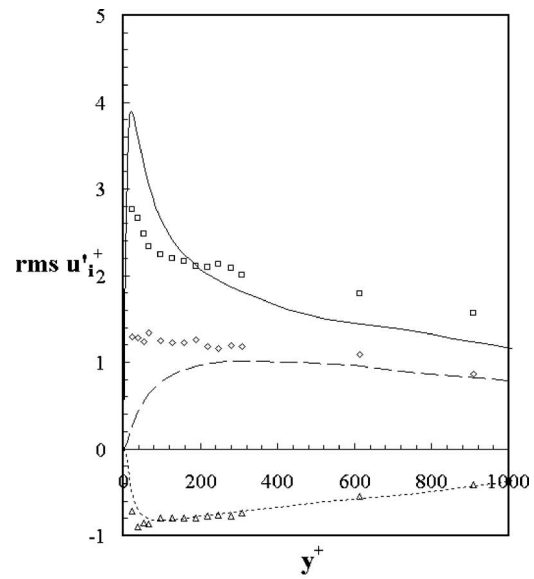


Fig. 10 Nondimensional rms values of $\bar{u}'^+$, $\bar{v}'^+$, and $\bar{u}'v'^+$ versus y^+ for $Re_\tau=1800$

Fig. 10 Nondimensional rms values of $\bar{u}'^+$, $\bar{v}'^+$, and $\bar{u}'v'^+$ versus y^+ for $Re_\tau=1800$

In order to further investigate this observation, the distribution of predicted Reynolds stresses should be taken into account. Figures 7–9 present the variation of the rms values of nondimensional resolved $\bar{u}'^+$, $\bar{v}'^+$, and $\bar{w}'^+$ with y^+ , for cases 1, 2, and 3, respectively. Figure 10 presents comparison with the nondimensional resolved value of $\bar{u}'v'^+$ instead of $\bar{w}'^+$ for case 4 since no experimental data were available for $\bar{w}'^+$.

Figures 7–10 show that the main differences in the predicted stresses are concentrated in the near-wall region, while, in general, further away from the wall and in the flow core all models gave the same level of accuracy and were in good agreement with the reference DNS and experimental data.

The general trend in the near-wall region was an overestimation of $\bar{u}'^+$ and an underestimation of $\bar{v}'^+$ and $\bar{w}'^+$, with the worst case being at $Re_\tau=1800$ and the best at $Re_\tau=180$ and $Re_\tau=395$. In cases 1 and 2, the Smagorinsky, k_{SGS} , and ESD models gave similar results, while their predictions began to separate as Re_τ was

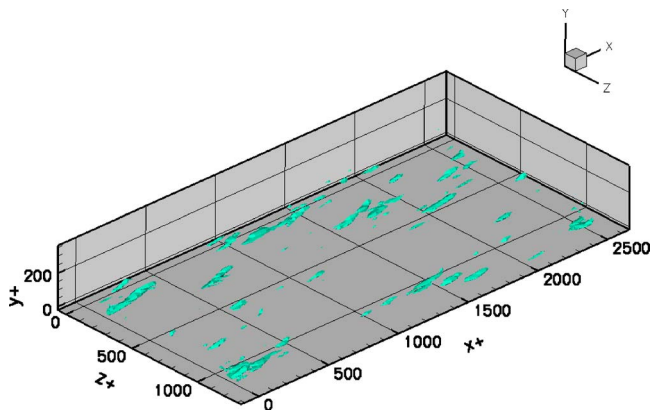


Fig. 11 Discriminant isocontour for the Smagorinsky model (case 2)

increased, with the ESD approach giving the best overall predictions. The k_{SGS} -equation model returned accurate results for cases 1 and 2, indicating a very good agreement with reference data, particularly for $\bar{u}'^+$ and $\bar{w}'^+$. However, in cases 3 and 4, a considerable overprediction of the streamwise stress was present.

The effect of the SG modeling on the development of the turbulent flow field can also be seen through a visualization of the coherent structures present in the flow. Using the Chong et al. proposal [36], the core of such structures may be identified using isosurfaces of the discriminant of the characteristic equation of the velocity-gradient tensor to locate regions where this parameter is positive. Figures 11–13 show coherent structures using this technique for the $Re_\tau=395$ flow case.

The structures shown in these figures are aligned with the streamwise direction and inclined to the wall surface at an angle of 30–45 deg. Their average length in wall units is 200–300, and their width is approximately 100. These observations are in good agreement with the description of coherent structures in channel flow made by Sagaut [37].

Comparing the three figures, it is clear that the coherent structures predicted using the Smagorinsky model were significantly fewer than in the other two cases. This is an indication that the dissipative character of the Smagorinsky model resulted in a flow field with lower large scale turbulence. The ESD and k_{SGS} -equation models predicted a similar number of coherent structures, with the ESD model producing slightly wider formations. Considering the degree of sensitivity of C_f to the near-wall

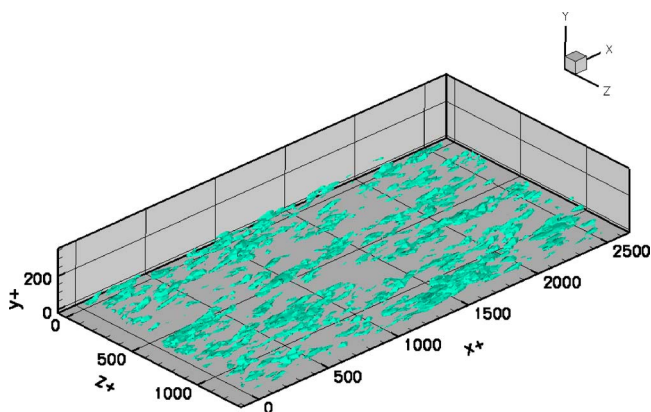


Fig. 12 Discriminant isocontour for the k_{SGS} -equation model (case 2)

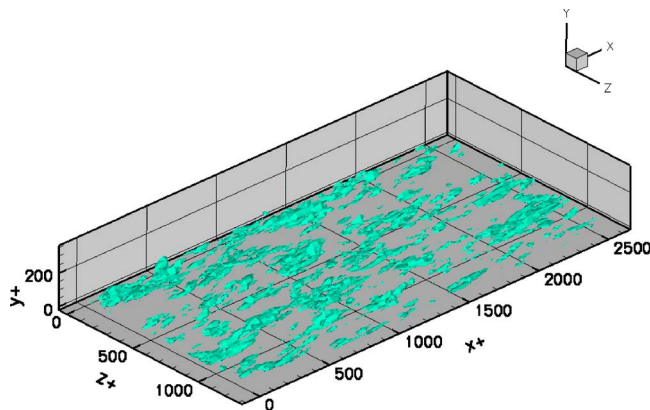


Fig. 13 Discriminant isocontour for the ESD model (case 2)

fluid structures' activity, it can be suggested that there is a direct link between the predicted C_f and the coherent structures predicted in the flow.

5 Conclusions

The formulation and implementation of a new SGS model has been described—the ESD model. The model was used in a finite-volume code and tested against the Smagorinsky model [24] and the k_{SGS} -equation model proposed by Yoshizawa and Horiuti [10] on four fully developed channel flow cases at $Re_\tau=180, 395, 640$, and 1800. The computational cost of the ESD model was found to be slightly increased in comparison with the other two models, but still less expensive than using a further transport equation for ε .

In terms of mean velocity distribution, the ESD model indicated some discrepancy in capturing accurately the linear sublayer and the lower part of the buffer layer, in all probability due to the model spectrum employed in the current implementation. However, in all cases, predictions of the log-law region were in very good agreement with the reference data.

The ESD model showed superior performance at high Re on a given grid compared to the other two SGS models, which did not manage to predict accurately the skin friction coefficient in the higher Reynolds number cases, and as a result, they overpredicted the variation of U^+ in the log-law region. In contrast, the proposed ESD model was less sensitive to grid resolution at high Re, giving the most accurate C_f predictions.

Regarding the prediction of normal and shear stresses, the general trend was an overprediction of the streamwise stress and an underprediction of the other normal stresses. The main differences were in the near-wall region, while further away all three models gave similar results, in good agreement with reference data. The ESD model produced good results in all cases tested, indicating that it is a good candidate for further testing in more complex cases. This was also confirmed via the visualization of the coherent structures in the instantaneous flow field, which indicated that the ESD model produced a realistic turbulent field, especially compared to the Smagorinsky model.

In conclusion, this first series of tests of the ESD model approach has provided promising results, indicating the validity of the idea and the potential of this new SGS model. Further studies should concentrate on four main areas: the use of a more realistic energy spectrum, the investigation of the near-wall damping effect on the model behavior, the application of the model to more complex, higher Re number flow cases, and the investigation of a possible reduction of computational cost via the use of an alternative numerical implementation of the root identification algorithm.

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Analytical Model of the Confined Compression Test Used to Characterize Brittle Materials

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Numerical and analytical simulations of projectiles penetrating brittle materials such as ceramics and glasses are a very challenging problem. The difficulty comes from the fact that the yield surface of brittle materials is not well characterized (or even defined), and the failure process may change the material properties. Recently, some works have shown that it is possible to characterize and find the constitutive equation for brittle materials using a confined compression test, i.e., a test where a cylindrical specimen, surrounded by a confining sleeve, is being compressed axially by a mechanical testing machine. This paper focuses on understanding the confined compression test by presenting an analytical model that explicitly solves for the stresses and strains in the sample and the sleeve, assuming the sleeve is elastic and the specimen is elastoplastic with a Drucker–Prager plasticity model. The first part of the paper briefly explains the experimental technique and how the stress-strain curves obtained during the test are interpreted. A simple and straightforward approach to obtain the constitutive model of the material is then presented. Finally, a full analytical model with explicit solution for displacements, strains, and stresses in the specimen and the sleeve is described. The advantage of the analytical model is that it gives a full understanding of the test, as well as information that can be useful when designing the test (e.g., displacements of the outer radius of the specimen). [DOI: 10.1115/1.2775501]

1 Introduction

To be a useful tool, analytical and numerical simulations of impact into brittle materials, such as glass or silicon carbide, require accurate constitutive equations. These are needed for both the projectile and the target. The added complication with brittle materials under impact is that their strength depends on the hydrostatic pressure, an effect often modeled with a Drucker–Prager [1] model or other similar plasticity models (e.g., Johnson–Holmquist [2]).

Low pressure confinement experiments are relatively simple and well understood; see, for example, static and dynamic tests by Chen and Ravichandran [3,4] and, more recently, Nielsen et al. [5]. Confinement pressures found in the literature are rarely above 100 MPa because triaxial tests at high pressures are very expensive. High confinement pressures are only achieved in plate impact experiments (e.g., Partom [6]) at very high strain rates. Material properties are usually backed out from numerical simulations of the experiment.

Recently, a modification of the confinement sleeve technique, which allowed attainment of moderately high confinement pressures of up to 1 GPa, has been presented by Dannemann et al. [7] based on the work of Walker et al. [8]. The technique consists of testing a specimen in compression while being surrounded by a thick elastic sleeve. A strain gage on the sleeve measures hoop strain during the test indirectly. The confinement pressure is then determined with a simple elastic calculation. The same technique has been recently used by Forquin et al. [9]. Forquin et al. used numerical simulations to relate the hoop strain and the pressure in the specimen.

This paper presents a detailed analytical model of the experiment, which helps in the interpretation of the results. The model assumes an elastoplastic specimen and an elastic sleeve. It is

shown that displacements, strains, and stresses can be solved explicitly, and relatively simple expressions are obtained. The model can also be used to determine the elastic and plastic constants of the specimen material being tested. It is remarkable that, when applied to glass, the strength found using this technique is very similar to that found in Partom's paper where a plate impact technique was used.

2 Experimental Technique

The experimental technique is carefully described in Ref. [10] but is briefly explained in this paper for completeness. The specimen to be tested is inserted into a Vascomax steel sleeve (Fig. 1) honed to fit the specimen. The sleeve outer diameter is 12.70 mm.

An axial compressive stress is applied to the specimen with a mechanical testing servohydraulic (MTS) machine by means of two SiC–N platens. The platens are not shown in Fig. 1. The variables recorded during the test are axial stress in the specimen measured by a load cell in the MTS machine, axial strain in the specimen measured by a clip gage placed on the top and bottom platens, and axial and hoop strain in the sleeve measured, respectively, by a vertical and annular strain gage on the sleeve.

The objective of the experiments is to obtain data for determination of a constitutive model (elastic and plastic parts) for the specimen. It is shown in this paper that by measuring the elastic and plastic slopes of the stress versus axial strain and stress versus hoop-strain curves obtained in the tests, it is possible to calculate the desired constants.

3 Interpretation

The result of a typical test is shown in Fig. 2. This particular sample (borosilicate glass that was predamaged prior to testing by using a thermal shock) underwent ten load-unload cycles where the maximum load was gradually incremented.

Figure 3 is an enlargement of the first cycle of a typical test to facilitate its interpretation. There are four distinctive parts in the load curve of Fig. 3 that are common to most of the tests and cycles.

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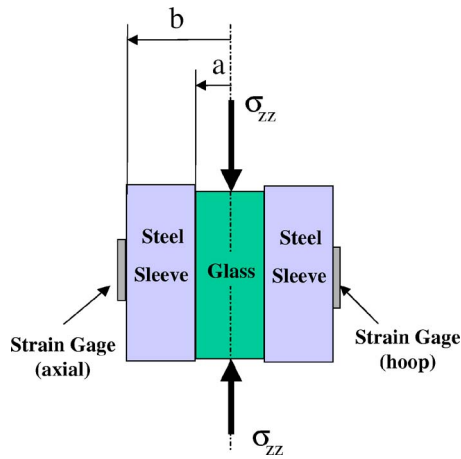


Fig. 1 Experimental setup

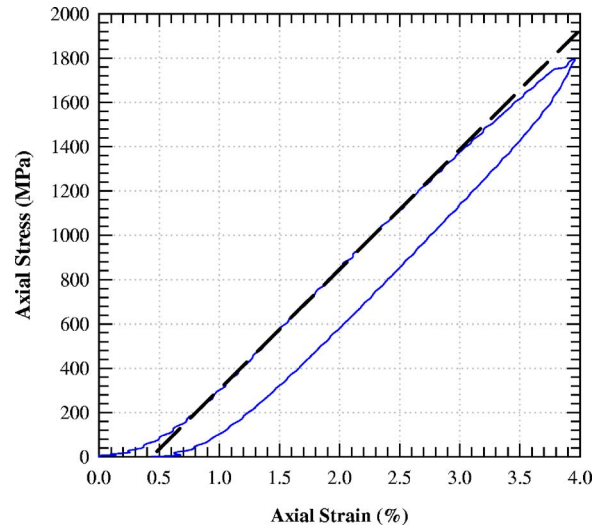
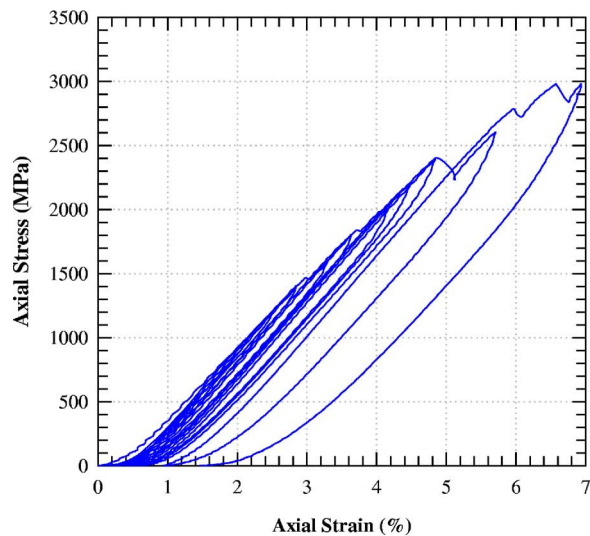
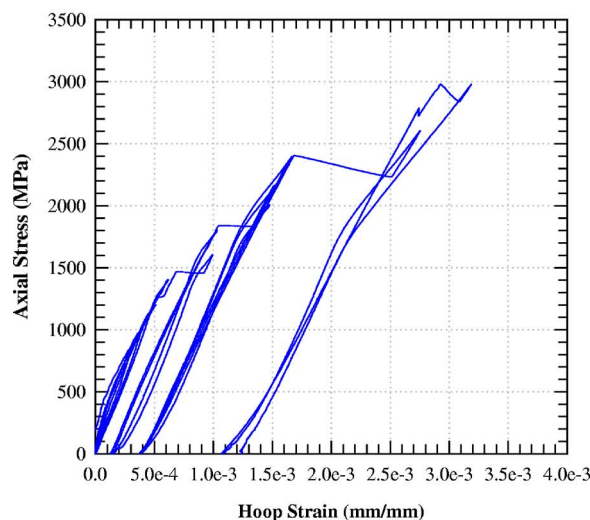


Fig. 3 First loading cycle in test BF-14



(a)



(b)

Fig. 2 (a) Stress versus axial strain obtained in a typical test with many load-unload cycles. (b) Stress versus hoop-strain curve for the same test.

1. First is a small nonlinear start, due probably to gaps between the specimen and the sleeve or between the anvils and the specimen and other small misalignments at the beginning of the test. This part is ignored in the analysis of the test since it does not provide any useful information about the specimen itself.
2. Second is a linear elastic ramp where, if unloading occurs, there will be no permanent deformation. The linear part is emphasized with a dashed straight line.
3. A small plastic deformation region is also apparent in most of the tests. Brittle materials are known to be able to undergo a brittle-ductile transition when tested under confinement, see, for example, Horii and Nemat-Nasser [11] and, more recently, Lankford et al. [12]. When unloading occurs in this part of the curve, the specimen deformation becomes permanent in both the hoop and axial directions.
4. Sudden jumps in axial or hoop strain are clearly seen in Fig. 2. These jumps are interpreted as the specimen faulting and slipping. This mechanism would create a big jump in hoop strain, as observed, while the load remains approximately constant.

A graphical summary of how the data from the test are interpreted (for a single cycle) is shown in Fig. 4. Slopes 1 and 1' are the elastic part of the load curve and slopes 2 and 2' the plastic part. Failure with slippage occurs when the strain increases suddenly while the load remains constant.

4 Simple Method for Direct Determination of the Drucker–Prager Equation

By using the classical solution for a thick tube with an internal pressure p , it is possible to easily calculate the confinement pressure P_c on the specimen as a function of hoop strain, see Ref. [13], p. 59:

$$P_c = \frac{E}{2} \frac{b^2 - a^2}{a^2} \epsilon_\theta \quad (1)$$

where E is the Young's modulus of the sleeve, a is the internal radius of the sleeve, and b its outer radius. ϵ_θ is the hoop strain measured on the surface of the sleeve. The specimen is then under a known stress state since $\sigma_r = \sigma_\theta = -P_c$ and σ_z is the stress applied by the MTS machine. Equivalent stress and pressure in the specimen, P , are then easily derived, see, for example, Ref. [14]:

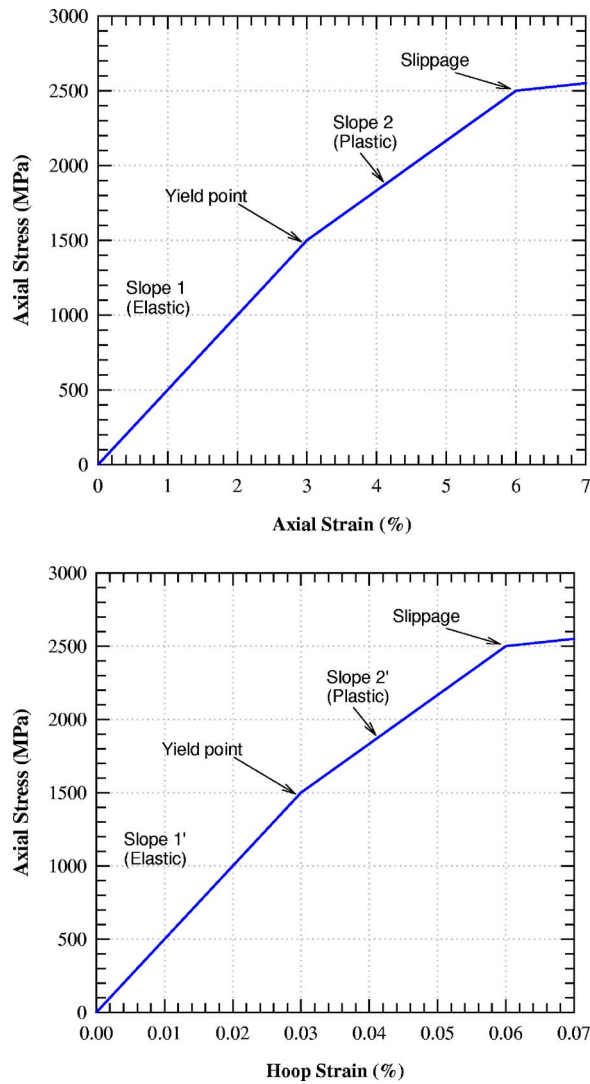


Fig. 4 “Idealized” stress-strain curves used for the interpretation of the results

$$\sigma_{eq} = \frac{1}{\sqrt{2}} [(\sigma_r - \sigma_\theta)^2 + (\sigma_r - \sigma_z)^2 + (\sigma_\theta - \sigma_z)^2]^{1/2} = |\sigma_r - \sigma_z| \quad (2)$$

where the shear stresses were assumed to be zero,

$$P = -\frac{1}{3}(\sigma_r + \sigma_\theta + \sigma_z) = -\frac{1}{3}(2\sigma_r + \sigma_z) \quad (3)$$

Using Eqs. (2) and (3), a plot of equivalent stress versus pressure can be constructed from the data recorded during the test. Figure 5(a) shows a typical σ_{eq} versus P plot.

Our main interest is to determine the yield strength of the specimen as a function of pressure and, particularly, a maximum strength or cap, if it exists at all. By taking the envelope of the σ_{eq} versus P plot, it is possible to “hide” the large jumps that are due to internal fractures and slippage. The envelope is formed by connecting the stresses, for each load cycle, where a hoop-strain jump is apparent. Figure 5(b) is the result of taking the envelope for all six tests shown. The plot implies that there is a cap around 1.8–2.2 GPa, a value in agreement with that published by Partom [6] (which cites a later publication by Bourne et al. [15]).

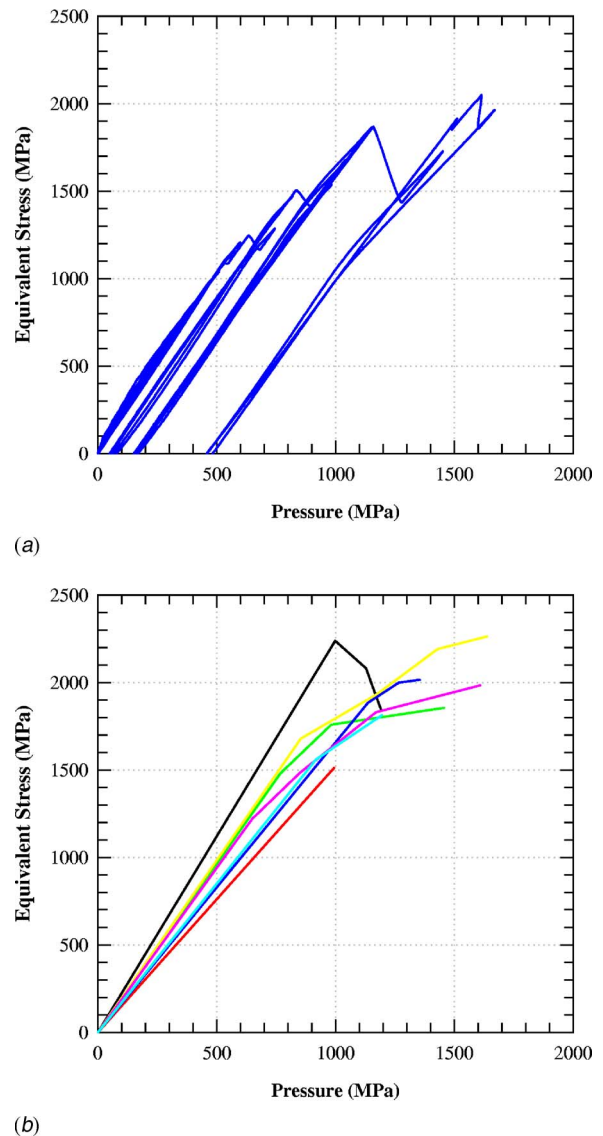


Fig. 5 (a) Equivalent stress versus pressure plot for a typical test. (b) Envelopes for six different tests showing that sample strength is limited. All the tests shown are confined compression of predamaged samples.

4.1 Different Test to Determine the Strength at Zero Pressure Y_0 . Unfortunately, the test presented above does not allow us to determine Y_0 , the strength at zero pressure, with reasonable accuracy. Y_0 is a small number of the order of 100 MPa. However, it is important to determine Y_0 because hydrocode calculations of penetrators impacting brittle materials show that the final depth of penetration is very sensitive to this parameter. The reason for the sensitivity is that if Y_0 is assumed to be zero, a failed target then has no strength. The result is that the projectile cannot be stopped.

The test proposed is compression of the specimen under *constant* confinement pressure. With this test, determining Y_0 is straightforward. If the specimen is yielding, then the equivalent pressure ($\sigma_{eq} = |\sigma_r - \sigma_z|$) is equal to the yield strength $Y = Y_0 + \beta P$ and, since $|\sigma_z| > |\sigma_r|$,

$$\sigma_r - \sigma_z = Y_0 + \beta P = Y_0 + \frac{\beta}{3}(\sigma_r + \sigma_\theta + \sigma_z) \quad (4)$$

where σ_z is the known applied load, σ_r is the confinement stress, and $\sigma_\theta = \sigma_r$, see Sec. 5. Since the confinement pressure is, for the discussion in this section, constant and known (i.e., controlled by

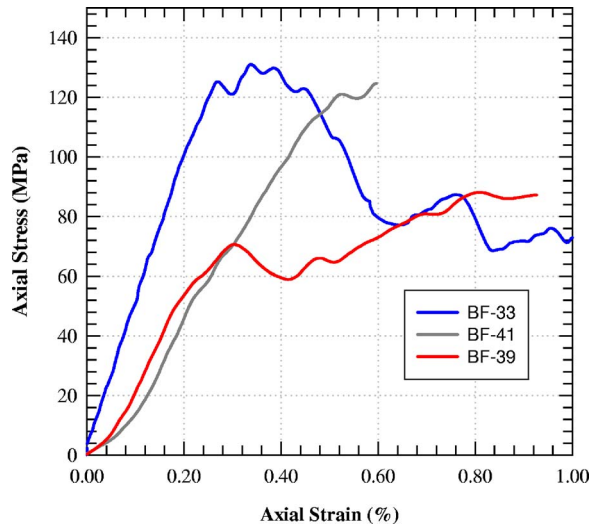


Fig. 6 Stress versus strain curves for compression of unconfined-predamaged samples. Confinement pressure in these samples is zero.

a hydraulic system), let us call it $P_c^0 \equiv -\sigma_r$. It is then possible to write Eq. (4) as

$$\sigma_z|_{\text{yield}} = \frac{Y_0}{\beta/3 - 1} + \frac{2\beta + 3}{\beta - 3} P_c^0 \quad (5)$$

Equation (5) shows that, if the confinement pressure is constant, yielding will appear as a plateau in the axial stress versus axial strain curve. It also shows that β and Y_0 can be determined by testing the specimen at two different confinement pressures. Figure 6 shows the results of three experiments as an example. In this case, the experiments were only used to determine Y_0 since all were performed at zero confinement pressure. Strength values found were 130 MPa, 120 MPa, and 70 MPa, which gave, respectively, Y_0 values of 52 MPa, 48 MPa, and 28 MPa if the slope of the Drucker–Prager is assumed to be $\beta = 1.8$ (a typical slope, as shown in Fig. 5). Therefore, the average value and standard deviation of Y_0 are 43 ± 13 MPa. This value agrees (at least in order of magnitude) with the 100 MPa value assumed, because of lack of experimental data, by Walker and Anderson [16] when developing their penetration model into glass.

5 Full Analytical Model of the Confined Test With Explicit Elastoplastic Solution

One of the objectives of the analytical model is to obtain the elastic and plastic constants for the damaged borosilicate specimen by using the axial stress versus axial strain and axial stress versus hoop-strain curves obtained during the characterization tests. Obtaining the equivalent stress versus pressure curve is also desired. This is relatively straightforward as was shown in the previous section. The analytical model also helps us to provide a complete understanding of the test, in the design of new tests and in checking the assumptions and interpretation of the results.

The assumptions used in the model are described first. Then, the incremental equations to completely solve stresses and strains are presented. Finally, it will be shown how the equations can be solved explicitly as a function of ϵ_z and how the elastic and plastic constants can be derived from them.

5.1 Assumptions for the Specimen. The specimen is assumed to have an elastoplastic behavior. The elastic part is characterized by the two Lamé constants, λ and μ (or E and ν), which, in principle, are the constants we want to determine with the characterization tests. The plastic part is assumed to be a Drucker–

Prager constitutive model with a cap, where the strength is given by $Y = Y_0 + \beta P$. Y_0 is the tensile strength and P is the hydrostatic pressure. The strength is capped by the value $\bar{Y}$.

The displacement field in the specimen is typical of a solid cylinder under axisymmetric loading: $u_r = A_{sp} r$, where A_{sp} is a constant (that depends on the load applied) and r is the radial coordinate. This implies that the radial stress in the specimen is constant and that the hoop stress is equal to the radial stress, see Timoshenko Ref. [13].

There is no friction between the specimen and the sleeve, i.e., the only interaction with the sleeve is through radial stresses.

5.2 Assumptions for the Confining Sleeve. The sleeve is assumed to be elastic and characterized by the two known Lamé constants λ' and μ' for Vascomax steel. The displacement field in the sleeve is typical of a hollow cylinder under axisymmetric loading: $u_r = Ar + B/r$, where A and B are constants that depend on the load applied to the specimen.

5.3 Incremental Equations for the Deformation of the Specimen. In this section, incremental equations are written that will explicitly solve all the stresses and strains (both elastic and plastic) in the specimen while it is deforming.

Given that hoop stress and radial stress are the same in the specimen, Hooke's law can be written incrementally as

$$d\sigma_r = 2(\lambda + \mu)d\epsilon_r^e + \lambda d\epsilon_z^e \quad (6)$$

$$d\sigma_z = (\lambda + 2\mu)d\epsilon_z^e + 2\lambda d\epsilon_r^e$$

where the superscript e denotes elastic strain.

Since the sleeve is assumed to remain elastic, the confinement pressure that the specimen sees is proportional to the increase in the radius of the specimen. The proportionality constant, which only depends on the elastic constants of the sleeve and the geometry, is called C' and will be calculated in the next section. For now, since this simplifies the math, let us just write the equation that relates confinement pressure and radial displacement, assuming C' is known. Recall that the initial radius of the specimen is a ,

$$d\sigma_r = C' \frac{d\bar{u}_r}{a} \quad (7)$$

The radial displacement of the outer radius ($\bar{u}_r \equiv u_r(r=a)$) of the specimen is related to the radial strain through the following equation:

$$d\bar{u}_r = a d\epsilon_r|_{r=a} \quad (8)$$

This equation comes from differentiating the engineering definition of strain, so it assumes small strains:

$$\epsilon_r = \frac{r_0 + \bar{u}_r}{r_0} - 1 \quad (9)$$

where r_0 is the initial radius and $r_0 + \bar{u}_r$ the final radius.

Now, let us write the plasticity part of the equations. Since a Drucker–Prager model is assumed, the yield strength is given as a function of the hydrostatic pressure in the specimen by

$$Y = Y_0 + \beta P \quad (10)$$

where Y_0 and β are the plasticity constants of the damaged specimen. If the specimen is flowing plastically, then the equivalent stress equals the yield strength so that

$$\sigma_{eq} = |\sigma_r - \sigma_z| = Y_0 - \frac{\beta}{3}(\sigma_r + \sigma_\theta + \sigma_z) \quad (11)$$

and by differentiating this last equation, given that $|\sigma_z| > |\sigma_r|$,

$$d\sigma_r - d\sigma_z = -\frac{\beta}{3}(2d\sigma_r + d\sigma_z) \quad (12)$$

Finally, the only equations that remain to be written are the conservation of volume during plastic deformation (so nonassociated flow is assumed) and the fact that the total strain is found by adding the elastic (superscript e) and plastic strains (superscript p):

$$2d\epsilon_r^p + d\epsilon_z^p = 0 \quad (13)$$

$$d\epsilon_r = d\epsilon_r^e + d\epsilon_r^p \quad (14)$$

$$d\epsilon_z = d\epsilon_z^e + d\epsilon_z^p$$

Equations (6)–(8) and (12)–(14) constitute a set of eight equations with eight unknowns: $d\epsilon_r$, $d\epsilon_r^e$, $d\epsilon_r^p$, $d\epsilon_z^e$, $d\epsilon_z^p$, $d\sigma_r$, $d\sigma_z$, and $d\bar{u}_r$. Note that $d\epsilon_z$, the total strain in the z direction, is the strain applied to the specimen by the MTS machine and is consequently known.

Calculation of the Proportionality Constant C' . Since the sleeve is elastic (and there is no friction), the relationship between the confinement pressure provided by the sleeve and the displacement of its inner radius is linear: $d\sigma_r = C' d\bar{u}_r / a$. In this section, the value of C' is derived.

Again, the departure point is the classical displacement field; now, in the sleeve, $u_r = Ar + B/r$. A and B are constants that need to be calculated using the boundary conditions. Since the boundary conditions change at every increment of $d\epsilon_z$, they need to be recalculated at every increment. The boundary conditions for the sleeve are as follows.

- The outer surface of the sleeve is a free surface:

$$\sigma_r(r=b) = 0 \quad (15)$$

- The radial stress at the specimen/sleeve interface should be equal on both sides:

$$\sigma_r(r=a)|_{\text{sleeve}} = \sigma_r(r=a)|_{\text{specimen}} \quad (16)$$

- The radial displacement at the specimen/sleeve interface should be equal on both sides:

$$u_r(r=a)|_{\text{sleeve}} = u_r(r=a)|_{\text{specimen}} \quad (17)$$

- The axial stress in the sleeve is zero because there is no friction between the specimen and the sleeve:

$$\sigma_z|_{\text{sleeve}} = 0 \quad (18)$$

The strain tensor is derived from the displacement field with

$$\begin{aligned} \epsilon_r &= \frac{\partial u_r}{\partial r} = A - \frac{B}{r^2} \\ \epsilon_\theta &= \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} = A + \frac{B}{r^2} \\ \epsilon_z &= \frac{\partial u_z}{\partial z} \end{aligned} \quad (19)$$

Using Eqs. (19) in Hooke's law for the sleeve,

$$\sigma_r = (\lambda' + 2\mu')\epsilon_r + \lambda'\epsilon_\theta + \lambda'\epsilon_z = 2(\lambda' + \mu')A + (\lambda' - 2\mu')(B/r^2) + \lambda'\epsilon_z$$

$$\begin{aligned} \sigma_\theta &= \lambda'\epsilon_r + (\lambda' + 2\mu')\epsilon_\theta + \lambda'\epsilon_z = 2(\lambda' + \mu')A + (-\lambda' + 2\mu') \\ &\times (B/r^2) + \lambda'\epsilon_z \end{aligned} \quad (20)$$

$$\sigma_z = (\lambda' + 2\mu')\epsilon_z + 2\lambda'\epsilon_r = 0$$

The superscript e has been dropped from the equations because the sleeve is assumed to stay always elastic.

Since $\sigma_z = 0$, the axial strain is easily found as a function of the radial strain:

$$\epsilon_z = -\frac{2\lambda'}{\lambda' + 2\mu'}\epsilon_r \quad (21)$$

For the sake of this derivation, $\bar{u}_r$ is considered a known variable. Later in the derivation, when coupling of sleeve and specimen stresses/displacements is done, this assumption will be removed. Using Eq. (15) and the displacement field assumption, we can obtain A and B as a function of $\bar{u}_r$ and the elastic constants of the sleeve:

$$\sigma_r(r=b) = 2(\lambda' + \mu')A + (\lambda' - 2\mu')\frac{B}{b^2} + \lambda'\left(-\frac{2\lambda'}{\lambda' + 2\mu'}\right)\epsilon_r = 0 \quad (22)$$

$$\bar{u}_r \equiv u_r(r=a) = Aa + \frac{B}{a^2}$$

Hence, solving the above two equations with two unknowns (A, B) yields

$$A = A'\frac{\bar{u}_r}{a} \quad \text{where } A' \equiv \frac{(\lambda' + 2\mu')a^2}{b^2(3\lambda' + 2\mu') + a^2(\lambda' + 2\mu')} \quad (23)$$

$$B = B'\frac{\bar{u}_r}{a} \quad \text{where } \frac{B'}{a^2} \equiv \frac{(3\lambda' + 2\mu')b^2}{b^2(3\lambda' + 2\mu') + a^2(\lambda' + 2\mu')} \quad (24)$$

Now that A and B are known, the radial stress in the inner diameter of the sleeve can be written. From the first equation in Eq. (20), C' is then easily derived by using Eq. (7):

$$\sigma_r(r=a) = C'\frac{\bar{u}_r}{a} = 2(\lambda' + \mu')A + (\lambda' - 2\mu')\frac{B}{a^2} \quad (25)$$

So,

$$C' = 2\left[(\lambda' + \mu')A' - \mu'\frac{B'}{a^2}\right] \quad (26)$$

In summary, in this section, C' (as well as A' and B') has been written as a function of the elastic constants of the sleeve and its geometry. Therefore, it can be considered a known constant that will be used when convenient. The following sleeve variables will also be known as soon as the system of eight equations presented in Sec. 5.3 is solved so that $\bar{u}_r$ is found:

$$\sigma_r(r=a) = C'\frac{\bar{u}_r}{a} \quad \text{or} \quad d\sigma_r(r=a) = C'\frac{d\bar{u}_r}{a} \quad (27)$$

$$\epsilon_r = \frac{\partial u_r}{\partial r} = \left(A' - \frac{B'}{r^2}\right)\frac{\bar{u}_r}{a} \quad \text{or} \quad d\epsilon_r = \left(A' - \frac{B'}{r^2}\right)\frac{d\bar{u}_r}{a} \quad (28)$$

$$\epsilon_\theta = \frac{u_r}{r} = \left(A' + \frac{B'}{r^2}\right)\frac{\bar{u}_r}{a} \quad \text{or} \quad d\epsilon_\theta = \left(A' + \frac{B'}{r^2}\right)\frac{d\bar{u}_r}{a} \quad (29)$$

5.5 Explicit Solution When the Specimen is Deforming Elastically. We are now ready to write the explicit solution when the specimen is deforming elastically. Hooke's law for the specimen is

$$\begin{aligned} d\sigma_r &= 2(\lambda + \mu)d\epsilon_r + \lambda d\epsilon_z \\ d\sigma_\theta &= d\sigma_r \end{aligned} \quad (30)$$

$$d\sigma_z = (\lambda + 2\mu)d\epsilon_z + 2\lambda d\epsilon_r$$

Recall that σ_r is not a function of r in the specimen; it is uniform throughout the specimen. By substituting $d\sigma_r = C'\bar{u}_r/a$ and (see Eq. (9)) $d\bar{u}_r = ad\epsilon_r$ in Eq. (30), we can explicitly write

$$d\epsilon_r = \frac{\lambda}{C' - 2(\lambda + \mu)} d\epsilon_z \quad (31)$$

which does not depend on r , as expected from the displacement field assumed. Since the slope $d\sigma_z/d\epsilon_z$ is used in the analysis of the data, let us write it explicitly as

$$\left(\frac{d\sigma_z}{d\epsilon_z}\right)_{\text{elastic}} = \frac{2\lambda^2}{C' - 2(\lambda + \mu)} + \lambda + 2\mu \quad (32)$$

One of the variables that will be needed is the hoop strain in the outer surface of the sleeve given the axial strain of the specimen. At this point, it is straightforward to compute it by using Eqs. (7) and (29):

$$d\epsilon_{\theta|_r}^{\text{sleeve}} = \left(A' + \frac{B'}{r^2}\right) \frac{d\bar{u}_r}{a} = \left(A' + \frac{B'}{r^2}\right) d\epsilon_{r|_{r=a}}^{\text{specimen}} \quad (33)$$

So,

$$d\epsilon_{\theta|_{r=b}}^{\text{sleeve}} = \left(A' + \frac{B'}{b^2}\right) d\epsilon_{r|_{r=a}}^{\text{specimen}} \quad (34)$$

5.6 Explicit Solution When the Specimen is Deforming Plastically. The specimen starts to deform plastically when the equivalent stress reaches the yield strength. From Eq. (7), $d\sigma_r = C'd\epsilon_r$, and, from Eq. (13) and the second equation in Eq. (14),

$$d\epsilon_r = d\epsilon_r^e + \frac{d\epsilon_z^e}{2} - \frac{d\epsilon_z}{2} \quad (35)$$

By using the first equation of Eq. (6), we get

$$C'd\epsilon_r = C'd\epsilon_r^e + \frac{d\epsilon_z^e}{2} - \frac{d\epsilon_z}{2} = 2(\lambda + \mu)d\epsilon_r^e + \lambda d\epsilon_z^e \quad (36)$$

From Eq. (12), $d\sigma_z = \beta'd\sigma_r$, where $\beta' = (1 + 2\beta/3)/(1 - \beta/3)$. So, the second equation of Eq. (6) becomes

$$(\lambda + 2\mu)d\epsilon_z^e + 2\lambda d\epsilon_r^e = \beta'C'd\epsilon_r = 2(\lambda + \mu)d\epsilon_r^e + \lambda d\epsilon_z^e \quad (37)$$

Equations (36) and (37) allow us to find the relation between $d\epsilon_r^e$ and $d\epsilon_z^e$:

$$d\epsilon_r^e = -\frac{\lambda(\beta' - 1) - 2\mu}{\lambda(\beta' - 1) + \beta'\mu} \frac{d\epsilon_z^e}{2} \equiv -\delta d\epsilon_z^e \quad (38)$$

where a new variable δ , which only depends on material properties, has been defined to abbreviate notation. Using Eq. (36) again, it is possible to find $d\epsilon_z^e$, the elastic deformation in the z direction as a function of the total deformation $d\epsilon_z$, which is known:

$$d\epsilon_z^e = -\frac{C'}{2[2\delta(\lambda + \mu) - \lambda + C'(1/2 - \delta)]} d\epsilon_z \equiv \gamma d\epsilon_z \quad (39)$$

Again, a new variable (γ) was used to abbreviate notation and highlight the proportionality between all the unknowns and the strain applied $d\epsilon_z$. All the other unknowns are easily derived from the last equation, for example, by using Eqs. (38) and (39), we get

$$d\epsilon_r^e = -\gamma\delta d\epsilon_z \quad (40)$$

and with Eq. (35),

$$d\epsilon_r = \left[\gamma\left(\frac{1}{2} - \delta\right) - \frac{1}{2}\right] d\epsilon_z \quad (41)$$

The last three unknowns follow:

$$d\sigma_r = C'd\epsilon_r = C'\left[\gamma\left(\frac{1}{2} - \delta\right) - \frac{1}{2}\right] d\epsilon_z \quad (42)$$

$$d\sigma_z = \beta'd\sigma_r = \beta'C'\left[\gamma\left(\frac{1}{2} - \delta\right) - \frac{1}{2}\right] d\epsilon_z \quad (43)$$

$$d\bar{u}_r = a d\epsilon_r = a\left[\gamma\left(\frac{1}{2} - \delta\right) - \frac{1}{2}\right] d\epsilon_z \quad (44)$$

6 Summary: Determination of the Elastic and Plastic Constants

The analytical model presented in Sec. 5 allows one to easily calculate the elastic constants of the specimen, E and ν (or λ and μ), and the Drucker–Prager parameter β by solving three equations with the material constants being the three unknowns. In fact, if the experimental results are interpreted the way they are shown in Fig. 4, they provide four slopes, 1, 1', 2, and 2', to match the four equations provided by the model.

- 1 Slope 1 is the elastic slope of the axial stress versus axial strain curve:

$$\text{slope 1} = \left(\frac{d\sigma_z}{d\epsilon_z}\right)_{\text{elastic}} = \frac{2\lambda^2}{C' - 2(\lambda + \mu)} + \lambda + 2\mu \quad (45)$$

where λ and μ are the Lamé constants for the specimen and C' is a constant that depends on the elastic properties of the sleeve and its geometry. See Sec. 5, Eq. (32), for a complete explanation.

- 2 Slope 1' is the elastic slope of the axial stress versus hoop-strain curve:

$$\text{slope 1'} = \left(\frac{d\sigma_z}{d\epsilon_{\theta}}\right)_{\text{elastic}, r=b} = \frac{2\lambda^2 + (\lambda + 2\mu)[C' - 2(\lambda + \mu)]}{\lambda(A' + B'/b^2)} \quad (46)$$

where the constants A' and B' are explicitly written in Eqs. (23) and (24).

- 3 Slope 2 is the plastic slope of the axial stress versus axial strain curve:

$$\text{slope 2} = \left(\frac{d\sigma_z}{d\epsilon_z}\right)_{\text{plastic}} = \beta'C'\left[\gamma\left(\frac{1}{2} - \delta\right) - \frac{1}{2}\right] \quad (47)$$

where β' , γ , and δ are explicitly written in Sec. 5.6.

- 4 Slope 2' is the plastic slope of the axial stress versus hoop-strain curve:

$$\text{slope 2'} = \left(\frac{d\sigma_z}{d\epsilon_{\theta}}\right)_{\text{plastic}, r=b} = \frac{\beta'C'}{(A' + B'/b^2)} \quad (48)$$

7 Example: Results Obtained With Damaged Borosilicate Glass

Full detail of the following results and how they were obtained will be given in a separate paper. In this paper, the results are just briefly presented.

Slopes 1, 1', 2, and 2' were measured for a series of loading cycles and tests. Equations (45)–(48) were then solved for λ , μ , and β . Since there are three unknowns and four equations, the equations were solved in groups of 3. There were three different possibilities (two of them give identical solutions) that gave three slightly different values for the elastic and plastic constants.

The average values and standard deviations are shown in Table 1. Intact and ultrasound measurements are also provided for comparison. The modulus changes very little from intact to predamaged (thermally shocked) glass. Even when the glass is severely damaged, as, for example, in a multiple cycle load-unload test, the average modulus decreases less than 10% when compared to the modulus of the material determined from MTS data.

The results indicate that the elastic modulus and Poisson's ratio are not significantly affected by damage. Post-test evaluation of the samples [17] shows that after a few cycles, the glass is completely failed, almost powdered. Stating that highly comminuted glass has a similar modulus to intact glass is an unexpected result. Again, it should be emphasized that this is applicable if the ma-

Table 1 Average material properties inferred from the tests and the model for predamaged glass. Intact properties are provided for comparison

	E (GPa)	ν	β	$\bar{Y}$ (GPa)	Y_0 (MPa)
Damaged (MTS)	56 ± 5	0.20 ± 0.04	1.8 ± 0.2	1.8 ± 0.3	43 ± 13
Predamaged (ultrasonic)	60 ± 2	0.18 ± 0.02	—	—	
Intact (MTS)	59 ± 2	0.18 ± 0.01	—	—	

material is confined, i.e., *the material does not have any space to move* and, consequently, voids are not allowed to open and decrease the elastic modulus. During ballistic penetration, inertia and confinement keep the target material in place for some time. It can then be argued that even if the material right in front of the projectile is failed, its elastic modulus is almost unchanged (as well as its Poisson's ratio, sound speed, etc.).

8 Conclusions

Compression of brittle specimens inside a confinement sleeve is becoming a test commonly found in the literature. The objective of the test is to find constitutive parameters of brittle materials at high pressures. The constants can afterward be used in hydrocodes to predict ballistic behavior (e.g., depth of penetration or ballistic limit). An analytical model of the confined test with an explicit elastoplastic solution was developed to aid in the interpretation of compression test results and to infer the constants needed. The analytical model was applied to borosilicate glass. The constants determined here are in agreement with values previously published by other authors using different techniques. Average values for the elastic modulus and Poisson's ratio for the glass were determined, as well as the Drucker–Prager constitutive law.

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Modeling Surface Electrodes on a Piezoelectric Layer

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This paper considers a piezoelectric ceramic layer with a surface electrode. It focuses on the effect of the layer thickness on the electrode tip fields. A closed-form solution for the electromechanical fields at the electrode tip is obtained and is expressed in terms of the applied electric field intensity factor, which can be obtained exactly for infinite layer thickness and numerically for finite layer thickness. The stress, electric displacement, and electric field are plotted to show the effect of layer thickness. It is found that the stresses and field intensities at the electrode tip can be reduced considerably by decreasing the thickness of the piezoelectric layer, confirming the previous finding. The paper also gives a solution for two identical and collinear surface electrodes. The relative distance between the electrodes is observed to have significant influence on the electromechanical field in the piezoelectric layer. [DOI: 10.1115/1.2775504]

Keywords: piezoelectric materials, electrode, fracture mechanics

1 Introduction

Electroelastic interaction of ceramic materials with electrodes is of importance in smart material and devices. The electric field in these materials and devices are usually applied through thin electrodes. Stress and electric field concentrations near crack tips or electrode tips in piezoelectric ceramics can result in electromechanical degradation. Hence, it is important to study the electric fields and stress distributions at the edge of electrodes and to understand the mechanical and electric failure phenomena of the materials. It has been observed that cracking ahead of the electrode is a common case of failure in many electromechanical devices [1]. A significant body of work has been developed to model the piezoelectric ceramics with electrodes. Many theoretical analysis and numerical results have shown that the edge of the electrode is a potential position of high stress and electric field concentrations [2–5]. For piezoelectric materials containing multilayered internal electrodes, electric field and elastic fields induced by quadratic electrostriction have been obtained via the analytical approach and the finite element method [6,7]. Experimental observation and 3D finite element simulation were made to explore the electroelastic field concentrations ahead of the electrodes in multilayer piezoelectric actuators [8].

As shown in Fig. 1 for the electrode configurations investigated in references, a lot of relevant problems have been solved [9–19]. However, some results for the straight electrode problem presented in literature contain some mistakes. Moreover, the actual structures should always have finite size. This means that study on a finite electrode on a finite piezoelectric layer is important. Therefore, it is highly necessary to establish a general model for the finite layer thickness, from which results for the infinite layer can also be derived.

This paper focuses on the remaining unsolved problem shown in Fig. 2, in which an electrode is attached on a piezoelectric layer of finite thickness (the case of infinite h has been investigated in Refs. [20,21]). The influence of the layer thickness is studied in detail. The effect of air surrounding the piezoelectric layer is also explored. An electric field intensity factor is defined to characterize the electrode tip fields. The electrode tip electromechanical fields are expressed in terms of the electric field intensity factor. The field intensity factor at the electrode tip, and the electric dis-

placement and electric field on the electrode plane are shown graphically for different values of layer thickness. Besides the single electrode configuration, the problem of two collinear surface electrodes is also investigated.

2 Description of the Problem

Let us consider the piezoelectric ceramic layer in Fig. 2. The poling direction of the medium is parallel to the positive y axis, where (x, y) is the coordinate system. The center of the electrode is directed through the y axis. We investigate a 2D problem such that all field variables are functions of x and y only. We denote the displacement along the x and y directions as u and v , respectively, and the electric potential as ϕ . Constitutive equations for piezoelectric materials polarized along the y direction can be written as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ D_x \\ D_y \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{13} & 0 & 0 & -e_{31} \\ c_{13} & c_{33} & 0 & 0 & -e_{33} \\ 0 & 0 & c_{44} & -e_{15} & 0 \\ 0 & 0 & e_{15} & \epsilon_{11} & 0 \\ e_{31} & e_{33} & 0 & 0 & \epsilon_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \\ E_x \\ E_y \end{Bmatrix} \quad (1)$$

where σ_{ij} and D_i ($i, j = x, y$) are stresses and electrical displacements; c_{ij} , e_{ij} , and ϵ_{ii} are elastic constants, piezoelectric constants, and dielectric permittivities, respectively; and ε_{ij} and E_i are, respectively, strains and electrical fields, which are related to the displacements and electric potential through

$$\varepsilon_{ij} = (u_{,i} + u_{,j})/2 \quad E_i = -\phi_{,i} \quad (2)$$

in which $(i, j = x, y)$. In the absence of body forces and body charges, the equilibrium equations for the piezoelectric media are

$$\sigma_{ij,j} = 0 \quad D_{i,i} = 0 \quad (3)$$

Equation (3) can be expressed in terms of displacements and electric potential with the substitution of Eqs. (1) and (2).

2.1 Electric Boundary Conditions. As shown in Fig. 2, the medium at $y=0$ is surrounded by free space (vacuum or air). The normal component of the electric displacement vector outside the electrode region at $y=0$ and the electric potential in the entire horizontal axis are continuous so that

$$D_y(x, 0) = D_y^a(x, 0) \quad x > a \quad (4a)$$

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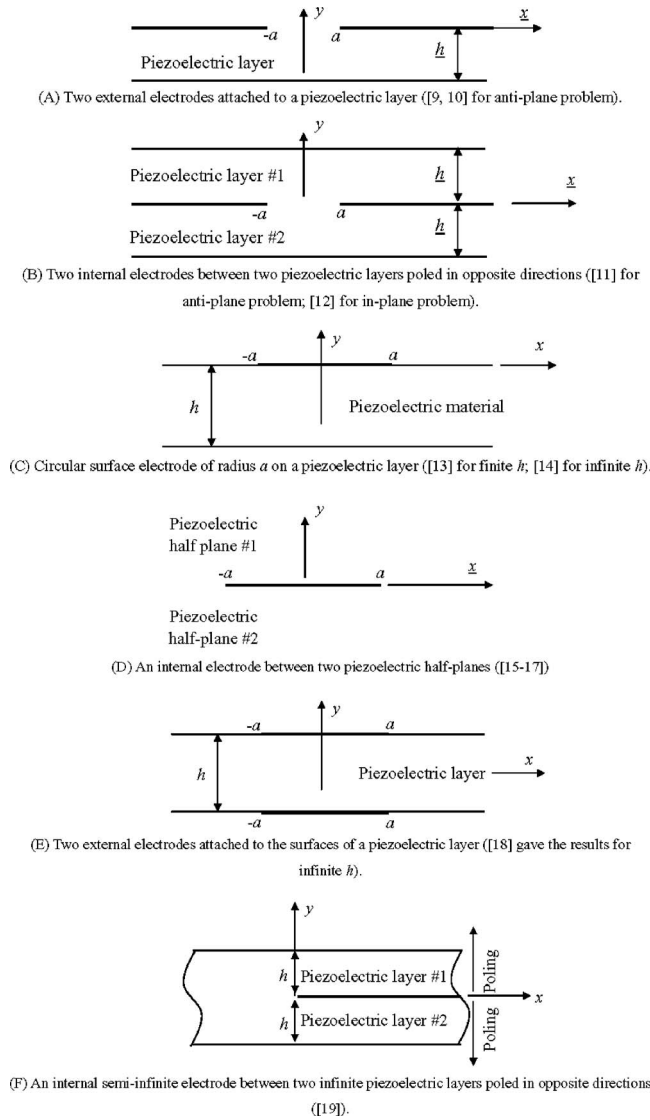


Fig. 1 Various electrode configurations of reference papers (the bold lines represent electrodes). (A) Two external electrodes attached to a piezoelectric layer (Refs. [9,10] for anti-plane problem). (B) Two internal electrodes between two piezoelectric layers poled in opposite directions (Ref. [11] for anti-plane problem; Ref. [12] for in-plane problem). (C) Circular surface electrode of radius a on a piezoelectric layer (Ref. [13] for finite h ; Ref. [14] for infinite h). (D) An internal electrode between two piezoelectric half-planes (Refs. [15–17]). (E) Two external electrodes attached to the surfaces of a piezoelectric layer (Ref. [18] gave the results for infinite h). (F) An internal semi-infinite electrode between two infinite piezoelectric layers poled in opposite directions (Ref. [19]).

$$\phi(x, 0) = \phi^a(x, 0) \quad x > 0 \quad (4b)$$

in which the superscript a denotes the field quantities in air. Under an applied voltage, the electric charge will be accumulated on the electrode but the electric field inside the electrode should be zero. We denote the total charge accumulated on the electrode as $2Q_0$. Thus, we have

$$\int_0^a [D_y(x, 0) - D_y^a(x, 0)] dx = Q_0 \quad x < a \quad (5a)$$

$$E_x(x, 0) = E_x^a(x, 0) = 0 \quad x < a \quad (5b)$$

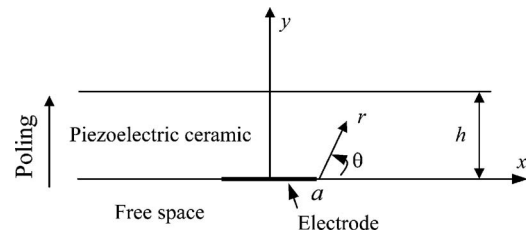


Fig. 2 A surface electrode on a piezoelectric layer

The assumption of Eq. (5a) is similar to those made in Refs. [15,16,21].

In addition, it is assumed that the surface $y=h$ of the piezoelectric medium is grounded so that

$$\phi(x, h) = 0 \quad (6)$$

2.2 Mechanical Boundary Conditions. As for the mechanical boundary conditions, we always assume that the boundaries $y=0$ and $y=h$ are free from horizontal shear stress. That is,

$$\sigma_{xy}(x, 0) = 0 \quad \sigma_{xy}(x, h) = 0 \quad (7)$$

Furthermore, the electrode plane of the piezoelectric layer (the $y=0$ plane) is free of normal stress. In addition, it is assumed that the entire medium does not bend on the $z=h$ plane. These conditions can be written as

$$\sigma_{yy}(x, 0) = 0 \quad v(x, h) = 0 \quad (8)$$

3 Solution

The governing equations (3) for the displacements and electric potential can be expressed in terms of unknown functions $F_m(s)$ as follows:

$$\begin{cases} u(x, y) \\ v(x, y) \\ \phi(x, y) \end{cases} = \frac{2}{\pi} \int_0^\infty \sum_{m=1}^6 \begin{cases} a_{1m} \sin(sx) \\ a_{2m} \cos(sx) \\ a_{3m} \cos(sx) \end{cases} \exp(s\lambda_m y) F_m ds \quad (9)$$

where $\text{sgn}(\xi)$ equals 1 for positive values of ξ and -1 for negative values of ξ ; λ_m are eigenvalues, and (a_{1m}, a_{2m}, a_{3m}) are eigenvectors of the following characteristic equation:

$$\begin{bmatrix} c_{11} - c_{44}\lambda_m^2 & (c_{13} + c_{44})\lambda_m & (e_{31} + e_{15})\lambda_m \\ (c_{13} + c_{44})\lambda_m & c_{33}\lambda_m^2 - c_{44} & e_{33}\lambda_m^2 - e_{15} \\ (e_{31} + e_{15})\lambda_m & e_{33}\lambda_m^2 - e_{15} & \epsilon_{11} - \epsilon_{33}\lambda_m^2 \end{bmatrix} \begin{Bmatrix} a_{1m} \\ a_{2m} \\ a_{3m} \end{Bmatrix} = 0 \quad (10)$$

In Eq. (10), there are six roots for λ_m . It can be shown that if $[\lambda_m, (a_{1m}, a_{2m}, a_{3m})^T]$ is an eigensolution of Eq. (10), then $[-\lambda_m, (-a_{1m}, a_{2m}, a_{3m})^T]$ is also an eigensolution of Eq. (10). In what follows, the order of the roots λ_m are arranged such that $\text{Re}(\lambda_1) < 0$, $\text{Re}(\lambda_2) < 0$, $\text{Re}(\lambda_3) < 0$ and $\lambda_4 = -\lambda_1$, $\lambda_5 = -\lambda_2$, $\lambda_6 = -\lambda_3$. Substituting Eq. (9) into the constitutive equations (1) through Eq. (2), the stresses and electric displacements can be obtained as

$$\begin{cases} \sigma_{xx}(x, y) \\ \sigma_{yy}(x, y) \\ D_y(x, y) \end{cases} = \frac{2}{\pi} \sum_{m=1}^6 \int_0^\infty s \begin{cases} C_{1m} \\ C_{2m} \\ C_{3m} \end{cases} \exp(s\lambda_m y) \cos(sx) F_m ds \quad (11a)$$

$$\begin{Bmatrix} \sigma_{xy}(x,y) \\ D_x(x,y) \end{Bmatrix} = \frac{2}{\pi} \sum_{m=1}^6 \int_0^\infty s \begin{Bmatrix} D_{1m} \\ D_{2m} \end{Bmatrix} \exp(s\lambda_m y) \sin(sx) F_m ds \quad (11b)$$

in which the coefficients C_{im} and D_{im} ($i=1, \dots, 3; m=1, \dots, 6$) are given in Appendix A.

For the free space (air), $y < 0$, the electric potential and the normal component of the electric displacement vector can be expressed as

$$\phi^a(x, y) = \frac{2}{\pi} \int_0^\infty A(s) \cos(sx) \exp(sy) ds \quad (12)$$

and

$$D_z^a(x, y) = -\epsilon_0 \frac{2}{\pi} \int_0^\infty s A(s) \cos(sx) \exp(sy) ds \quad (13)$$

respectively, where ϵ_0 is the permittivity of free space (vacuum or air) and $A(s)$ is an unknown function of s . Equations (12) and (13) satisfy the condition that the field quantities vanish as y approaches negative infinity (see Fig. 1).

From the fact that the electric potential is continuous along the entire $y=0$ plane (Eq. (4b)), we obtain

$$A(s) = \sum_{m=1}^6 a_{3m} F_m(s) \quad (14)$$

we define the following electric displacement discontinuity:

$$g(x) = D_y(x, 0) - D_y^a(x, 0) \quad (15)$$

Thus, $g(x)$ represents the electric charge accumulated on the $y=0$ plane. It follows from Eqs. (11a), (13), and (14) that

$$g(x) = \frac{2}{\pi} \sum_{m=1}^6 (C_{3m} + \epsilon_0 a_{3m}) \int_0^\infty s \cos(sx) F_m(s) ds \quad (16)$$

Since the normal component of the electric displacement vector is continuous (the first equation of Eq. (4)), the function $g(x)$ is zero out of the electrode region. The inversion of Eq. (16) yields

$$\sum_{m=1}^6 (C_{3m} + \epsilon_0 a_{3m}) F_m = s^{-1} \int_0^a g(r) \cos(sr) dr \quad (17)$$

Through the boundary conditions (6)–(8) and with Eq. (17), it can be shown that the unknowns $F_m(s)$ are related to the function $g(r)$ as

$$F_m(s) = B_m(s) s^{-1} \int_0^a g(r) \cos(sr) dr \quad (18)$$

where $m=1, \dots, 6$ and $(B_1, B_2, B_3, B_4, B_5, B_6)^T$ is the first column of 6×6 matrix $[Z]$:

$$[Z] = \begin{bmatrix} C_{3j} + \epsilon_0 a_{3j} \\ C_{2j} \\ D_{1j} \\ a_{3j} \exp(s\lambda_j h_j) \\ a_{2j} \exp(s\lambda_j h_j) \\ D_{1j} \exp(s\lambda_j h_j) \end{bmatrix}^{-1} \quad (j=1, \dots, 6) \quad (19)$$

In order to use the boundary condition for the electric field inside the electrode, we write the electric field on the $y=0$ plane by differentiating the electric potential (Eq. (9)) with x and substituting Eq. (18) into it. This gives

$$E_x(x, 0) = \frac{2}{\pi} \int_0^a g(r) \cos(sr) dr \int_0^\infty G(s) \sin(sx) ds \quad (20)$$

where

$$G(s) = \sum_{m=1}^6 a_{3m} B_m(s) \quad (21)$$

When s approaches infinity, the function $G(s)$ has an asymptotic value G_0 . Equation (20) can be rewritten as

$$E_x(x, 0) = \frac{2}{\pi} \int_0^a g(r) \cos(sr) dr \int_0^\infty G_0 \sin(sx) ds + \frac{2}{\pi} \int_0^a \Omega(x, r) g(r) dr \quad (22)$$

where the integral kernel $\Omega(x, r)$ is

$$\Omega(x, r) = \int_0^\infty [G(s) - G_0] \sin(sx) \cos(sr) ds \quad (23)$$

Equation (22) can be further transferred to

$$E_x(x, 0) = \frac{G_0}{\pi} \int_0^a \left(\frac{1}{x+r} + \frac{1}{x-r} \right) g(r) dr + \frac{2}{\pi} \int_0^a \Omega(x, r) g(r) dr \quad (24)$$

This is a standard singular integral equation whose solution can be expressed as [22]

$$g(r) = \bar{g}(r) / \sqrt{a^2 - r^2} \quad (25)$$

where $\bar{g}(r)$ is a bounded function in the interval $[0, a]$. It can be seen from Eq. (25) that the electric charge accumulated on the electrode is singular at the electrode tips $x = \pm a$. Because the medium is finite, $g(r)$ can only be obtained numerically. We express $g(r)$ in terms of the Chebyshev polynomial of the first kind $T_m(r/a)$ as follows:

$$\bar{g}(r) = \sum_{m=0,2,4,\dots}^\infty C_m T_m\left(\frac{r}{a}\right) \quad (26)$$

where C_m are unknown coefficients to be determined. For $x < a$ the boundary condition (5b) results in a singular integral. By substituting Eq. (25), truncated with the first $M+1$ terms in Eq. (26), into Eq. (24) and using the integral formulas [23]

$$\frac{1}{\pi} \int_{-1}^1 \frac{T_m(r)}{(r-x)\sqrt{1-r^2}} dr = \begin{cases} U_{m-1}(x) & m \geq 1 \quad |x| < 1 \\ -\frac{\text{sgn}(x)}{\sqrt{x^2-1}} [x - \text{sgn}(x)\sqrt{x^2-1}]^m & m \geq 0 \quad |x| > 1 \\ 0 & m=0 \quad |x| < 1 \end{cases} \quad (27)$$

we obtain

$$-\frac{G_0}{a} \sum_{m=2,4,\dots}^{2M} C_m U_{m-1}\left(\frac{x}{a}\right) + \sum_{m=0,2,4,\dots}^{2M} C_m V_m(x) = 0 \quad x < a \quad (28)$$

where U_{m-1} is a Chebyshev polynomial of the second kind and V_m is

$$V_m(x) = \frac{2}{\pi} \int_0^a \frac{\Omega(x, r) T_m(r/a)}{\sqrt{a^2 - r^2}} dr \quad (29)$$

In order to obtain a nonzero solution from Eq. (28), the as yet unused Eq. (5a) can be considered. It follows that

$$\int_0^a g(r) dr = Q_0 \quad (30)$$

With the substitution of Eqs. (25) and (26), Eq. (30) gives

$$C_0 = \frac{2}{\pi} Q_0 \quad (31)$$

Submitting Eq. (31) into Eq. (28), we obtain a system of algebraic equations for the remaining coefficients $C_2, C_4, \dots$,

$$\frac{G_0}{a} \sum_{m=2,4,\dots}^{2M} C_m U_{m-1} \left(\frac{x}{a} \right) - \sum_{m=2,4,\dots}^{2M} C_m V_m(x) = \frac{2}{\pi} Q_0 V_0(x) \quad x < a \quad (32)$$

The algebraic equation (32) for C_m can be solved numerically by standard computer procedures (e.g., the collocation technique). Once C_m are determined, the function $g(r)$ can be found from Eqs. (25) and (26). Then the functions F_m can be determined from Eq. (18). To this end, the full field solution is obtained.

From Eqs. (24) and (27), we can obtain the horizontal electric field out of the electrode region at the $y=0$ plane:

$$E_x(x, 0) = G_0 \frac{1}{\sqrt{x^2 - a^2}} \sum_{m=0,2,4,\dots}^{2M} C_m \left[\frac{x}{a} - \sqrt{(x/a)^2 - 1} \right]^m + \sum_{m=0,2,4,\dots}^{2M} C_m V_m(x) \quad (33)$$

The above equation is obtained by substituting Eqs. (25)–(27) into Eq. (24). Obviously, the horizontal electric field is singular ahead of the electrode tip. If we define the electric field intensity factor

$$K_E = \lim_{x \rightarrow a+0} \sqrt{2\pi(x-a)} E_x(x, 0) \quad (34)$$

then

$$K_E = G_0 \sqrt{\pi/a} \sum_{m=0,2,4,\dots}^{2M} C_m \quad (35)$$

or

$$K_E = \frac{2}{\sqrt{\pi a}} G_0 Q_0 + \sqrt{\frac{\pi}{a}} G_0 \sum_{m=2,4,\dots}^{2M} C_m \quad (36)$$

Having obtained the auxiliary function $g(r)$, the coefficient $F_m(s)$ can be determined from Eq. (18). As a result, the full field solution is determined. For the infinite medium, the problem can be solved in closed form as given in Appendix B. The electrode tip field has been expressed in terms of the electric field intensity factors by Eqs. (B8) and (B9). Since the piezoelectric layer boundary $y=h$ will not affect the qualitative character of the state of affairs near the electrode border, the local stresses, electric displacements, and electric fields for $r \rightarrow 0$ (see Fig. 2) should be the same as those given by Eqs. (B8) and (B9) for an infinite piezoelectric medium. The electric field intensity factor, however, should now read as Eq. (35).

4 Two Collinear Electrode

In formulating the problem, the condition of symmetry with respect to $x=0$ was assumed with regard to the electrode geometry. Thus, the electric field equation (24) derived in Sec. 3 is valid basically for any number of electrodes in the $y=0$ plane symmetri-

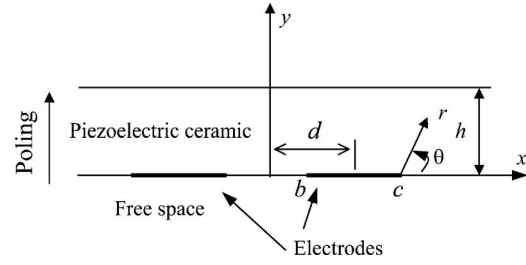


Fig. 3 Two symmetric surface electrodes; electrode length $2a=c-b$; electrode center of the right electrode is at $d=(c+b)/2$

cally located with respect to $x=0$. That is, the electrodes are defined by $(y=0, b_j < x < c_j)$ and $(y=0, -c_j < x < -b_j)$, in which $j = 1, \dots, n$, with the following additional condition:

$$\int_{-c_j}^{-b_j} [D_y(x, 0) - D_y^a(x, 0)] dx = \int_{b_j}^{c_j} [D_y(x, 0) - D_y^a(x, 0)] dx = 2Q_{j0} \quad (37)$$

As an example, we consider the case of two symmetrically located collinear electrodes (Fig. 3). That is, we assume that $b_1 = b, c_1 = c, b_2 = -c, c_2 = -b, b < x < c$. Each electrode is charged with $2Q_0$. In this case, using the symmetry conditions, Eq. (24) should be modified as

$$E_x(x, 0) = \frac{G_0}{\pi} \int_b^c \left(\frac{1}{x+r} + \frac{1}{x-r} \right) g(r) dr + \frac{2}{\pi} \int_b^c \Omega(x, r) g(r) dr \quad (38)$$

Note that in this case, the auxiliary function $g(r)$ is defined in the interval $[b, c]$. Equation (38) can be normalized to

$$E_x(x, 0) = \frac{G_0}{\pi} \int_{-1}^1 \left(\frac{1}{\bar{x} - \bar{r}} \right) g(\bar{r}) d\bar{r} + \frac{2a}{\pi} \int_{-1}^1 \Omega_1(x, r) g(\bar{r}) d\bar{r} \quad (39)$$

where

$$x = a\bar{x} + (c+b)/2, \quad r = a\bar{r} + (c+b)/2, \quad a = (c-b)/2 \quad (40)$$

$$\Omega_1(x, r) = \frac{G_0}{2} \frac{1}{x+r} + \Omega(x, r) \quad (41)$$

In this case, the solution of $g(r)$ can be expressed as

$$g(r) = \frac{1}{a\sqrt{1-\bar{r}^2}} \sum_{m=0,1,2,\dots}^{\infty} C_m T_m(\bar{r}) \quad (42)$$

By substituting Eq. (42), truncated with the first $M+1$ terms, into Eq. (39) and using the integral formulas in Eq. (27), we obtain

$$-\frac{G_0}{a} \sum_{m=1,2,\dots}^M C_m U_{m-1}(\bar{x}) + \sum_{m=0,1,2,\dots}^M C_m V_m(x) = 0 \quad x < a \quad (43)$$

where U_{m-1} is a Chebyshev polynomial of the second kind and V_m is

$$V_m(x) = \frac{2}{\pi} \int_{-1}^1 \frac{\Omega_1(x, r) T_m(\bar{r})}{\sqrt{1-\bar{r}^2}} d\bar{r} \quad (44)$$

The fact that the electrode is charged with $2Q_0$ can be written as

Table 1 Elastic constants (10^{10} N/m²), piezoelectric constants (C/m²), and dielectric permittivities (10^{-10} C/V m)

Materials	Elastic constants (10^{10} N/m ²)					Piezoelectric constants (C/m ²)			Dielectric permittivities (10^{-10} C/V m)	
	c_{11}	c_{12}	c_{13}	c_{44}	c_{33}	e_{31}	e_{33}	e_{15}	ϵ_{11}	ϵ_{33}
PZT-4	13.9	7.78	7.43	2.56	11.3	-6.980	13.84	13.44	60.0	54.7

$$\int_b^c g(r)dr = 2Q_0 \quad (45)$$

With the substitution of Eq. (42), Eq. (45) gives

$$C_0 = \frac{2}{\pi} Q_0 \quad (46)$$

Substituting Eq. (46) into Eq. (43), we obtain a system of algebraic equations for the remaining coefficients $C_1, C_2, \dots$,

$$\frac{G_0}{a} \sum_{m=1,2,\dots}^M C_m U_{m-1}(\bar{x}) - \sum_{m=1,2,\dots}^M C_m V_m(x) = \frac{2}{\pi} Q_0 V_0(x) \quad b < x < c \quad (47)$$

The algebraic equations for C_m can be solved numerically in a similar way as Eq. (32). Once C_m are determined, the function $g(r)$ can be found from Eq. (42). From Eq. (39), one can obtain the horizontal electric field out of the electrode region on the x axis:

$$E_x(x, 0) = G_0 \frac{\text{sgn}(\bar{x})}{a\sqrt{\bar{x}^2 - 1}} \sum_{m=0,1,2,\dots}^M C_m [\bar{x} - \text{sgn}(\bar{x})\sqrt{(\bar{x})^2 - 1}]^m + \sum_{m=0,1,2,\dots}^M C_m V_m(x) \quad (48)$$

The electric field intensity factor in this case is defined as

$$K_E(c) = \lim_{x \rightarrow c+0} \sqrt{2\pi(x-c)} E_x(x, 0) \quad \text{for electrode tip } x = c \quad (49a)$$

$$K_E(b) = - \lim_{x \rightarrow b-0} \sqrt{2\pi(b-x)} E_x(x, 0) \quad \text{for electrode tip } x = b \quad (49b)$$

The negative sign in Eq. (49b) is introduced so that the electric field intensity factor is positive for an electric field directing to the negative x axis at the left electrode tip. From Eqs. (48) and (49), we have

Table 2 Ratios of the material properties $C_{3j} + \epsilon_0 a_{3j}$ and G_0 considering the existence of air (vacuum) and ignoring air (vacuum)

$(C_{3j} + \epsilon_0 a_{3j}) / C_{3j}$	0.999583-0.000658i 0.999583+0.000658i 1.000326 1.000413+0.000658i 1.000413-0.000658i 0.999673
$\frac{G_0(\text{with air})}{G_0(\text{without air})}$	0.999226

$$K_E(b) = \frac{2}{\sqrt{\pi a}} G_0 Q_0 + \sqrt{\frac{\pi}{a}} G_0 \sum_{m=1,2,\dots}^M (-1)^m C_m \quad (50a)$$

$$K_E(c) = \frac{2}{\sqrt{\pi a}} G_0 Q_0 + \sqrt{\frac{\pi}{a}} G_0 \sum_{m=1,2,\dots}^M C_m \quad (50b)$$

5 Numerical Example

Numerical results have been obtained for PZT-4 piezoelectric ceramics. The material constants are given in Table 1 [24]. The poling axis of the material is aligned in the positive y direction. In Sec. 5.1 and 5.2, the problems of single electrode and two collinear electrodes will be considered separately.

5.1 Single Electrode. From the formulation of the problem, it is clear that air (or vacuum) surrounding the $y=0$ plane influences the solution of the response of the piezoelectric medium only through the quantities G_0 and $C_{3j} + \epsilon_0 a_{3j}$ ($j=1, \dots, 6$). These quantities change the solution of the auxiliary function $g(x)$ through Eq. (24). They further change the solution of the expressions $F_m(s)$ through Eq. (17). Hence, to understand the effect of air (or vacuum), it suffices to study the values of G_0 and $C_{3j} + \epsilon_0 a_{3j}$ for two cases: (1) taking into account the air (or vacuum) and (2) ignoring the air (or vacuum). Table 2 lists the ratios of (1) to (2). It can be seen that these ratios are almost equal to 1 (relative error less than 0.1%). This means that air (vacuum) has little effect on the response of its surrounding piezoelectric ceramics.

The vertical component of the electric displacement vector on the electrode plane immediately above the electrode (that is, $D_y(x, 0)$) is plotted in Fig. 4 for selected values of layer thickness. Note that $D_y(x, 0)$ is identical to zero for $x > a$ if the effect of air (vacuum) is ignored. However, even if air (vacuum) is considered, $D_y(x, 0)$ is still almost equal to zero for $x > a$. It can be seen that on the electrode plane behind the electrode tip (that is, $x < a$), $D_y(x, 0)$ increases quickly as x approaches a and exhibits singularity when $x \rightarrow a$. This means that charges are highly accumulated at the electrode tip. Further, as the layer thickness decreases, the distribution of D_y along the electrode plane becomes more uniform, resulting in less severity of the electric charge concentration

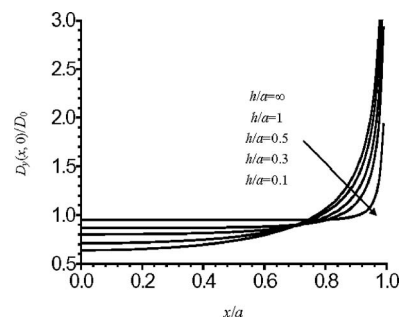


Fig. 4 Electric displacement $D_y(x, 0)$ on the electrode plane above the electrode, D_y is zero for $x > a$; $D_0 = Q_0/a$

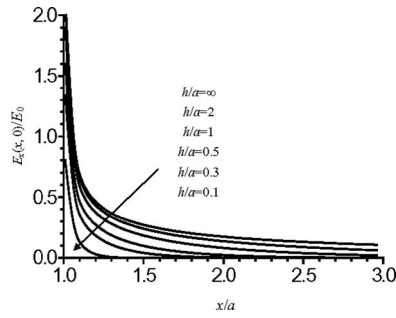


Fig. 5 Electric field $E_x(x,0)$ on the electrode plane ahead of the electrode; E_x is zero for $x < a$; $E_0 = Q_0/(a\epsilon_{33})$

at the electrode tip.

Some results for the horizontal electric field on the electrode plane are displayed in Fig. 5 for selected values of layer thickness. The electric field is zero on the electrode plane behind the electrode tip (that is $x < a$). For any given electrode spacing, the electric field is a continuous function of x ahead of the electrode. As expected, $E_x(x,0)$ becomes infinity ahead of the electrode tip. $E_x(x,0)$ decreases monotonously with increasing x and becomes zero when x is very far away from the electrode tip. It can be seen that the electric field for the finite medium is always smaller than that for the infinite medium. Decrease of the layer thickness can significantly reduce the field quantities near the electrode tip.

Figure 5 has clearly shown that the electric field is infinite at the electrode tip. This behavior can be described by the electric field intensity factor, defined by Eq. (34). The electric field intensity factor K_E as a function of layer thickness is plotted in Fig. 6. The results are normalized with $2G_0Q_0/\sqrt{\pi a}$, which is the electric field intensity factor for infinite layer thickness. It is evident that K_E decreases rapidly with decreasing layer thickness. Therefore, from the standpoint of fracture mechanics, thin electrode-piezoelectric layer is more reliable than thick electrode-piezoelectric layer.

In fracture evaluation of materials, the “cleavage” stress $\sigma_{\theta\theta}$ is important since it can be used to assess possible crack initiation. Given the stress components σ_{xx} , σ_{xy} , and σ_{yy} near the electrode edge, the cleavage stress $\sigma_{\theta\theta}$ is expressed as

$$\sigma_{\theta\theta} = \sigma_{yy} \cos^2 \theta - 2\sigma_{xy} \cos \theta \sin \theta + \sigma_{xx} \sin^2 \theta \quad (51)$$

Figure 7 displays the variation of $\sigma_{\theta\theta}$ with angle θ near the electrode tip. The maximum value of $\sigma_{\theta\theta}$ does not occur perpendicular to the electrode plane (the plane $\theta=90$ deg is perpendicular to the electrode plane). Obviously, the stress at the electrode tip decreases significantly with decreasing layer thickness. The infinite medium solution reported in the literature is the limited case. In actual structures, the stress should be less than that for the infinite medium solution. Once again, Fig. 7 supports the fact that reduc-

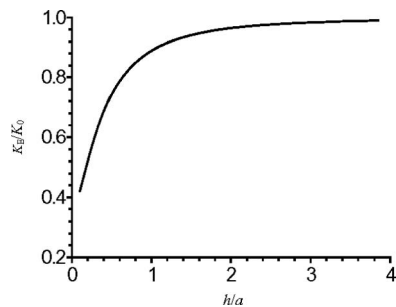


Fig. 6 Electric field intensity factor at the electrode tip as a function of layer thickness; $K_0 = 2G_0Q_0/\sqrt{\pi a}$ is the intensity factor for infinite layer thickness

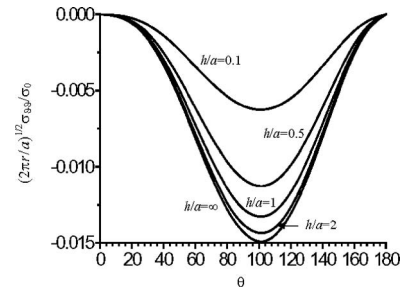


Fig. 7 Distributions of cleavage stress $\sigma_{\theta\theta}$ with angle; $\sigma_0 = e_{33}Q_0/\epsilon_{33}a$

tion in layer thickness if favorable to the reliability of the structure. Structural reliability design based on infinite layer thickness is too conservative.

5.2 Two Collinear Electrodes. For the collinear electrode problem of Fig. 3, the main concern is the effect of the electrode spacing. In Fig. 8, the normal component of the electric displacement vector immediately above the electrode is plotted for selected values of electrode spacing $b/a = 2, 1, 0.5, 0.2$, and 0.02 . It can be seen that D_y near the inner tip ($x=b$) is always smaller than that near the outer tip. This means that the electric charges are more accumulated at the outer electrode tip. As a result, the electric field intensity factor at the outer tip of the electrode should be higher than that at the inner tip. This is confirmed by the results of Fig. 9 where the electric field intensity factors are plotted as a function of electrode spacing for selected values of layer thickness. When the two electrodes are very close, there are considerable differences between the results for the outer tip and the inner tip. When the two electrodes are almost contacted (i.e., $b \rightarrow 0$), the electric field intensity factor for the inner tips approaches zero and the electric field intensity factor for the outer tips becomes the

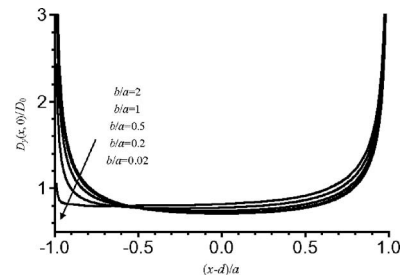


Fig. 8 Electric displacement $D_y(x,0)$ above the right electrode for the collinear electrode problem shown in Fig. 3. Out of the electrode region on the $y=0$ plane, D_y is zero. $D_0 = Q_0/a$ and $h = a$.

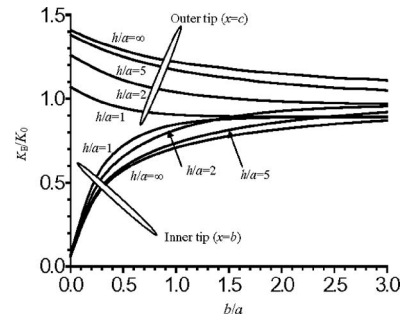


Fig. 9 Electric field intensity factor at the electrode tips as functions of the layer thickness for the collinear electrodes shown in Fig. 3; $K_0 = 2G_0Q_0/\sqrt{\pi a}$

value for a single electrode of length $4a$. On the other hand, if the two electrodes are sufficiently separated from each other (i.e., $b \gg a$), the inner tip and the outer tip of the electrode have the same electric field intensity factor, which is equal to the value of a single electrode of length $2a$. As an example, we check the results for infinite layer thickness. As b approaches zero, the electric field intensity factor for the inner tip becomes zero and that for the outer tip becomes the known value $K_E(c)/K_0 = \sqrt{2}$. As b approaches infinity, the electric field intensity factors for the inner and the outer tips have the same value of $K_E(b)/K_0 = K_E(c)/K_0 = 1$. The curves shown in Fig. 9 support these facts.

6 Conclusion

The linear electroelastic problem for a piezoelectric ceramic layer with a surface electrode and two collinear surface electrodes are studied. The stresses, electric fields, and electric displacements near the electrode tip are obtained. The electrode tip singularity is characterized by the electric field intensity factor. The analysis shows that the stresses and electric field intensities decrease with the decreasing layer thickness. On the electrode plane, the vertical electric displacement is singular behind the electrode tip and vanishes ahead of the electrode front. Conversely, the horizontal electric field on the electrode plane is singular ahead of the electrode tip but becomes zero behind the electrode tip. The relative distance between two collinear electrodes has a significant influence on the electrode tip field. The decreasing distance weakens the quantities near the inner tips of the electrodes but strengthens the quantities near the outer tips of the electrodes. It is also found that the effect of air (vacuum) surrounding the piezoelectric layer on the electromechanical fields inside the piezoelectric layer is negligible.

Acknowledgment

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Appendix A: Material Constants

The coefficients C_{im} and D_{im} ($i=1,2,3$; $m=1,\dots,6$) are

$$\begin{Bmatrix} C_{1m} \\ C_{2m} \\ C_{3m} \\ D_{1m} \\ D_{2m} \end{Bmatrix} = \begin{Bmatrix} c_{11} \\ c_{13} \\ e_{31} \\ c_{44}\lambda_m \\ e_{15}\lambda_m \end{Bmatrix} a_{1m} + \begin{Bmatrix} c_{13}\lambda_m \\ c_{33}\lambda_m \\ e_{33}\lambda_m \\ -c_{44} \\ -e_{15} \end{Bmatrix} a_{2m} + \begin{Bmatrix} e_{31}\lambda_m \\ e_{33}\lambda_m \\ -e_{33}\lambda_m \\ -e_{15} \\ e_{11} \end{Bmatrix} a_{3m} \quad (A1)$$

These coefficients have the relationships

$$C_{i(3+j)} = -C_{ij} \quad D_{i(3+j)} = D_{ij} \quad (A2)$$

in which $i, j=1,2,3$.

Appendix B: Electrode Tip Field

In order to obtain the quantities very near the electrode tip, it suffices to consider the infinite medium, that is, $h \rightarrow \infty$. In this situation, the functions B_m obtained from Eq. (19) vanish for $m=4,5,6$ and become constants B_{m0} for $m=1,2,3$. Consequently, $G(s)$ defined in Eq. (21) becomes a constant G_0 , which is $G_0 = \sum_{m=1}^3 a_{3m} B_{m0}$. As a result, the integral kernel $\Omega(x,r)=0$. It follows from Eq. (32) that $C_m=0$ for $m>0$. Thus,

$$g(r) = \frac{2}{\pi} Q_0 / \sqrt{a^2 - r^2} \quad (B1)$$

Substituting Eq. (B1) into Eq. (18), we obtain

$$F_m(s) = s^{-1} B_{m0} Q_0 J_0(sa) \quad (B2)$$

where J_0 is the zero-order Bessel function of the first kind. Substituting Eq. (B2) into Eq. (11) and using the known results

$$\int_0^\infty \exp(s\lambda_m y) J_0(sa) \cos(sx) ds = \frac{\sqrt{l_2^2 - x^2}}{l_2^2 - l_1^2} \quad (B3a)$$

$$\int_0^\infty \exp(s\lambda_m y) J_0(sa) \sin(sx) ds = \frac{\sqrt{x^2 - l_1^2}}{l_2^2 - l_1^2} \quad (B3b)$$

in which $\text{Re}(\lambda_m y) < 0$ and

$$l_1 = \frac{1}{2} [\sqrt{(x+a)^2 + \lambda_m^2 y^2} - \sqrt{(x-a)^2 + \lambda_m^2 y^2}] \quad (B4a)$$

$$l_2 = \frac{1}{2} [\sqrt{(x+a)^2 + \lambda_m^2 y^2} + \sqrt{(x-a)^2 + \lambda_m^2 y^2}] \quad (B4b)$$

we obtain the full expressions for the stress and electric displacement field inside the medium,

$$\begin{Bmatrix} \sigma_{xx}(x,y) \\ \sigma_{yy}(x,y) \\ D_y(x,y) \end{Bmatrix} = \frac{2}{\pi} Q_0 \sum_{m=1}^3 B_{m0} \begin{Bmatrix} C_{1m} \\ C_{2m} \\ C_{3m} \end{Bmatrix} \frac{\sqrt{l_2^2 - x^2}}{l_2^2 - l_1^2} \quad (B5a)$$

$$\begin{Bmatrix} \sigma_{xy}(x,y) \\ D_x(x,y) \end{Bmatrix} = \frac{2}{\pi} Q_0 \sum_{m=1}^3 B_{m0} \begin{Bmatrix} D_{1m} \\ D_{2m} \end{Bmatrix} \frac{\sqrt{x^2 - l_1^2}}{l_2^2 - l_1^2} \quad (B5b)$$

In addition, the electric field is obtained as follows:

$$\begin{aligned} E_x(x,y) &= \frac{2}{\pi} Q_0 \sum_{m=1}^3 B_{m0} a_{3m} \frac{\sqrt{x^2 - l_1^2}}{l_2^2 - l_1^2} \quad E_y(x,y) \\ &= -\frac{2}{\pi} Q_0 \sum_{m=1}^3 \lambda_m B_{m0} a_{3m} \frac{\sqrt{l_2^2 - x^2}}{l_2^2 - l_1^2} \end{aligned} \quad (B6)$$

The physical quantities near the electrode front are particularly interesting. We introduce the polar coordinate system (r, θ) in Fig. 2 in the following manner:

$$x = a + r \cos \theta, \quad y = r \sin \theta \quad (B7)$$

Substituting this into Eqs. (B5) and (B6), the quantities for a very small value of r are obtained as

$$\begin{Bmatrix} \sigma_{xx}(r, \theta) \\ \sigma_{yy}(r, \theta) \\ D_y(r, \theta) \end{Bmatrix} = \frac{K_E}{\sqrt{\pi r G_0}} \sum_{m=1}^3 B_{m0} \begin{Bmatrix} C_{1m} \\ C_{2m} \\ C_{3m} \end{Bmatrix} \frac{\sqrt{\cos^2 \theta + \lambda_m^2 \sin^2 \theta - \cos \theta}}{2\sqrt{(\cos^2 \theta + \lambda_m^2 \sin^2 \theta)}} \quad (B8a)$$

$$\begin{Bmatrix} \sigma_{xy}(r, \theta) \\ D_x(r, \theta) \end{Bmatrix} = \frac{K_E}{\sqrt{\pi r G_0}} \sum_{m=1}^3 B_{m0} \begin{Bmatrix} D_{1m} \\ D_{2m} \end{Bmatrix} \frac{\sqrt{\cos^2 \theta + \lambda_m^2 \sin^2 \theta + \cos \theta}}{2\sqrt{(\cos^2 \theta + \lambda_m^2 \sin^2 \theta)}} \quad (B8b)$$

and

$$E_x(r, \theta) = \frac{K_E}{\sqrt{\pi r}} \frac{\sqrt{\cos^2 \theta + \lambda_m^2 \sin^2 \theta + \cos \theta}}{2\sqrt{(\cos^2 \theta + \lambda_m^2 \sin^2 \theta)}} \quad (B9a)$$

$$E_y(r, \theta) = -\frac{K_E}{\sqrt{\pi r} G_0} \sum_{m=1}^3 \lambda_m B_{m0} a_{3m} \frac{\sqrt{\cos^2 \theta + \lambda_m^2 \sin^2 \theta} - \cos \theta}{2\sqrt{(\cos^2 \theta + \lambda_m^2 \sin^2 \theta)}} \quad (\text{B9b})$$

where

$$K_E = \lim_{r \rightarrow 0} \sqrt{2\pi r} E_x(r, 0) = \frac{2}{\sqrt{\pi a}} G_0 Q_0 \quad (\text{B10})$$

is the electric field intensity factor for infinite piezoelectric medium. The singularity at the electrode tip is obvious. For example, the stresses σ_{xx} and σ_{yy} , the electric displacement D_y , and the electric field E_y are singular at the electrode plane behind the electrode tip (i.e., $\theta = \pi$) but are not singular at the electrode plane ahead of the electrode tip (i.e., $\theta = 0$). In contrast, the stress σ_{xy} , the electric displacement D_x , and the electric field E_x are not singular at the electrode plane behind the electrode tip (i.e., $\theta = \pi$) but singular at the electrode plane ahead of the electrode tip (i.e., $\theta = 0$).

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Multiscale Modeling of Adsorbed Molecules on Freestanding Microfabricated Structures

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This paper outlines a multiscale model to quantitatively describe the chemomechanical coupling between adsorbed molecules and thin elastic films. The goal is to provide clear, quantitative connections between molecular interactions, adsorption distribution, and surface stress, which can be integrated with conventional thin film mechanics to quantify device performance in terms of molecular inputs. The decoupling of molecular and continuum frameworks enables a straightforward analysis of arbitrary structures and deformation modes, e.g., buckling and plate/membrane behavior. Moreover, it enables one to simultaneously identify both chemical properties (e.g., binding energy and grafting density) and mechanical properties (e.g., modulus and film geometry) that result in chemically responsive devices. We present the governing equations for scenarios where interactions between adsorbed molecules can be described in terms of pair interactions. These are used to quantify the mechanical driving forces that can be generated from adsorption of double-stranded DNA and C₆₀ (fullerenes). The utility of the framework is illustrated by quantifying the performance of adsorption-driven cantilevers and clamped structures that experience buckling. We demonstrate that the use of surface-grafted polyelectrolytes (such as DNA) and ultracompliant elastomer structures is particularly attractive since deformation can be tuned over a very wide range by varying grafting density and chemical environment. The predictions illustrate that it is possible to construct (1) adsorption-based tools to quantify molecular properties such as polymer chain flexibility and (2) chemically activated structures to control flow in microfluidic devices.

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Keywords: adsorption, pair potentials, chemical actuation, microfabrication

1 Introduction

Adsorption of molecules onto freestanding microfabricated cantilevers is an emerging technique for chemical sensing, in particular, with regard to biomolecules such as DNA, antibodies, and biotinylated species (e.g., see reviews [1–4]). The central concept is to induce mechanical deformation by molecular interactions on the surface, as illustrated in Fig. 1. Using highly compliant freestanding structures as substrates, ultrasensitive detectors can be obtained. Depending on the chosen surface chemistry, they can either be broadly responsive, or highly chemically selective, indicating the presence of an analyte by mechanical deflection. For example, if single strands of DNA (ss-DNA) are end adsorbed onto the surface of a compliant structure, the system becomes selective for complementary strands in solution. Hybridization leads to a conformation change (i.e., randomly coiled ss-DNA to helical double-stranded DNA (ds-DNA)), which changes the interaction of adsorbed molecules and leads to a measurable deformation of the cantilever. The degree of chemomechanical coupling is naturally quite sensitive to grafting density and the details of the molecular interactions, such as hydration repulsion, screened electrostatic repulsion, and steric fluctuations. [1].

While much of the underlying physics has been explored, the translation of molecular surface interactions to a generally applicable continuum framework has yet to be established. A complete modeling of the phenomenon poses a multiscale challenge: While

the interactions among adsorbed groups and the adsorption process demand a molecular description, the deformation of the elastic substrate is most effectively treated in terms of continuum mechanics. This paper outlines the general relationships between molecular interactions and structural deformation in such a way that it decouples the chemical and mechanical frameworks needed to quantify behavior. The approach provides a clear path to translating molecular interaction parameters into quantities that are compatible with conventional continuum mechanics models to predict general device performance.

As such, the framework presented here subsumes previous work limited to cantilever geometry (e.g., Refs. [5,6]) and exploits the generality of the modeling approach to quantify other types of adsorption and device geometry. For example, if the chemomechanical coupling observed for sensing applications can be enhanced, one can envision using molecular adsorption to create chemically activated fluidic control elements in micro-total-analysis-systems (μ -TAS). Figures 1(a) and 1(b) are a schematic illustration of a chemically activated microfluidic check valve, i.e., the use of a cantilever suspended at the junction of two microchannels. The design of such devices requires a complete and general quantitative framework to predict their performance, flexible enough to accommodate a broad range of molecular interaction types as well as any conceivable device geometry.

In order for chemically activated fluidic control elements of this type to be realized, the submicron displacements typical of silicon-based cantilevers must be enhanced considerably. This can be achieved by increasing the compliance of the cantilever via the use of either polymers [7] or nanoporous metals [8,9]. Moreover, specific sensing strategies that generate particularly large surface stresses can be developed, for example, by using self-assembled

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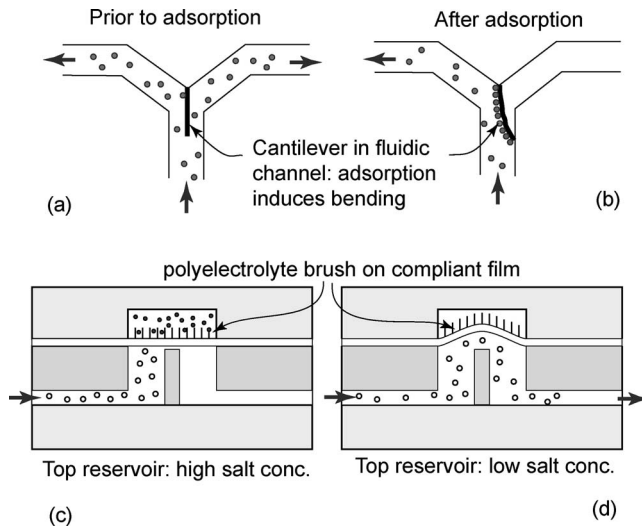


Fig. 1 Schematic illustrations of chemically activated structures to control fluid motion [(a) and (b)]. Top view of a cantilever suspended at a junction of two microfluidic channels. [(c) and (d)]. Side view of a conventional microfluidic valve spanning two chambers: The valve is functionalized with polyelectrolyte surface and actuated by controlling electrostatic screening in the top reservoir.

polymer brushes [10,11].¹ Figures 1(c) and 1(d) illustrate a possible implementation of both concepts; here, a clamped elastomer film is coated with a polyelectrolyte brush. At high salt concentrations surrounding the brush, electrostatic interactions are heavily screened, such that surface stress is low. As the salt concentration is reduced, the surface stresses rise and deform the membrane to allow fluid exchange between adjacent networks.

As will be detailed, the present framework allows one to establish the molecular parameters, material properties, and dimensions needed to achieve this functionality. Thus, one can simultaneously establish quantitative goals for *both* surface functionalization and microfabrication. The first half of this paper describes the molecular concepts and parameters needed to quantify the surface stress generated via molecular adsorption. We first demonstrate basic relationships between stresses generated via adsorption and binding energy, molecular interactions, grafting density, and molecular arrangement. The second part of the paper then utilizes those relationships to explore device performance, using well-established models of molecular interactions between double-stranded DNA, on the one hand, and buckminsterfullerene (C_{60}), on the other. We demonstrate that the use of microfabricated freestanding elastomeric structures opens new avenues to chemically activated microdevices, as well as new tools for molecular characterization.

2 Mechanics Framework

2.1 Molecular Description and Adsorption. The free energy change upon adsorption of a molecule from solution can be written as follows (Begley et al. [14]):

$$\Delta\mu = kT \ln \left(\frac{\sigma(1-c)}{c(1-\sigma)} \right) - \Delta\mu_B + \Delta\mu_I \quad (1)$$

where k is Boltzmann's constant, T is the temperature, c is the volume fraction of molecules in solution,² σ is the fraction of

¹For charged (i.e., polyelectrolyte) brushes, the surface stress in the adsorbed layer driving deformation increases with increasing molecular length, as will be illustrated later.

²For ds-DNA in water, the relationship between volume fraction and molarity is roughly $c \approx (0.64 \text{ l/mol}) N_{bp} M$, where N_{bp} is the number of base pairs and M is the number of moles of DNA per liter.

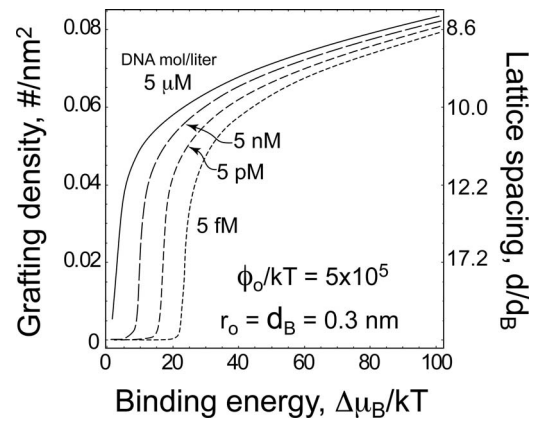


Fig. 2 Predictions of grafting density versus binding energy for nearest-neighbor repulsion using parameters appropriate for Debye screening of short DNA (40 bp). The molecular pair potential is given by $\phi(r) = \phi_o e^{-dr/r_o} / \sqrt{r/r_o}$.

occupied surface, $\Delta\mu_B$ is the binding energy of an isolated molecule, and $\Delta\mu_I$ is the energy that arises from the interaction between adsorbed species. In equilibrium, $\Delta\mu = 0$; thus, Eq. (1) can be used to estimate the grafting density as a function of solution concentration, binding free energy, and molecular interactions.

As an illustration, consider a scenario where (i) the binding sites and subsequent adsorption have a hexagonal arrangement and (ii) molecular interactions are governed by a short-range potential, such that considering only nearest-neighbor interactions is reasonable. For simplicity, the molecular pair potential is assumed to be $\phi(r) = \phi_o \exp(-r/r_o) / \sqrt{r/r_o}$, where r is the separation between molecules, and ϕ_o and r_o are the parameters controlling the amplitude and decay length of the potential. For the hexagonal geometry, the number density of potential binding sites is $\rho_b = 1/\sqrt{3}d_B^2$, where d_B is the distance between adjacent sites in the lattice. Similarly, the number density of adsorbed molecules is $\rho = 1/\sqrt{3}d^2$, where d is the distance between adsorbed molecules.

Considering only nearest neighbors on a hexagonal grid implies three pair interactions per molecule, such that the equilibrium surface coverage is governed by the solution to

$$\ln \left(\frac{d^2(1-c)}{c(d^2-d_B^2)} \right) - \left(\frac{\Delta\mu_B}{kT} \right) + 3 \left(\frac{\phi_o}{kT} \right) \exp(-d/r_o) = 0 \quad (2)$$

Figure 2 illustrates the resulting relationship between grafting density and binding energy for concentrations that span nine orders of magnitude. Here, we have chosen to plot grafting density (as opposed to effective spacing), as this quantity is often measured via fluorescence spectroscopy (e.g., Ref. [12]).

The chosen molecular interaction parameters, ϕ_o and r_o , are based on the empirical constants determined by Strey et al. [13] for nematically ordered ds-DNA in 1M monovalent salt solution. The interaction between DNA strands is well described by the sum of two repulsion terms of the above form, one representing hydration and the other electrostatic repulsion. In 1M salt solution, the length scales associated with these two terms are about equal (Strey et al. [13]). As will be discussed in Sec. 3, the parameters of Strey et al. must be multiplied by the length of the molecules: Here, we have chosen ds-DNA with 40 base pairs. The range of grafting densities shown in Fig. 2 is typical of DNA adsorption studies, i.e., 0.01–0.1 nm⁻² (e.g., Refs. [5,12]). For reference, the binding energy associated with Au–S bonds utilized in gold-thiol chemistry is ~ 160 kJ, or ~ 76 kT at room temperature. For the chosen parameters, the adsorption spacing is much larger than binding site spacing; this implies that the adsorption is not limited by the availability of binding sites.

The results in Fig. 2 predict that for sufficiently low solution

concentrations, there is a critical binding energy required to overcome entropic effects. For sufficiently large binding energies, dense surface coverage is achieved once a critical concentration of molecules in solution is reached. Section 3 provides a more detailed examination of adsorption density for interactions that involve screened electrostatic repulsion, hydration repulsion, and thermal bending fluctuations of the DNA strands. (See Fig. 7, which predicts grafting density for ds-DNA as a function of salt concentration in the solution.)

2.2 Translation to Continuum Variables: Surface Stress and Adsorbed Modulus. Begley et al. described how the interaction energy $\Delta\mu_l$ can be used to derive the surface stress induced by molecular adsorption [14]. The adsorbed molecules comprise a two-dimensional elastic solid with the constitutive relation,

$$\gamma_{ij} = (\gamma_0 + \lambda_0 \epsilon_{kk}^S) \delta_{ij} + 2(\gamma_0 + \lambda_0) \epsilon_{ij}^S \quad (3)$$

where ϵ_{ij}^S is the strain tensor describing deformation of the surface onto which the molecules are adsorbed. For a pairwise interaction between the adsorbed molecules, the isotropic surface stress contribution is given by

$$\gamma_0 = \frac{\pi \rho^2}{4} \int_0^\infty r^2 g(r) \phi'(r) dr \quad (4)$$

where ρ is the number density (molecules per unit area) of adsorbed groups, $g(r)$ is a pair correlation function that describes the molecular arrangement of particles,³ and $\phi(r)$ is a pair potential that describes the interaction between two molecules. The prime denotes differentiation with respect to r . The second constant relates to the effective modulus of the adsorbed groups and is given by

$$\lambda_0 = \frac{\pi \rho^2}{16} \int_0^\infty r^3 g(r) \left[\phi''(r) - \frac{\phi'(r)}{2r} \right] dr \quad (5)$$

For short-range potentials that decay sufficiently rapidly over lengths comparable to the characteristic spacing of the molecules, it is reasonable to assume that only nearest-neighbor interactions factor into the integrals in Eqs. (4) and (5). This limit can be treated by replacing the pair correlation function with a single delta peak at the characteristic spacing, d . In this approach, the “nearest-neighbor correlation function” $g_n(r)$ is defined according to

$$\int_0^\infty 2\pi \rho g_n(r) r dr = N \quad (6)$$

where N is the number of nearest neighbors (or coordination number). This yields the normalization,

$$g_n(r) = \frac{N}{2\pi \rho d} \delta(r - d) \quad (7)$$

Of course, there is a relationship between the surface density ρ and the nearest-neighbor spacing d . Generally, $\rho = 1/(\beta d^2)$; the proportionality factor β is of order unity and depends on the geometry of the lattice. The proportionality factor is found by considering the Voronoi polygon around each molecule, which is defined as the set of points in the plane that are closer to the molecule under consideration than any other. Table 1 gives the relevant parameters for lattices of different symmetries. A random arrangement may be treated by choosing a coordination number between (5) and (6), and β close to unity.

Thus, in the nearest-neighbor approach, the use of Eq. (7) with Eqs. (4) and (5) yields

Table 1 Nearest-neighbor correlation function parameters

Lattice	Coordination number N	Proportionality factor β
Trigonal	3	$4/3\sqrt{3}=0.77$
Cubic	4	1
Hexagonal	6	$\sqrt{3}/2=0.87$

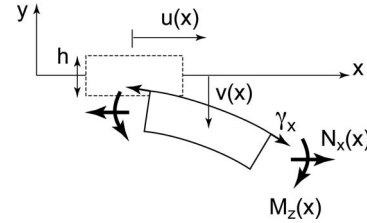


Fig. 3 Elementary beam models with continuum surface stress on the upper surface

$$\gamma_0 = \frac{N}{8\beta d} \phi'(d) \quad \lambda_0 = \frac{N}{64\beta d} [2d\phi''(d) - \phi'(d)] \quad (8)$$

2.3 Thin Film Mechanics. The properties of the adsorbed molecules derived above can be simply inserted into conventional continuum mechanics models of a freestanding film. For small deflections,⁴ the governing equations can be derived from elementary equilibrium models, such as those shown in Fig. 3. For plane strain deformation (i.e., $\epsilon_{zz}=0$), the surface stress is given by

$$\gamma_{xx} = \gamma_0 + (3\lambda_0 + 2\gamma_0) \epsilon_{xx}^S = \gamma_0 + \lambda' \epsilon_{xx}^S \quad (9)$$

where ϵ_{xx}^S is the strain on the surface of the film with adsorbed molecules and λ' is the effective modulus of the adsorbed groups. Note that λ' involves a contribution from the isotropic surface stress.

The strain in the film can be written generally as the superposition of stretch of the centerline, $\epsilon_0(x)$, and that arising from bending, $-\kappa(x)y$, where $\kappa(x)$ is the curvature of the film. In the following, we consider adsorption on a single side of a freestanding structure. Using Fig. 3 and Eq. (9), the net axial force resultant per unit width and acting at the centerline of the beam is given by

$$N_x(x) = (\bar{E}h + \lambda') \epsilon_0(x) - \lambda' \frac{h}{2} \kappa(x) + \gamma_0 \quad (10)$$

while the net moment per unit width acting at the centerline of the beam is given by

$$M_z(x) = -\frac{h}{2} \lambda' \epsilon_0(x) + \left(\frac{\bar{E}h^3}{12} + \frac{h^2}{4} \lambda' \right) \kappa(x) - \frac{h\gamma_0}{2} \quad (11)$$

These expressions can then be used with appropriate boundary conditions to relate deformation to adsorption parameters in combination with other external loads.

For example, consider the vertical deflection of a cantilever structure with adsorption (on the positive y face) that is loaded with a point load P at $x=L$, such as may be applied with an atomic force microscope (AFM). The net moment is $M_z(x)=P(L-x)$, while $N_x(x)=0$. Using these relationships, the vertical deflection at the tip of the cantilever is given by

³The pair correlation function is defined such that $2\pi \rho g(r) dr$ yields the number of particles in an annulus of width dr and radius r from a given particle.

⁴Begley et al. [14] illustrated this for cases of deformation involving large displacements, encompassing both plate (i.e., bending-dominated) and membrane (i.e., stretch-dominated) regimes.

$$\delta = 3 \frac{L^2}{h} \left(\frac{\gamma_0}{\bar{E}h + 4\lambda'} \right) + \frac{4PL^3}{\bar{E}bh^3} \left(\frac{\bar{E}h + \lambda'}{\bar{E}h + 4\lambda'} \right) \quad (12)$$

where b is the width of the cantilever. The first term is the static deflection due to adsorption in the absence of point loads,⁵ and the second term is the deflection due to the application of the tip load. It is noteworthy that the relative effects of the surface adsorption parameters involve different scalings with the beam thickness h . This has critical implications for device development and is discussed in subsequent sections.

As a second example, one can easily apply this framework to scenarios involving clamped films. For simplicity's sake, consider cases where $\lambda' = 0$, which is a good approximation for the short-range potentials presented later for DNA. For compressive surface stress, the deflection is identically zero below a critical value that leads to buckling. Solving the eigenvalue problem yields the critical surface stress to induce buckling (for single-sided adsorption),

$$\frac{-\gamma_0^c}{\bar{E}h} > \frac{\pi^2}{3} \left(\frac{h}{L} \right)^2 + \varepsilon_p \quad (13)$$

where ε_p represents a prestrain (prior to buckling), which may arise during fabrication. The negative sign on the left reflects that buckling is only possible with compressive (negative) surface stresses. Scenarios with $\varepsilon_p < 0$ represent a compressive fabrication stress that lowers the adsorption buckling threshold; $\varepsilon_p > 0$ implies residual tension, which lowers (or eliminates) the likelihood of buckling. The postbuckling displacements are given by

$$\delta = \frac{2L}{\pi} \sqrt{\frac{\gamma}{\gamma_0^c} - 1} \quad (14)$$

This expression is derived by integrating the axial strains produced by out-of-plane deflection, which must yield displacements consistent with imposed boundary conditions [15]. For double-sided adsorption, a factor of 2 should premultiply the surface stress in Eqs. (13) and (14).

3 Surface Stresses Generated by Adsorption: Illustrative Cases

3.1 Two-Dimensional Adsorption of Buckminsterfullerene (C_{60} Molecules). C_{60} represents an excellent opportunity to quantify adsorption-induced stresses and explore their impact on microdevice response for two reasons. First, the adsorption of C_{60} molecules onto metal and oxide surfaces is well established (e.g., Ref. [16]). Second, the pair potential governing molecular interaction has been extensively characterized (e.g., Refs. [17–19]). The pair interaction is well described by Lennard–Jones-type potentials that lead to both attractive and repulsive intermolecular forces, depending on the separation. Of the many forms of the C_{60} – C_{60} potential that have been set forth, we adopt the following as it is simple and captures the essential behavior:

$$\phi(r) = \phi_0 \left[\left(\frac{r_0}{r} \right)^{12} - 2 \left(\frac{r_0}{r} \right)^6 \right] \quad (15)$$

where $\phi_0 \approx 0.25$ eV $\approx 3. \times 10^{-20}$ J is the depth of the potential well and $r_0 \approx 1.04$ nm is the equilibrium separation for a pair of C_{60} molecules.

The nature of this potential is such that the surface properties will be very sensitive to the distribution of molecules on the surface, particularly for adsorption spacing near the pair equilibrium spacing r_0 . This is because small changes in spacing near the potential minimum can dramatically alter the contribution of a given pair to the overall surface stress. As such, a meaningful prediction of surface stresses requires an accurate pair correlation

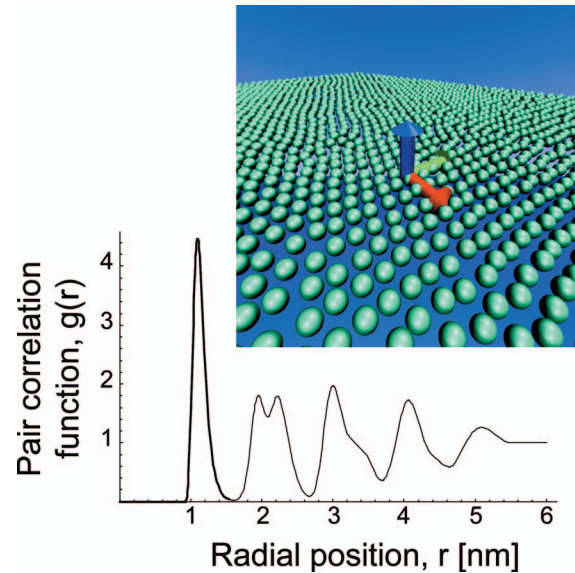


Fig. 4 Simulation snapshot (top) and pair correlation function (bottom) determined from MC simulations of C_{60} adsorption. Only the details of the first peak factor into surface stresses.

function to describe the arrangement of molecules on the surface.

In order to obtain a realistic pair correlation function, we have conducted Monte Carlo (MC) simulations (Fig. 4) of the molecular distribution on a 2D surface using the above potential. The simulations depart from an initially perfect 2D hexagonal structure of 4096 particles in a periodic box with an initial edge length of 66.56 nm. The particle positions were then equilibrated for 5×10^4 MC cycles at a temperature of $T = 700$ K and at a constant line pressure of 0.1 N/m. An isothermal-isobaric ensemble was simulated; i.e., the simulation box was allowed to fluctuate in size during the simulation. In the present case, no substrate-molecular interactions are included—i.e., it is assumed that the binding energy and availability of binding sites do not influence the spatial arrangement.

The distribution of particles and the resultant pair correlation function are shown in Fig. 3. The pair correlation exhibits peaks at roughly the spacing of the pair equilibrium position; however, it is important to emphasize that the minute details of the distribution near the first peak play a critical role in dictating the surface stress due to the sharpness of the potential. The surface stresses can be obtained either directly from the MC simulation or by using the results in Fig. 3 in conjunction with Eq. (4). In order to explore the magnitude of the surface properties as a function of characteristic spacing, the pair correlation function can be scaled radially. Figure 5 shows the outcome of this scaling procedure, illustrating

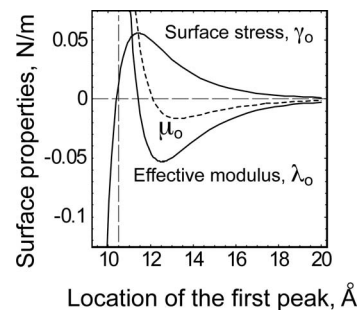


Fig. 5 Surface stresses generated by adsorption of C_{60} fullerenes as a function of characteristic separation in a hexagonal array

⁵This is identical to results published elsewhere, except for the modification due to the effective modulus of the adsorbed groups, λ' .

the surface properties as a function of the location of the first peak in the pair correlation function.

It is noteworthy that significant surface stresses can be generated by C_{60} adsorption, both in the tensile and compressive regimes—provided the effective adsorption spacing can be controlled to within 1 nm or 2 nm. Moreover, the stiffness of the adsorbed layer can be comparable to the in-plane stiffness of microdevices, as discussed in Sec. 4.

3.2 Semiflexible Polyelectrolyte Brushes: DNA. ds-DNA is a stiff polyelectrolyte. In concentrated solution, it forms a nematically ordered liquid crystalline phase, and similarly ordered arrangements are expected to arise from end-grafting ds-DNA onto surfaces at sufficiently high density. Measuring the interchain spacing in nematically ordered ds-DNA solutions as a function of the osmotic pressure, Strey et al. derived and experimentally validated an effective pair interaction potential between ds-DNA molecules [13]. In the following, we demonstrate the use of this potential to describe chemomechanical coupling. Naturally, with a suitable adjustment of parameters, the same potential is also applicable to other semiflexible charged polyelectrolytes.

For long semiflexible polymers subject to electrostatic interactions and hydration forces, the pair potential in a general form is written as

$$\phi_o(r) = \phi_D \frac{\exp(-r/r_D)}{\sqrt{r/r_D}} + \phi_H \frac{\exp(-r/r_H)}{\sqrt{r/r_H}} \quad (16)$$

where r is the separation between two molecules. r_D is the Debye screening length, which controls the scale over which electrostatic repulsion is relevant,

$$r_D = \frac{1}{\sqrt{4\pi l_B \sum_i \rho_c^i z_i}} \quad (17)$$

where $l_B \approx 0.7$ nm is the Bjerrum length (for water at room temperature), and ρ_c^i and z_i are the number density and valence of ionic species in the solution surrounding the polyelectrolytes. For monovalent salt solutions in water, $r_D = 0.308$ nm / $\sqrt{M/(mol/l)}$. r_H is the length that controls the scale of hydration repulsion; for charged DNA, $r_H \approx 0.3$ nm.

The two amplitudes of the potential, ϕ_D and ϕ_H , are empirical constants (with units of energy). It should be noted that Strey et al. derived the interaction potential on a per length basis, as it was motivated by osmotic pressure measurements of microscale liquid crystals [13]. Here, we have chosen to recast the potential in terms of total interaction energy, such that the molecular length of the adsorbed groups remains as an explicit parameter. As such, one must multiply the Strey et al. coefficients by molecular length to obtain ϕ_D and ϕ_H .

For long molecules, conformational fluctuations along their length result in additional repulsion. This can be accounted for by modifying the bare interaction potential according to Strey et al. [13],

$$\phi(r) = \phi_0 + \alpha \frac{(kTL)^{3/4}}{(l_p)^{1/4}} \left(\frac{\partial^2 \phi_{D,H}}{\partial r^2} - \frac{1}{r} \frac{\partial \phi_{D,H}}{\partial r} \right)^{1/4} \quad (18)$$

where α is a dimensionless constant of order unity. Note that the bracketed fluctuation term involves the bare interaction parameters ϕ_D , ϕ_H , r_D , and r_H . To understand when the fluctuation term is important, one may consider a large separation ($> \sim 1$ nm), such that screened electrostatic repulsion dominates hydration repulsion. In this regime, the interaction potential can be written as

$$\phi(r) \approx \phi_D \frac{\exp(-r/r_D)}{\sqrt{r/r_D}} \left[1 + \beta \left(\frac{kT}{\phi_D} \right)^{3/4} \left(\frac{L^3}{l_p r_D^2} \right)^{1/4} f(r/r_D) \right] \quad (19)$$

where

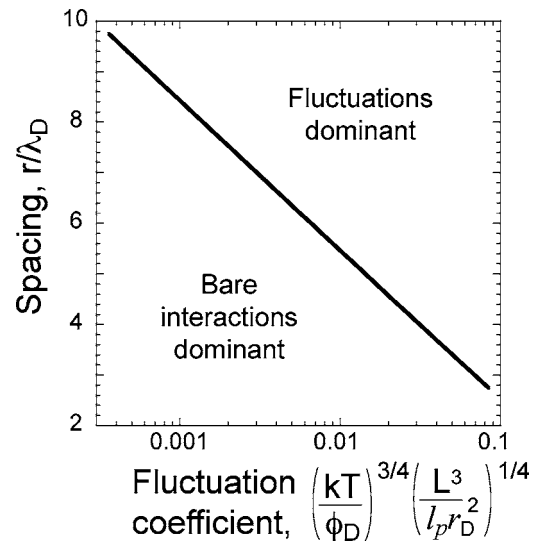


Fig. 6 Critical adsorption spacing for a hexagonal array of semiflexible polymers (e.g., DNA), where bare interactions and thermal fluctuations are equivalent.

$$f(r/r_D = \hat{r}) = \left(\frac{[5 + 8\hat{r} + 4\hat{r}^2](\exp \hat{r})^3}{\hat{r}} \right)^{1/4} \quad (20)$$

Written in this manner, the second term in brackets in Eq. (19) reflects the *relative* size of the fluctuation contribution to the bare interaction. Retaining the molecule length in the potential formulation clearly illustrates that the fluctuation contribution depends on the length of the molecule relative to its persistence length and the Debye length.

The increasing function of spacing given by Eq. (20) indicates that fluctuation terms quickly dominate at large spacing; a rough estimate for the crossover between bare interactions and fluctuation interactions can be determined by setting $\alpha=1$ and equating the bracketed terms in Eq. (19) Figure 6 shows the critical spacing at which fluctuation terms become larger than the bare interaction, as a function of the controlling dimensionless parameter. For large spacing, the interaction potential asymptotes to

$$\phi_F(r) \approx \phi_D \left(\frac{kT}{\phi_D} \right)^{3/4} \left(\frac{L^3}{l_p r_D^2} \right)^{1/4} \frac{\exp(-r/4r_D)}{\sqrt{r/4r_D}} \quad (21)$$

This form clearly indicates the competition between molecular length, persistence length, and Debye length, as well as the relative importance of electrostatic energy and thermal energy. Moreover, it illustrates the point made by Strey et al., i.e., that the additional term due to conformational fluctuations quadruples the scale over which interactions occur [13].

Figure 7 shows the implications of using the pair potential of Strey et al. to describe the role of molecular interactions in the adsorption energy balance (Eq. (1)). Here, the full potential of Strey et al. (including electrostatic, hydration, and fluctuation terms) is used with Eq. (1) to predict the grafting density of ds-DNA as a function of salt concentration. A hexagonal arrangement is assumed. The chosen binding energy is appropriate for Au–S bonds. The relevant potential parameters are listed in Table 2. These parameters are approximately equal to those identified by Strey et al. [13] multiplied by the molecular length, with variations on ϕ_D to illustrate the relative effect of Debye screening. For salinities greater than unity, the hydration repulsion term dominates and dictates the maximum possible grafting density.

The data from the adsorption measurements of Castellino et al. on 40 bp of ss-DNA [12] are superimposed on the predictions of Fig. 7. The comparison is not meant to be strictly quantitative since the use of a semiflexible polymer pair potential is clearly

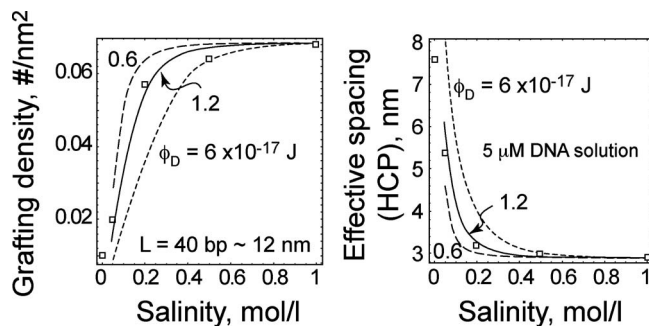


Fig. 7 Grafting density and hcp lattice spacing as a function of salt concentration under which adsorption occurs using the full potential for DNA developed by Strey et al. [13].

objectionable for ss-DNA. Nevertheless, it is clear that a pair potential that involves two different interaction length scales is capable of capturing the trend in grafting density with salinity—using molecular interaction parameters that are in agreement with existing measurements. It is likely that any strongly repulsive potential with similar interaction length scales and amplitudes will reproduce the behavior shown in Fig. 7.

The surface stresses generated by ds-DNA adsorption are shown in Fig. 8 as a function of adsorption spacing and salinity. The molecular parameters used in the prediction are given in Table 3. Here, we have chosen to plot effective adsorption spacing (as opposed to grafting density) as one can clearly observe the transition from hydration-dominated to fluctuation-dominated surface stresses. For small spacing, the surface stress scales linearly with the number of base pairs because the bare interaction amplitudes ϕ_H and ϕ_D scale with the molecular length. For larger spacing, the surface stress scales as $L^{7/4}$ due to the interplay between bare interactions and fluctuation terms.

The results in Fig. 8 are in agreement with the microcantilever experiments reported elsewhere (e.g., Ref. [5,12]). This is to be expected, of course, given the similarity of the present approach and that outlined by Hagan et al. [5]; the key point here is that the surface stresses can be derived independently of the structural deformation. Moreover, the present formulation clearly delineates

Table 2 Pair potential parameters for grafting density prediction (Fig. 7) ($\Delta\mu_B=160$ kJ, $c=1.2 \times 10^{-4}$ – 5×10^{-6} μ M, $T=293$ K)

Hydration repulsion	Electrostatic repulsion	Fluctuation ($\beta=0.16$)
$\phi_H=3.6 \times 10^{-15}$ J $r_H=0.3$ nm	$\phi_D=(0.4, 1.2, 6) \times 10^{-17}$ J $r_D=0.3$ nm/ $\sqrt{M/(mol/l)}$	$L=40$ bp ~ 12 nm $l_p=44$ nm

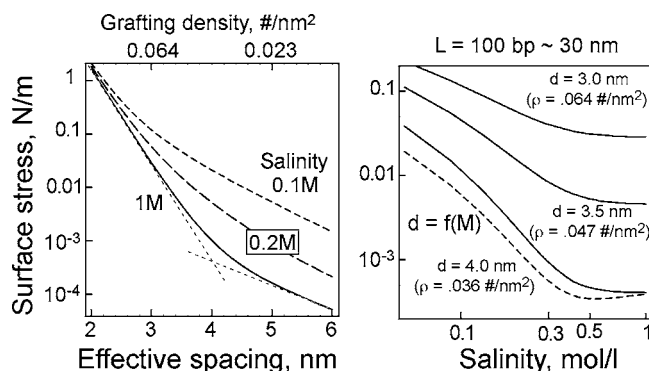


Fig. 8 Surface stress predictions for hcp grafting of DNA developed by assuming nearest-neighbor interactions and the full potential for DNA developed by Strey et al. [13]

Table 3 Pair potential parameters for surface stress prediction (Fig. 8) ($T=293$ K)

Hydration repulsion	Electrostatic repulsion	Fluctuation terms ($\alpha=0.13$)
$\phi_H=3.3 \times 10^{-15}$ J $r_H=0.3$ nm	$\phi_D=1.2 \times 10^{-17}$ J $r_D=0.3$ nm/ $\sqrt{M/(mol/l)}$	$L=100$ bp ~ 30 nm $l_p=44$ nm

between regimes where fluctuation terms are not important. Strictly quantitative comparisons will require calculations for equivalent numbers of base pairs and, mostly likely, alteration of the interaction parameters to reflect the effects of end grafting on the pair potential outlined by Strey et al. [13].

The present framework can be used to predict surface stresses as a function of the salt concentration maintained during adsorption. In this approach, the grafting density is not an independent parameter, but rather dictated by the energy balance given as Eq. (1). The chosen concentration of ds-DNA in solution is 5 μ M. The bottom curve in Fig. 8 illustrates the results, wherein the same pair potential used to predict surface stress is also used to predict grafting density. In this case, the surface stress is lower than that of fixed grafting density because decreasing the amount of electrostatic screening (by decreasing salinity) will decrease grafting density. It is important to note that despite the fact that the grafting density is decreasing, it is still possible to generate significant surface stress.

4 Performance of Devices Utilizing Molecular Adsorption

In this section, we couple the molecular and continuum frameworks in Sec. 2 to predict device behavior, with the purpose of illustrating how the present approach facilitates integrated design of surface chemistry and microdevice geometry.

4.1 Cantilevers as a Path Toward Characterizing Molecular Potentials. The behavior of cantilevers with adsorbed C_{60} molecules on a single side is illustrated in Fig. 9 by plotting mechanical response as a function of characteristic adsorption spacing. Here, two 500 μ m long cantilevers are compared: a 250 nm thick silicon cantilever and an 8 μ m thick elastomer cantilever. The dimensions are chosen such that the bare cantilever bending stiffnesses, which scale with Eh^3 , are equivalent. Figure 9 plots separately the static deflection due to adsorption and the spring constant of the cantilever when subjected to a point load at its tip, as a function of effective spacing between adsorbed groups. In order to obtain the plots, the pair correlation function resulting from MC simulation at a specific density has been scaled with the location of the first peak as a parameter. The deflection of both cantilevers changes sign at an effective adsorption spacing of ~ 1 nm, due to the switch from attractive to repulsive molecular

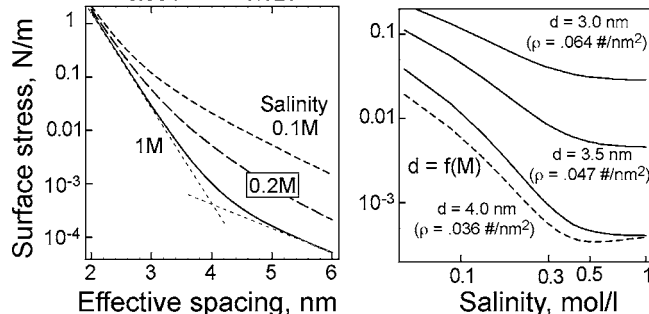


Fig. 9 Cantilever deflections and stiffness as a function of C_{60} adsorption spacing, comparing several silicon and PDMS microcantilevers

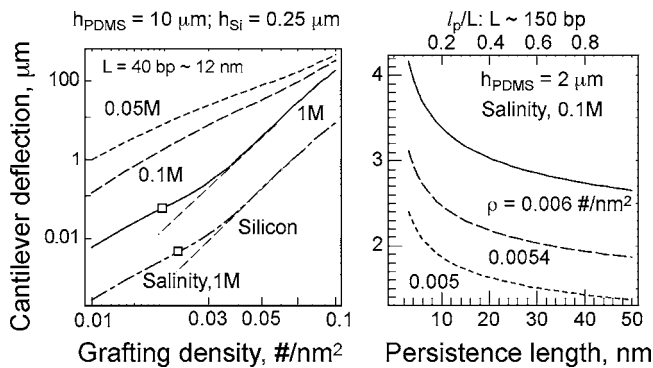


Fig. 10 Predicted cantilever deflections as a function of the persistence of semiflexible polymers for several adsorption spacings (hexagonal nearest-neighbor interactions using the Strey potential)

interactions.

Adsorption-induced deflection is much more dramatic in the elastomer cantilever. This example clearly illustrates the difference between bending stiffness and *surface compliance*. Surface compliance refers to the sensitivity of the device to surface stress and scales with $\bar{E}h^2$ rather than with $\bar{E}h^3$. This distinction is critical to the development of ultrasensitive sensors. The surface compliance governs the sensitivity of the device to surface adsorption, while the bending stiffness governs the sensitivity of the device to external loading, such as pressure fluctuations created by differences in fluid velocity above and below the cantilever. Thus, despite the vastly different size and dramatically increased surface stress sensitivity of the elastomer film, the elastomer is not necessarily more sensitive to external perturbations.

The increase in surface compliance arising from low modulus materials opens up avenues to molecular characterization that are not feasible with stiff materials. Figure 9 clearly indicates that, unlike a silicon cantilever, a PDMS structure will be highly sensitive to effective adsorption spacing. Both the static deflection and point-load bending stiffness of a PDMS cantilever will vary significantly for small perturbations in effective spacing. This is a direct result of the scaling argument outlined above. Poly(dimethylsiloxane) (PDMS) cantilevers create the possibility of characterizing both the surface stresses arising from adsorption and the effective modulus of the adsorbed groups. This implies that both the depth of the potential well and the pair equilibrium spacing can be estimated since measurements of static deflection and bending stiffness represent two independent measures of adsorption effects.

The possibility of constructing ultracompliant polymer cantilevers to quantify molecular interactions is further supported by the implications of Fig. 10, which depicts cantilever deflection as a function of DNA characteristics. Here, we have chosen to plot cantilever deflection as a function of grafting density since this quantity can be directly compared with existing fluorescence measurements. The fluctuation contributions to deflection are only dominant for relatively low grafting densities (large adsorption spacing). The square markers on the left of Fig. 10 represent the grafting density for which the fluctuation contribution is equal to that arising from bare interactions.

In this regime, the overall magnitude of surface stress is relatively low—so low, in fact, that silicon devices are incapable of quantifying fluctuation-related interactions. Figure 10 illustrates that it should be possible to directly quantify fluctuation effects by using polymer cantilevers; the increase in surface compliance implies that significant deflections can be achieved even for low grafting densities. Figure 10(b) shows that in order for the deflection to be a strong function of persistence length, the adsorption spacing has to be relatively high. Otherwise, electrostatic and hy-

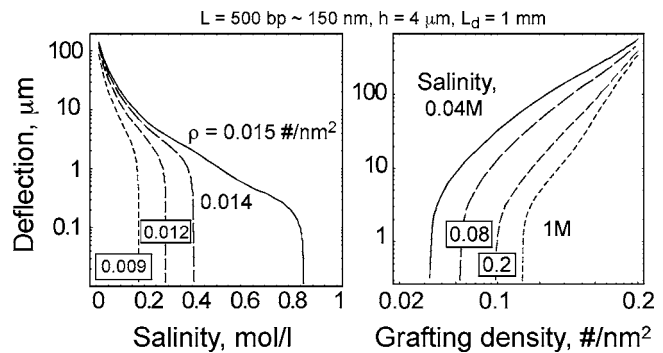


Fig. 11 Predicted clamped PDMS film deflections as a function of DNA adsorption spacing at several salt concentrations. At a fixed concentration, there is a critical adsorption spacing that triggers out-of-plane buckling.

dration forces dominate the response. By utilizing ultracompliant elastomer structures, one can achieve measurable deflections with very low grafting densities.

It should be emphasized that despite the low modulus of the PDMS cantilever, the bending stiffness (which dictates the structures' response to ambient noise) is actually higher than that of the silicon. Figure 10 also demonstrates that it is possible to achieve deflections on the order of $100 \mu\text{m}$ for a very dense surface coverage. Such deflections are large enough to alter fluid motion inside a microfluidic device and realize the concept illustrated in Fig. 1.

4.2 Clamped Films. Cantilevers will obviously maximize displacement for a given adsorption scenario. However, if sufficiently compliant structures can be fabricated, even clamped films can exhibit significant deformation. In this scenario, silicon or metal cantilevers will not exhibit sufficient compliance to allow out-of-plane buckling. Hence, we again consider the performance of a PDMS or an elastomer film. Figure 11 plots the displacement of a clamped film as a function of adsorption spacing and salt concentration for the case of single-sided adsorption. It should be noted that unlike cantilevers, clamped films will still deform out of plane in response to double-sided adsorption; in fact, in this scenario, the driving force doubles. The fact that the film is clamped implies that beneath the critical surface stress, no deformation takes place. Figure 11 shows that there is a critical adsorption spacing for a given salinity that will trigger buckling: Beneath the critical spacing, the deflection will increase as grafting density increases. Similarly, for a given adsorption spacing, there is a critical salinity above which there is no deformation. Below the critical salinity, the deflection increases with decreasing salt concentration.

The results in Fig. 11 clearly indicate that it should be possible to create adsorption-driven films that experience large deformation accommodated via buckling. In particular, the results illustrate that this functionality can be achieved with relatively large adsorption spacing and fairly moderate changes in salt concentration. Moreover, there is the possibility of "tuning" devices to the onset of the buckling instability by controlling compressive prestrain. Adsorption would then trigger buckling, with the prestrain acting to amplify adsorption-induced deformation. Of course, this requires microfabrication procedures capable of precisely modulating prestrain in the film.

5 Concluding Remarks

The rational design of microdevices that exploit chemomechanical coupling requires that choices regarding chemical surface functionalization be made in concert with choices of device materials and geometry. The present framework enables device developers to identify suitable nano- and microfabrication targets

that incorporate both device-level and molecular-level phenomena. For example, the binding energy required to produce sufficient grafting density for a strongly coupled behavior can be identified in terms of microfabrication limits. Conversely, microfabrication targets that would enable new types of molecular characterization can be defined for known functionalization pathways.

The calculations presented for DNA—using experimentally verified pair interactions—clearly demonstrate the potential to construct highly sensitive structures via self-assembly of polyelectrolyte brushes. The ability to modulate surface stress via electrostatic screening represents a powerful opportunity to create chemically activated structures to guide fluid flow in microfluidic devices. An important implication of the present calculations is that grafting density and surface stresses can be independently controlled; one can self-assemble polymer brushes under heavily screened conditions (where surface stresses are low), and then increase chemomechanical coupling by reducing electrostatic screening, simply by diluting the salt concentration of the surrounding fluid.

In the preceding, we have mainly limited the discussion to scenarios where molecular adsorption is irreversible; i.e., desorption is not explicitly considered. However, the present framework sets the stage to consider more complex interactions, notably the desorption of molecules driven by mechanical deformation. Using the framework presented here, one can explicitly calculate the balances between binding energy and deformation that lead to desorption.

The use of elastomeric structures dramatically broadens the range of applicability of chemomechanical coupling. The large displacements that are possible with such materials may ultimately eliminate the need for a sophisticated detection hardware, which is indispensable with stiff metallic or semiconductor materials. By pushing chemically induced deformation into the 10–100 μm range, optical and electronic (i.e., capacitance) transduction strategies, which can be integrated directly into lab-on-a-chip technology, are feasible.

The success of this strategy naturally depends critically on the development of surface functionalization pathways that do not comprise the surface compliance of PDMS-based materials. It is interesting to note that even nanoscale metallic coatings significantly compromise surface compliance and limit sensitivity [20]. To date, feasible strategies rely on the use of plasma oxidation to create a glassy layer of SiO_2 groups, which then serves as a platform for subsequent reactions, for example, the covalent bonding of organic molecules such as (3-aminopropyl)triethoxysilane (APTES) [21]. It may be possible to combine plasma oxidation with C_{60} grafting approaches developed for oxide surfaces (e.g., Ref. [16]).

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Spectral Stiffness Microplane Model for Quasibrittle Composite Laminates—Part I: Theory

The paper presents the spectral stiffness microplane model, which is a general constitutive model for unidirectional composite laminates, able to simulate the orthotropic stiffness, prepeak nonlinearity, failure envelopes, and, in tandem with the material characteristic length, also the post-peak softening and fracture. The framework of the microplane model is adopted. The model exploits the spectral decomposition of the transversely isotropic stiffness matrix of the material to define orthogonal strain modes at the microplane level. This decomposition is a generalization of the volumetric-deviatoric split already used by Bažant and co-workers in microplane models for concrete, steel, rocks, soils, and stiff foams. Linear strain-dependent yield limits (boundaries) are used to provide bounds for the normal and tangential microplane stresses, separately for each mode. A simple version, with an independent boundary for each mode, can capture the salient aspects of the response of a unidirectional laminate, although a version with limited mode coupling can fit the test data slightly better. The calibration of model parameters, verification by test data, and analysis of multidirectional laminates are postponed for the subsequent companion paper. [DOI: 10.1115/1.2744036]

Keywords: fiber composites, laminates, spectral methods, microplane model, fracture energy, crack band model, damage, failure criteria, finite elements

Dedicated to Professor Franz Ziegler of the Technical University of Vienna at the occasion of his 70th birthday.

Introduction

At an early stage of development of any field of science, different models exist for different situations. Such models have, of course, narrow applicability and cannot be extended beyond the limited range of their experimental support. Development of a general, unified theory which agrees with experiments of all kinds generally provides a far broader applicability and makes it possible to extrapolate to situations outside the range of the existing experimental support.

The present paper attempts to formulate a unified theory for the mechanical behavior of fiber-polymer laminates. A constitutive law will first be developed for laminates with unidirectional reinforcement. General laminates with multidirectional reinforcement will then be modeled as an overlay of lamina with unidirectional reinforcements of different orientations.

For fiber-polymer composites, efforts in the 1960s by Rosen [1,2], Adams and Doner [3,4], and Bažant (see Sec. 11.8 of [5]) led to highly accurate models for predicting the elastic moduli [6,7]. Many models also exist for the multiaxial strength criteria (or strength envelopes in the stress space), among which the quadratic criterion of Tsai and Wu [8] usually performs the best. The separate roles of matrix, fibers, and their interface have also been clarified.

These models, however, generally neglect the quasibrittle character of these materials. In both quasibrittle and brittle-ductile fracture, the crack tip is surrounded by a nonlinear zone that is not negligible compared to the cross-section dimension of the structures. While in brittle-ductile fracture, the fracture process zone

(FPZ) at crack tip occupies a negligibly small part of the nonlinear zone, which is plastic, the FPZ in quasibrittle fracture occupies almost the entire nonlinear zone and undergoes softening damage rather than plastic deformation. The stress along the FPZ is non-uniform and the stress decreases with crack opening gradually, due to discontinuous cracking in the FPZ, crack bridging by fibers, and frictional pullout of inhomogeneities.

Aside from polymer-fiber composites, the quasibrittle materials include concrete (the archetypical case studied the earliest), rocks, stiff soils, sea ice, wood, toughened ceramics, rigid foams, paper, etc. The quasibrittle materials generally exhibit not only the statistical size effect but also a strong energetic (nonstatistical) size effect, which often dominates and is caused by stress redistribution due to formation of a large crack or large FPZ before the maximum load (for laminates, this behavior has been experimentally demonstrated for various fracture types in [9–14]. As a consequence, the fracturing behavior of quasibrittle materials and, most importantly, the energetic size effect and the quasibrittleness effect associated with structure geometry, cannot be described by means of the classical linear elastic fracture mechanics.

The energetic size effect and geometric effect of brittleness due to large tensile mode I cracks or notches, with their characteristic material length, have been described by an orthotropic generalization of Bažant's size effect law [9]. However, this model applies neither to short cracks nor initiating fractures, for which the multiaxial strength criteria are intended. For kink-band compression failure of fiber composites and the associated size effect, an adaptation of the cohesive crack model has been formulated and experimentally justified [10,15] but the model has not been extended to general multiaxial and mixed-mode loadings.

The energetic size effect at fracture initiation, which is observed in flexural failure of laminates, the size of which represents the characteristic length of the material, has recently been modeled as a consequence of stress redistribution due to formation of a boundary layer of cracking (or FPZ) and received a strong experi-

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mental justification [11,12,16,17]. Nonetheless, the model does not apply to multiaxial loading and extends neither to post-peak crack propagation, nor to structures of negative geometry (i.e., structures in which the derivative of energy release rate with respect to crack length at constant load is negative).

The fracture energy and material characteristic length, which are most easily identified from size effect tests of notched specimens, should, in principle, allow computational predictions of damage localization and post-peak energy absorption under dynamic loadings such as impact, blast, and water or ground shock. However, such predictions are not feasible without a general model capturing also the anisotropic stiffness determined by microstructure and the multiaxial anisotropic strength criteria.

The analysis that follows is intended to make a step toward combining all the aforementioned phenomena into a single unified theory—a theory that can be used in general computer codes for predicting any behavior under general loadings. Much greater advances toward this lofty goal have already been made for other, perhaps less complicated, materials; e.g., for concrete, the microplane model, in the crack band or nonlocal setting, can approximately model the effects of random fiber reinforcement and steel reinforcing bars, the multiaxial strength criteria (with their history dependence), fracture propagation, post-peak softening, damage localization, energy absorption, and loading rate effects. For ductile metals, similar great advances have been made by Taylor models for polycrystals and Gurson's model for damage combined with plasticity theory and with material characteristic length.

It ought to be possible to achieve the same for fiber-polymer composites, despite the special complexities of their microstructure. Such an advance would allow a much more realistic analysis of large fracture-sensitive sandwich structures, such as the hulls, decks, bulkheads, masts, and antenna covers of very large and ultra-light ships made of composites, as well as load-bearing fuselage panels, wing box and rudder assemblies of large very light aircraft, or wind turbine blades, drive shafts, and containments or enclosures of various kinds. The goals and the problems are similar for carbon, glass, and aramid fibers in epoxy or vinylesther matrix, and for various manufacturing techniques.

Background of Microplane Model

Inspired by the slip theory of plasticity (pioneered by Taylor [18] and formulated in detail by Batdorf and Budiansky [19]), Bažant and Oh [20,21] introduced the microplane model to simulate materials such as concrete, exhibiting softening damage. The model captures the fact that almost all of the inelastic phenomena in the concrete microstructure, including crack opening, frictional slip and dilatancy, compressive splitting with lateral spreading, and fiber break or pullout, have distinct spatial orientations that cannot be captured by the classical tensorial constitutive models exemplified by the theories of plasticity.

The microplane model rests on two basic ideas: (1) describe idealized microstructural phenomena by a constitutive relation expressed not in terms of stress and strain tensors of the macroscopic continuum, but in terms of the stress vector and strain vector acting on planes of all possible orientations at a given point of the continuum; and (2) use a variational principle to relate the microplane vectors (the micro) to the continuum tensors (the macro).

Therefore, instead of a traditional tensorial constitutive model, the microplane model uses a constitutive law formulated in terms of stress and strain vectors acting on a generic microplane. This approach has a number of advantages [22], among which the main ones are as follows. (1) Various physical phenomena are easier to formulate. (2) Various combinations of loading and unloading on different microplanes provide a rich spectrum of path dependence and automatically produce the Bauschinger effect. (3) Thanks to a vectorial constitutive description, the spatial orientations of cracking and slip can be captured (note that, for example, a relation

between the first and second invariants of the stress tensor is generally a poor characteristic of internal friction because frictional slip typically occurs on planes of one or several distinct orientations). (4) The model automatically exhibits the vertex effect, which is not captured by any of the practically usable tensorial models. (5) Interaction of microplanes provides all the cross-effects such as shear dilatancy and pressure sensitivity.

The penalty to pay for these advantages is an increased amount of computations. However, this penalty is becoming insignificant for the latest, most powerful computers. Besides, the larger the system, the smaller the penalty, because large systems are computationally dominated by the structural stiffness matrix rather than the constitutive subroutine. Systems with >10 millions of finite elements are being solved with the microplane model for concrete [23].

Since its introduction in the early 1980s, the microplane model for concrete has evolved through five progressively improved versions labeled as M1 [20,21], M2 [24], M3 [25], M4 [22,26,27], and M5 [28,29]. Microplane models have also been developed for other complex materials such as rock [30], sand, clay, rigid foam, shape memory alloys [31–33], and fiber reinforced concrete [34]. The microplane model has been generalized for finite strain in [23]. A more fundamental finite strain formulation based on thermodynamic potentials was developed in [35]. Additional microplane model formulations can also be found in [36,37].

Spectral Decomposition of the Stiffness Matrix. The elastic stress-strain relation of an anisotropic material reads, in tensorial notation,

$$\sigma_{ij} = E_{ijkl} \varepsilon_{kl} \quad (1)$$

where the indices refer to Cartesian coordinates x_i ($i=1,2,3$); σ_{ij} and ε_{ij} are the second-order stress and strain tensors, respectively. They are symmetric and their symmetry enables their contraction into six-dimensional vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$. Similarly, the internal and external symmetries of the fourth-order stiffness tensor E_{ijkl} allow its contraction into a 6×6 matrix $\mathbf{E}$. The following rules contract a pair of indices into a single index: $11 \rightarrow 1$, $22 \rightarrow 2$, $33 \rightarrow 3$, $(23,32) \rightarrow 4$, $(13,31) \rightarrow 5$, and $(12,21) \rightarrow 6$. Therefore, in matrix notation we can write:

$$\boldsymbol{\sigma} = \mathbf{E} \boldsymbol{\varepsilon} \quad (2)$$

where

$$\boldsymbol{\sigma} = [\sigma_{11} \ \sigma_{22} \ \sigma_{33} \ \sqrt{2}\sigma_{23} \ \sqrt{2}\sigma_{13} \ \sqrt{2}\sigma_{12}]^T \quad (3)$$

$$\boldsymbol{\varepsilon} = [\varepsilon_{11} \ \varepsilon_{22} \ \varepsilon_{33} \ \sqrt{2}\varepsilon_{23} \ \sqrt{2}\varepsilon_{13} \ \sqrt{2}\varepsilon_{12}]^T \quad (4)$$

and the matrix $\mathbf{E}$ is defined accordingly. The foregoing definitions of six-dimensional vectors are known as the *Kelvin notation* [38,39]. The factor $\sqrt{2}$ assures that both the stiffness tensor and its column matrix have the same norm, given by the sum of the squares of their elements.

By exploiting the well known spectral decomposition theorem, we can decompose the stiffness matrix [40–43] as follows:

$$\mathbf{E} = \sum_I \lambda_I \mathbf{E}_I \quad (5)$$

where λ_I are the six eigenvalues of the stiffness matrix and $\mathbf{E}_I$ define a set of matrices constructed from the eigenvectors of $\mathbf{E}$:

$$\mathbf{E}_I = \sum_n \phi_{In} \phi_{In}^T \quad (6)$$

where ϕ_{In} is the eigenvector associated with the eigenvalue λ_I of multiplicity n and normalized such that $\phi_{In}^T \mathbf{E} \phi_{In} = \lambda_I$. Matrices $\mathbf{E}_I$ represent a partition of unity, i.e., $\sum_I \mathbf{E}_I = \mathbf{1}$, they are orthogonal,

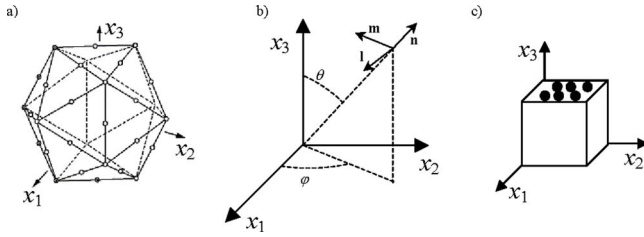


Fig. 1 (a) Microplane orientations (normals defined by radial lines through circled points); (b) spherical coordinate system; and (c) coordinate system for laminates

i.e., $\mathbf{E}_I \mathbf{E}_J = \mathbf{0}$ if $I \neq J$, and idempotent, i.e., $\mathbf{E}_I \mathbf{E}_I = \mathbf{E}_I$.

The same matrices $\mathbf{E}_I$ also decompose the stress and strain vectors into energetically orthogonal modes, which are called eigenstresses and eigenstrains and are defined as:

$$\boldsymbol{\sigma}_I = \mathbf{E}_I \boldsymbol{\sigma} \quad (7)$$

$$\boldsymbol{\varepsilon}_I = \mathbf{E}_I \boldsymbol{\varepsilon} \quad (8)$$

It is easy to show that $\boldsymbol{\sigma} = \sum_I \boldsymbol{\sigma}_I$, $\boldsymbol{\varepsilon} = \sum_I \boldsymbol{\varepsilon}_I$, and $\boldsymbol{\sigma}_I = \lambda_I \boldsymbol{\varepsilon}_I$. In the case of isotropic materials, the decomposition of stresses and strains in Eqs. (7) and (8) represents the well known volumetric-deviatoric decomposition.

The spectral decomposition theorem can be applied to the compliance matrix $\mathbf{C} = \mathbf{E}^{-1}$ with a similar result:

$$\mathbf{C} = \sum_I \lambda_I^{-1} \mathbf{E}_I \quad (9)$$

Note that in this case the eigenvalues are the inverse of the eigenvalues of the corresponding stiffness matrix, while the matrices $\mathbf{E}_I$ are the same.

Microplane Model Formulation With Spectral Decomposition. At the microstructural level of a material, nonlinear and inelastic phenomena often occur on planes of a certain specific orientation. Therefore, the constitutive law characterizing the mechanical behavior is best described through a relation between stress and strain vectors acting on a generic plane of arbitrary spatial orientation. These planes, called the microplanes [20,21], can be imagined as the tangent planes of a unit sphere surrounding every point in the three-dimensional space (Fig. 1(a)).

There are two different classes of microplane models: the kinematically constrained and the statically constrained. In the kinematically constrained microplane model (introduced by Bažant and Oh in [20,21]), the strain vector on each microplane is the projection of the macroscopic strain tensor. By using the Kelvin notation, we can write:

$$\boldsymbol{\varepsilon}_P = \mathcal{P} \boldsymbol{\varepsilon} \quad (10)$$

where $\boldsymbol{\varepsilon}_P = [\varepsilon_N \ \varepsilon_M \ \varepsilon_L]^T$ is the microplane strain vector, with ε_N =normal strain component, ε_M and ε_L =shear strain components, and

$$\mathcal{P} = \begin{bmatrix} N_{11} & N_{22} & N_{33} & \sqrt{2}N_{23} & \sqrt{2}N_{13} & \sqrt{2}N_{12} \\ M_{11} & M_{22} & M_{33} & \sqrt{2}M_{23} & \sqrt{2}M_{13} & \sqrt{2}M_{12} \\ L_{11} & L_{22} & L_{33} & \sqrt{2}L_{23} & \sqrt{2}L_{13} & \sqrt{2}L_{12} \end{bmatrix} \quad (11)$$

Matrix $\mathcal{P}$ collects the components of the tensors $N_{ij} = n_i n_j$, $M_{ij} = (m_i n_j + m_j n_i)/2$, and $L_{ij} = (l_i n_j + l_j n_i)/2$, where n_i , m_i , and l_i are local Cartesian coordinate vectors on the generic microplane, with n_i being normal. If the microplane orientation is defined by spherical angles ϑ and φ (Fig. 1(b)), then $n_1 = \sin \vartheta \cos \varphi$, $n_2 = \sin \vartheta \sin \varphi$, $n_3 = \cos \vartheta$, and one can choose $m_1 = \cos \vartheta \cos \varphi$, $m_2 = \cos \vartheta \sin \varphi$, $m_3 = -\sin \vartheta$, which gives $l_1 = -\sin \varphi$, $l_2 = \cos \varphi$,

and $l_3 = 0$.

By the spectral decomposition of the strain tensor and a separate projection of each eigenstrain, we can also decompose the microplane strain vector into microplane eigenstrains as:

$$\boldsymbol{\varepsilon}_P = \sum_I \boldsymbol{\varepsilon}_{PI} \quad (12)$$

where $\boldsymbol{\varepsilon}_{PI} = \mathcal{P} \boldsymbol{\varepsilon}_I = \mathcal{P}_I \boldsymbol{\varepsilon}$, with $\mathcal{P}_I = \mathcal{P} \mathbf{E}_I$. The split of the microplane strain vector introduced in Eq. (12) is an anisotropic generalization of the volumetric-deviatoric split introduced in the isotropic microplane model in [24,22].

From the microplane eigenstrains, the microplane eigenstresses $\boldsymbol{\sigma}_{PI}$ are calculated according to the constitutive relations for the normal and shear components of each eigenmode: $\sigma_{NI} = \mathcal{F}_{NI}(\boldsymbol{\varepsilon}_{P1}, \boldsymbol{\varepsilon}_{P2}, \dots)$, $\sigma_{MI} = \mathcal{F}_{MI}(\boldsymbol{\varepsilon}_{P1}, \boldsymbol{\varepsilon}_{P2}, \dots)$, and $\sigma_{LI} = \mathcal{F}_{LI}(\boldsymbol{\varepsilon}_{P1}, \boldsymbol{\varepsilon}_{P2}, \dots)$.

The macroscopic stress tensor may then be computed from the principle of virtual work, which reads:

$$\boldsymbol{\sigma} = \frac{3}{2\pi} \sum_I \int_{\Omega} \mathcal{P}_I^T \boldsymbol{\sigma}_{PI} d\Omega \quad (13)$$

where Ω is the surface of a unit hemisphere. This principle represents a weak variational constraint. In general, the projection of the stress tensor does not coincide with the microplane eigenstress: $\boldsymbol{\sigma}_{PI} \neq \mathcal{P}_I \boldsymbol{\sigma}$. Nevertheless, such a coincidence, called the double constraint, holds in the elastic regime if and only if the microplane eigenstress vector is proportional to the microplane eigenstrain vector through the associated eigenvalue:

$$\boldsymbol{\sigma}_{PI} = \lambda_I \boldsymbol{\varepsilon}_{PI} \quad (14)$$

In this particular case, recalling that $\boldsymbol{\varepsilon}_{PI} = \mathcal{P}_I \boldsymbol{\varepsilon}$ with $\mathcal{P}_I = \mathcal{P} \mathbf{E}_I$, from Eq. (13) we obtain:

$$\begin{aligned} \boldsymbol{\sigma}_{PI} &= \mathcal{P}_I \boldsymbol{\sigma} = \mathcal{P} \mathbf{E}_I \left[\frac{3}{2\pi} \sum_I \int_{\Omega} \mathbf{E}_I \mathcal{P}_I^T \lambda_I \mathcal{P} \mathbf{E}_I \boldsymbol{\varepsilon} d\Omega \right] \\ &= \mathcal{P} \mathbf{E}_I \sum_I \lambda_I \mathbf{E}_I \frac{3}{2\pi} \int_{\Omega} \mathcal{P}_I^T \mathcal{P} d\Omega \mathbf{E}_I \boldsymbol{\varepsilon} \end{aligned} \quad (15)$$

Moreover, it can be proven that:

$$\frac{3}{2\pi} \int_{\Omega} \mathcal{P}_I^T \mathcal{P} d\Omega = \mathbf{1} \quad (16)$$

Therefore, from Eq. (15) we obtain:

$$\begin{aligned} \mathcal{P}_I \boldsymbol{\sigma} &= \mathcal{P} \mathbf{E}_I \sum_J \lambda_J \mathbf{E}_J \mathbf{E}_I \boldsymbol{\varepsilon} = \mathcal{P} \lambda_I \mathbf{E}_I \mathbf{E}_I \boldsymbol{\varepsilon} = \lambda_I \mathcal{P} \mathbf{E}_I \boldsymbol{\varepsilon} = \lambda_I \mathcal{P}_I \boldsymbol{\varepsilon} \\ &= \lambda_I \boldsymbol{\varepsilon}_{PI} \end{aligned} \quad (17)$$

which proves the existence of a double constraint for elastic behavior.

In a statically constrained microplane model, the microplane stress vectors are the projections of the macroscopic stress tensor

$$\boldsymbol{\sigma}_P = \mathcal{P} \boldsymbol{\sigma} \quad (18)$$

or, equivalently, by introducing the spectral decomposition of stresses:

$$\boldsymbol{\sigma}_P = \sum_I \boldsymbol{\sigma}_{PI}, \quad \boldsymbol{\sigma}_{PI} = \mathcal{P} \boldsymbol{\sigma}_I = \mathcal{P}_I \boldsymbol{\sigma} \quad (19)$$

After formulating suitable constitutive relations at the microplane level, it is possible to obtain the microplane strains from the microplane stresses as: $\varepsilon_{NI} = \mathcal{G}_{NI}(\boldsymbol{\sigma}_{P1}, \boldsymbol{\sigma}_{P2}, \dots)$, $\varepsilon_{MI} = \mathcal{G}_{MI}(\boldsymbol{\sigma}_{P1}, \boldsymbol{\sigma}_{P2}, \dots)$, and $\varepsilon_{LI} = \mathcal{G}_{LI}(\boldsymbol{\sigma}_{P1}, \boldsymbol{\sigma}_{P2}, \dots)$. The macroscopic strain is then obtained by imposing the principle of complementary virtual work, which reads:

$$\boldsymbol{\varepsilon} = \frac{3}{2\pi} \sum_I \int_{\Omega} \mathcal{P}_I^T \boldsymbol{\varepsilon}_{PI} d\Omega \quad (20)$$

In this case the microplane strains are generally not the projections of the macroscopic strains: $\boldsymbol{\varepsilon}_{PI} \neq \mathcal{P}_I \boldsymbol{\varepsilon}$. Nevertheless, as seen earlier for the kinematically constrained microplane model, the double constraint holds in the elastic regime ($\boldsymbol{\varepsilon}_{PI} = \lambda_I^{-1} \boldsymbol{\sigma}_{PI}$).

Comparison With Alternative Formulations

Brocca et al. [33] proposed an alternative microplane formulation for anisotropic materials based on the assumption that the elastic moduli on the microplanes vary ellipsoidally as a function of the microplane orientation:

$$E_i(\varphi, \theta) = E_{i1} \sin \theta \cos \varphi + E_{i2} \sin \theta \sin \varphi + E_{i3} \cos \theta \quad (21)$$

where φ and θ are the angles characterizing the normal direction of a generic microplane in spherical coordinates (Fig. 1(b)). Subscript $i=N, M, L$ labels the components of the microplane strain and stress vectors, and E_{i1}, E_{i2}, E_{i3} are the microplane elastic moduli in the x_1 -, x_2 -, and x_3 -directions, respectively. This approach can be used to approximatively represent the behavior of a mildly anisotropic material such as a PVC foam. However, it cannot represent correctly the mechanical properties of strongly anisotropic materials such as fiber composite laminates. This can be easily proven by computing the elastic stiffness matrix arising from Eq. (21).

By integrating the microplane elastic energy over the unit hemisphere, we obtain:

$$\mathcal{W} = \frac{3}{2\pi} \int_{\Omega} \frac{1}{2} \boldsymbol{\sigma}_P^T \mathbf{E}_P \boldsymbol{\sigma}_P d\Omega = \frac{1}{2} \boldsymbol{\sigma}^T \left[\frac{3}{2\pi} \int_{\Omega} \mathcal{P}^T \mathbf{E}_P \mathcal{P} d\Omega \right] \boldsymbol{\sigma} \quad (22)$$

where matrix $\mathbf{E}_P = \text{diag}(E_i)$. The equivalent stiffness matrix is defined as:

$$\mathbf{E}^* = \frac{3}{2\pi} \int_{\Omega} \mathcal{P}^T \mathbf{E}_P \mathcal{P} d\Omega \quad (23)$$

The stiffness matrix $\mathbf{E}$ of a generic orthotropic material depends on nine independent constants that must be uniquely related to the nine parameters in Eq. (21) if the exact correspondence between

this microplane formulation and the tensorial formulation holds. To determine that relationship, we can try to solve the system of simultaneous equations obtained by equating each element of matrix $\mathbf{E}$ to the corresponding element of matrix $\mathbf{E}^*$ (note that the elements of $\mathbf{E}^*$ are a linear combination of the microplane elastic moduli E_{i1}, E_{i2} , and E_{i3}). Such a system of equations reads

$$\mathbf{Y} = \mathbf{A} \mathbf{X} \quad (24)$$

where

$$\mathbf{Y} = [E_{11}, E_{22}, E_{33}, E_{12}, E_{13}, E_{23}, E_{44}, E_{55}, E_{66}]^T \quad (25)$$

$$\mathbf{X} = [E_{N1}, E_{N2}, E_{N3}, E_{M1}, E_{M2}, E_{M3}, E_{L1}, E_{L2}, E_{L3}]^T \quad (26)$$

It can be shown that the rank of matrix $\mathbf{A}$ is 6, whereas 9 would be required to solve the system (24). Therefore, if an ellipsoidal formulation for the microplane moduli is used as indicated in Eq. (21), it is impossible to represent the material behavior in the elastic range since an exact correspondence with the tensorial model cannot be obtained. In particular, the lower rank of the matrix demonstrates that the ellipsoidal formulation implies a higher number of symmetries than those implied by material orthotropy. Only the microplane formulation based on spectral decomposition of the stiffness tensor guarantees that an exact correspondence in elasticity between tensorial macrostiffness and vectorial microstiffness can be established.

A similar limitation can be shown for the microplane models developed for mildly anisotropic clay in [31,44].

Transverse Isotropy: Analysis of Microplane Eigenmodes

Let us now consider the case of transverse isotropy, which is relevant to unidirectional laminates. We introduce a cartesian coordinate system whose axis x_3 coincides with the direction of the fibers (Fig. 1(c)). This axis is orthogonal to the plane of isotropy of the material. Let the *longitudinal* direction be the direction of the fibers (axis x_3), and let the *transverse* directions be all those orthogonal to the fibers. The elastic compliance matrix for a transversely isotropic material is given by:

$$\mathbf{C} = \begin{bmatrix} 1/E_T & -\nu_T/E_T & -\nu_L/E_T & 0 & 0 & 0 \\ -\nu_T/E_T & 1/E_T & -\nu_L/E_T & 0 & 0 & 0 \\ -\nu_L/E_T & -\nu_L/E_T & 1/E_L & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/(2G_L) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/(2G_L) & 0 \\ 0 & 0 & 0 & 0 & 0 & (1+\nu_T)/E_T \end{bmatrix} \quad (27)$$

where E_L, E_T =Young's moduli in the longitudinal and transverse directions, respectively, G_L =out-of-plane shear modulus, ν_L, ν_T =Poisson ratios in the longitudinal and transverse directions, respectively.

Following [43], the eigenvalues of the compliance matrix, which are the reciprocal of the eigenvalues λ_i of the stiffness matrix, can be expressed as:

$$\lambda_1^{-1} = \frac{1+\nu_T}{E_T} = \frac{1}{2G_T} \quad (28)$$

$$\lambda_2^{-1} = \frac{1-\nu_T}{2E_T} + \frac{1}{2E_L} - \left[\left(\frac{1-\nu_T}{2E_T} - \frac{1}{2E_L} \right)^2 + \frac{2\nu_L^2}{E_T^2} \right]^{1/2} \quad (29)$$

$$\lambda_3^{-1} = \frac{1-\nu_T}{2E_T} + \frac{1}{2E_L} + \left[\left(\frac{1-\nu_T}{2E_T} - \frac{1}{2E_L} \right)^2 + \frac{2\nu_L^2}{E_T^2} \right]^{1/2} \quad (30)$$

$$\lambda_4^{-1} = \frac{1}{2G_L} \quad (31)$$

and the idempotent matrices decomposing $\mathbf{E}$ are

Table 1 Microplane strain modes: Normal component

Mode	ε_N
I	$\sin^2 \theta [\alpha_1 (\cos^2 \varphi - \sin^2 \varphi) + 2\sqrt{2}\varepsilon_6 \sin \varphi \cos \varphi]$
II	$\alpha_2 (-\sin \omega \sin^2 \theta / \sqrt{2} + \cos \omega \cos^2 \theta)$
III	$\alpha_3 (\cos \omega \sin^2 \theta / \sqrt{2} + \sin \omega \cos^2 \theta)$
IV	$2\sqrt{2} \sin \theta \cos \theta (\varepsilon_4 \sin \varphi + \varepsilon_5 \cos \varphi)$

$$\mathbf{E}_1 = \begin{bmatrix} 1/2 & -1/2 & 0 & 0 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (32)$$

$$\mathbf{E}_2 = \begin{bmatrix} c^2/2 & c^2/2 & cs/\sqrt{2} & 0 & 0 & 0 \\ c^2/2 & c^2/2 & cs/\sqrt{2} & 0 & 0 & 0 \\ cs/\sqrt{2} & cs/\sqrt{2} & s^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (33)$$

$$\mathbf{E}_3 = \begin{bmatrix} s^2/2 & s^2/2 & -cs/\sqrt{2} & 0 & 0 & 0 \\ s^2/2 & s^2/2 & -cs/\sqrt{2} & 0 & 0 & 0 \\ -cs/\sqrt{2} & -cs/\sqrt{2} & c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (34)$$

$$\mathbf{E}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (35)$$

where $c = \cos \omega$, $s = \sin \omega$, and ω is the so-called eigenangle defined as:

$$\tan 2\omega = \frac{-2\sqrt{2}\nu_L/E_T}{(1 - \nu_T)/E_T - 1/E_L} \quad (36)$$

The projection of the strain modes defined in Eq. (8) on a generic microplane gives the microplane strain distribution in Tables 1 and 2, where $\alpha_1 = (\varepsilon_1 - \varepsilon_2)/2$, $\alpha_2 = -\sin \omega (\varepsilon_1 + \varepsilon_2)/\sqrt{2} + \varepsilon_3 \cos \omega$, and $\alpha_3 = \cos \omega (\varepsilon_1 + \varepsilon_2)/\sqrt{2} + \varepsilon_3 \sin \omega$.

Let us now consider the distribution of the normal strain components on the microplane sphere caused by a uniaxial strain applied at the macroscale in the longitudinal direction (i.e., along the

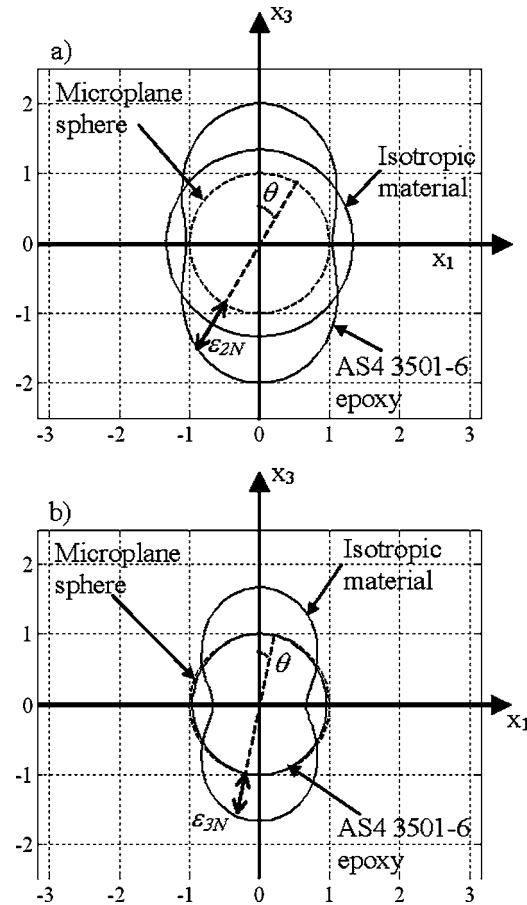


Fig. 2 Effect of a macroscopic strain applied in fiber direction on (a) mode II, and (b) mode III

x_3 -axis). Strain mode I and strain mode IV are exactly zero because they do not depend on ε_3 . Strain mode II and strain mode III are nonzero and they depend on the eigenangle ω . Figures 2(a) and 2(b) show the microplane normal strain distribution of modes II and III for isotropic materials ($\omega=0.615$) and for a highly anisotropic carbon-epoxy composite (AS4 3501-6/epoxy with $\omega=0.06$). For isotropic materials, mode II coincides with the volumetric strain mode, and mode III with the deviatoric strain mode. On the contrary, for the composite considered, mode II mostly loads the microplanes whose normal orientation is close to the direction of the fibers, and mode III almost vanishes.

A different picture arises if we consider a uniaxial strain at the macroscale applied in the transverse direction (x_1 -axis). In this case the only mode that is exactly zero is mode IV. Figures 3(a)–3(c) show the microplane normal strain distribution of modes II, III, and I, respectively. For the isotropic case, strain mode II coincides again with the volumetric strain mode, and the sum of mode I and mode III coincides with the deviatoric strain mode. Mode II vanishes and modes I and II mostly load the microplanes whose orientation is close to the direction orthogonal to

Table 2 Microplane strain modes: Tangential components

Mode	ε_M	ε_L
I	$\sin \theta \cos \theta [\alpha_1 (\cos^2 \varphi - \sin^2 \varphi) + 2\sqrt{2}\varepsilon_4 \sin \varphi \cos \varphi]$	$\sin \theta [\alpha_1 \sin \varphi \cos \varphi + \sqrt{2}\varepsilon_6 (\cos^2 \varphi - \sin^2 \varphi)]$
II	$\alpha_2 \sin \theta \cos \theta (-\sin \omega / \sqrt{2} - \cos \omega)$	0
III	$\alpha_3 \sin \theta \cos \theta (\cos \omega / \sqrt{2} - \sin \omega)$	0
IV	$\sqrt{2} (\cos^2 \theta - \sin^2 \theta) (\varepsilon_4 \sin \varphi + \varepsilon_5 \cos \varphi)$	$\sqrt{2} \cos \theta (\varepsilon_4 \cos \varphi - \varepsilon_5 \sin \varphi)$

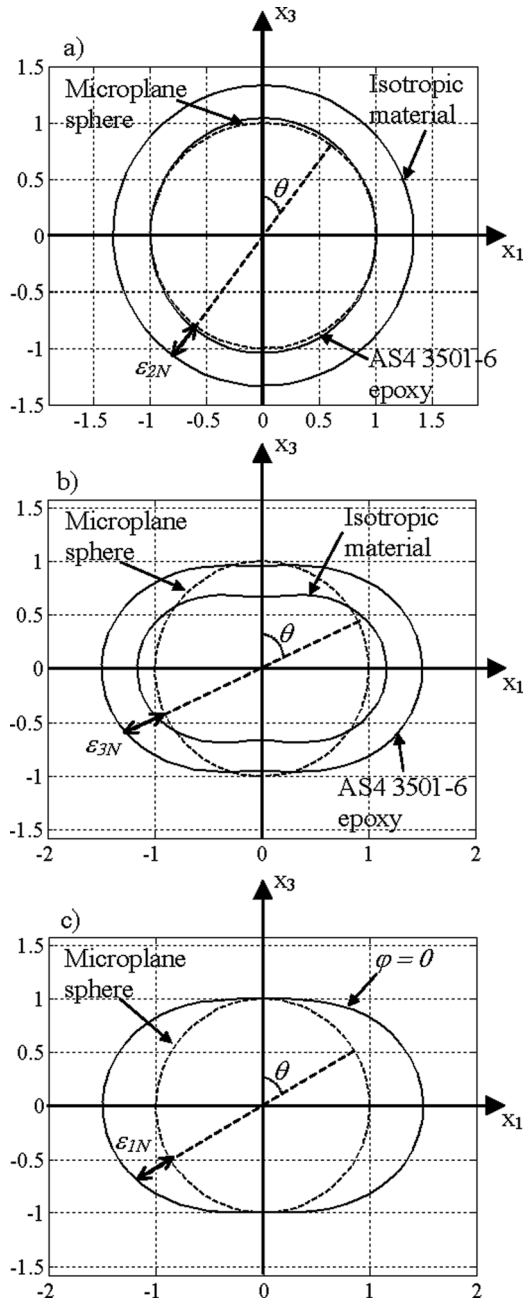


Fig. 3 Effect of a macroscopic strain applied in transverse direction on (a) mode II, (b) mode III, and (c) mode I

the fibers. Note that strain mode I consists of a deviatoriclike mode in the plane of isotropy of the unidirectional laminate (see Table 1).

This analysis shows that the response of the composite material subjected to strain mode II strongly depends on the behavior of the fibers whereas material response under modes I and III is dominated by the behavior of the matrix.

Strain mode IV is a pure-shear strain mode loading the microplanes in such a way that generally both the matrix and the fibers have an effect. Nevertheless, in the following formulation, we will assume the shear resistance of the fibers to vanish.

Nonlinear and Softening Behavior of Unidirectional Laminates. The macroscopic behavior of composite laminates, in both linear and nonlinear range, is governed by various phenomena occurring in a material microstructure with the three phases—

the fibers, matrix, and fiber-matrix interface. The mechanics of these phenomena, which include microcracking, stress redistribution among the phases, fiber pull-out, fiber-matrix debonding, fiber buckling, etc., should be included in formulating constitutive laws that aim at real predictive capability. However, in practice, the complexity and the lack of accurate knowledge of the details of microstructural behavior impede attaining this aim and the only option available is a phenomenological approach. That is particularly true for tensorial constitutive laws.

The microplane framework, in which the constitutive laws are formulated in terms of vectors acting on planes of many specified orientations, provides more flexibility than the classical tensorial framework and allows introducing in the formulation some microstructural features even in a phenomenological setting. In what follows, the constitutive law at the microplane level is formulated and physically justified on the basis of current knowledge of damage and failure of the composite laminate microstructure.

Exploiting the experience with microplane models for concrete, rocks, and soils developed by Bažant and co-workers since the early eighties, we introduce for the stress-strain constitutive relation at the microplane level a kinematic constraint of the microplanes, and we adopt the concept of stress-strain boundary (or strain-dependent yield limit) introduced in [25] and used in microplane models M4, M5, and M5f [22,27,28,34]. According to that concept, for each strain mode, each stress component at the microplane level must satisfy the following inequalities

$$\mathcal{F}_{NI} \leq \sigma_{NI} \leq \mathcal{F}_{NI}^+, \quad |\sigma_{MI}| \leq \mathcal{F}_{TI}, \quad |\sigma_{LI}| \leq \mathcal{F}_{TI} \quad (37)$$

where $\mathcal{F}_{NI}$ =compression boundary, $\mathcal{F}_{NI}^+$ =tension boundary, and $\mathcal{F}_{TI}$ =shear boundary.

Note that, from the viewpoint of invariance with respect to rotations of microplane coordinates about the normal to the microplane, the shear boundary would be expressed more accurately in terms of the total shear stress $\sigma_{TI} = (\sigma_{MI}^2 + \sigma_{LI}^2)^{1/2}$ instead of the separate components σ_{MI} and σ_{LI} . However, the boundaries in Eq. (37), expressed in terms of components σ_{MI} and σ_{LI} , lead to simpler and more stable numerical implementation and give about the same results on the macrolevel.

The boundaries for the normal component of each mode ($I = \text{I}, \dots, \text{IV}$) can be formulated as follows:

$$\mathcal{F}_{NI}^+ = t_I(\theta) \prod_{J=1}^4 f_{IJ}^+(\epsilon_{PJ}, \theta), \quad \mathcal{F}_{NI}^- = -c_I(\theta) \prod_{J=1}^4 f_{IJ}^-(\epsilon_{PJ}, \theta) \quad (38)$$

in which angle θ gives the position of a generic point along the meridians of the unit hemisphere (Fig. 1(b)). For $\theta=0$, Eqs. (38) characterize the nonlinear behavior in the direction of the fibers, and for $\theta=\pi/2$, in the direction of the matrix. Because of the transverse isotropy, Eqs. (38) do not involve the angle φ giving the position along the parallels of the unit hemisphere (Fig. 1(b)).

The products in Eqs. (38) describe the interaction between different strain modes. In particular, each function $f_{IJ}(\epsilon_{PJ})$ gives the effect of strain mode J on the response of mode I . Obviously, if $f_{IJ}(\epsilon_{PJ}) \equiv 1$ for $I \neq J$, then the strain modes are completely uncoupled. Furthermore, if $f_{II}(\epsilon_{PI}) \equiv 1$ also, the behavior is purely plastic. For the plastic case, the distribution of the microplane yield strengths in tension and compression is given by $t_I(\theta)$ and $c_I(\theta)$, respectively. Note that, in the general case, $t_I(\theta)$ and $c_I(\theta)$ do not necessarily represent the microplane strength and must be regarded only as reference strengths.

Similarly, the boundary for the shear components can be formulated as:

$$\mathcal{F}_{II} = s_I(\theta) \prod_{j=1}^4 g_{IJ}(\epsilon_{PJ}, \theta) \quad (39)$$

where $s_I(\theta)$ are the shear reference strengths for each mode and functions $g_{IJ}(\epsilon_{PJ}, \theta)$ have the same meaning as functions $f_{IJ}(\epsilon_{PJ}, \theta)$ featured by the normal boundaries.

Longitudinal Tension and Compression. Except for a minor effect of fiber undulations, the matrix and fibers work in parallel coupling when the composite is subjected to tension or compression in the direction of the fibers.

In tension, the nonlinear behavior starts when the elastic limit is reached in the matrix whose strength is significantly lower than the strength of the fibers. At this point, the fibers are still in the elastic phase and, consequently, stresses can be redistributed in the microstructure, allowing an overall increase of the applied load. The maximum carrying capacity of the unidirectional laminate is attained when the second phase (fibers) fails. However, the fibers do not fail simultaneously and then it is reasonable to assume a gradual reduction of the macroscopic stress (or softening) with increasing macroscopic strain. The existence of softening quasibrittle behavior has been experimentally confirmed by recent size effect studies [11].

In the case of compression, the failure is dominated by various instability modes at the microscopic level [2]. For a low volume fraction of fibers, compression usually causes fiber kinking due to microbuckling. For a relatively high volume fraction of fibers, microbuckling is typically associated with the formation of a kink band that propagates transversely to the fibers. Very high values of fiber volume fraction usually prevent microbuckling and the laminate suffers shear failure. Again, as for the case of tension during failure, a certain degree of stress redistribution occurs in the microstructure, and then the loss of carrying capacity is gradual as the strain increases.

As shown in the preceding sections, mode II is the strain mode that is activated mainly by loading the material in the longitudinal direction (i.e., the direction of fibers). Thus, the behavior highlighted above can be analytically approximated through the definition of boundaries on mode II normal and shear components. This is achieved by setting:

$$t_2(\theta) = t_{20} \cos^2 \theta, \quad c_2(\theta) = c_{20} \cos^2 \theta, \quad s_2(\theta) = s_{20} \quad (40)$$

$$f_{22}^+(\epsilon_{P2}) = 1 - \frac{\epsilon_2^+ - k_{2i}^+}{k_{2f}^+}, \quad f_{22}^-(\epsilon_{P2}) = 1 - \frac{|\epsilon_2^-| - k_{2i}^-}{k_{2f}^-} \quad (41)$$

$$g_{22}(\epsilon_{P2}) = f_{22}^+(\epsilon_{P2})f_{22}^-(\epsilon_{P2})$$

with the conditions $1 \leq f_{22}^+(\epsilon_{P2}) \leq 0$ and $1 \leq f_{22}^-(\epsilon_{P2}) \leq 0$.

The preceding equations represent linear softening laws for the normal component in tension and compression while the softening evolution of the shear component is assumed to be proportional to the normal boundaries. t_{20} , c_{20} , and s_{20} are mode II strengths in tension, compression and shear, respectively. ϵ_2^+ and ϵ_2^- are the equivalent strains in tension and compression, respectively, which couple the effect of shear and normal microplane strains. They are defined as $\epsilon_2^+ = \max(\epsilon_2)$ and $\epsilon_2^- = \min(\epsilon_2)$, where $\epsilon_2 = \text{sign}(\epsilon_{N2}) \times [\epsilon_{N2}^2 + \epsilon_{M2}^2 + \epsilon_{L2}^2]^{1/2} \times k_{2i}^+$ and k_{2f}^+ are two material parameters: k_{2i}^+ defines the tensile strain at which the behavior becomes softening, and $k_{2i}^+ + k_{2f}^+$ is the tensile strain at which the carrying capacity is completely exhausted. k_{2i}^- and k_{2f}^- have the same meaning as k_{2i}^+ and k_{2f}^+ but for compression.

Transverse Tension and Compression. Tensile behavior in the direction perpendicular to the fibers is mainly controlled by the fiber-matrix interface. Failure is usually caused by stress concentration at the interface followed by the propagation of cracks around the fibers [3]. Failure in compression in the transverse direction is induced by stress concentration in the microstructure,

which leads to the failure of one of the three phases of the composite. In particular, one of the following failure modes may occur: (1) crushing of the matrix, (2) crushing of the fibers, and (3) fiber-matrix debonding followed by a macroscopic shear failure.

The modes relevant to these phenomena are strain modes I and III because they mainly load microplanes in the transverse direction (Table 1). Since the failure modes described above are related mainly to the composite matrix, we can reasonably assume that the associated reference strengths, in both tension, compression, and shear, do not depend on the orientation (angle θ): $t_1(\theta) = t_{10}$, $t_3(\theta) = t_{30}$, $c_1(\theta) = c_{10}$, $c_3(\theta) = c_{30}$, $s_1(\theta) = s_{10}$, and $s_3(\theta) = s_{30}$.

As for the case of the strain mode II, the post-peak behavior of mode I and mode III is formulated through linear softening constitutive laws. Since strain mode I is quasideviatoric in the plane of symmetry of the unidirectional laminate, both compression and tension appear at the microplane level regardless of the sign of the macroscopic strains in the transverse direction. This does not help to simulate the lack of symmetry between matrix failures in tension and compression. The only way to distinguish between these two types of failure is to let the behavior of strain mode I be dependent on the mean stress in the plane of isotropy: $\bar{\sigma} = (\sigma_{11} + \sigma_{22})/2$. The mode I boundaries are formulated as follows:

$$f_{11}^+(\epsilon_{P1}) = 1 - \frac{\epsilon_1^+ - k_{1i\alpha}^+}{k_{1f\alpha}^+}, \quad f_{11}^-(\epsilon_{P1}) = 1 - \frac{|\epsilon_1^-| - k_{1i\alpha}^-}{k_{1f\alpha}^-}, \quad (\bar{\sigma} \geq 0) \quad (42)$$

and

$$f_{11}^+(\epsilon_{P1}) = 1 - \frac{\epsilon_1^+ - k_{1i\beta}^+}{k_{1f\beta}^+}, \quad f_{11}^-(\epsilon_{P1}) = 1 - \frac{|\epsilon_1^-| - k_{1i\beta}^-}{k_{1f\beta}^-}, \quad (\bar{\sigma} < 0) \quad (43)$$

always subjected to the conditions $1 \leq f_{11}^+(\epsilon_{P1}) \leq 0$ and $1 \leq f_{11}^-(\epsilon_{P1}) \leq 0$; ϵ_1^+ and ϵ_1^- are equivalent strains in tension and compression, respectively, as defined in the case of mode II, and $k_{1i\alpha}^+, \dots, k_{1f\beta}^+$ are material parameters.

The boundaries for mode III are formulated as follows:

$$f_{33}^+(\epsilon_{P3}) = h_3 - \frac{\epsilon_3^+ - h_3 k_{3i}^+}{k_{3f}^+}, \quad f_{33}^-(\epsilon_{P3}) = 1 - \frac{|\epsilon_3^-| - k_{3i}^-}{k_{3f}^-} \quad (44)$$

with the conditions $h_3 \leq f_{33}^+(\epsilon_{P3}) \leq 0$ and $1 \leq f_{33}^-(\epsilon_{P3}) \leq 0$. Again, ϵ_3^+ and ϵ_3^- are equivalent strains, and $k_{3i}^+, \dots, k_{3f}^+$ are material parameters; h_3 is a function of the normal strain of mode II and it is introduced in the formulation of mode III in tension in order to prevent an unrealistic splitting-like failure associated with longitudinal compression: $h_3 = 1$ for $\epsilon_{N2} > 0$ and $h_3 = 1 + h_{30}\epsilon_2$ for $\epsilon_{N2} < 0$.

For both mode I and mode III, the shear boundary is formulated through the product of tensile and compressive normal boundaries: $g_{11}(\epsilon_{P1}) = f_{11}^+(\epsilon_{P1})f_{11}^-(\epsilon_{P1})$ and $g_{33}(\epsilon_{P3}) = f_{33}^+(\epsilon_{P3})f_{33}^-(\epsilon_{P3})$.

In-Plane Shear. When the composite is subjected to in-plane shear, high stress concentrations at the matrix-fiber interface usually lead to shear failure in the matrix or to fiber-matrix debonding, or to both [4]. The fibers have, in this situation, little effect and so the relevant microplane boundary (mode IV) can be assumed not to depend on the orientation. Consequently, the tensile and compressive microplane strengths read $t_4(\theta) = t_{40}$, $c_4(\theta) = c_{40}$, $s_4(\theta) = s_{40}$. Similar to the other cases, the softening evolution can be expressed as

$$f_{44}^+(\epsilon_{P4}) = 1 - \frac{\epsilon_4^+ - k_{4i}^+}{k_{4f}^+}, \quad f_{44}^-(\epsilon_{P4}) = 1 - \frac{|\epsilon_4^-| - k_{4i}^-}{k_{4f}^-} \quad (45)$$

$$g_{44}(\epsilon_{P4}) = f_{44}^+(\epsilon_{P4})f_{44}^-(\epsilon_{P4})$$

with $1 \leq f_{44}^+(\epsilon_{P4}) \leq 0$ and $1 \leq f_{44}^-(\epsilon_{P4}) \leq 0$. The variables and pa-

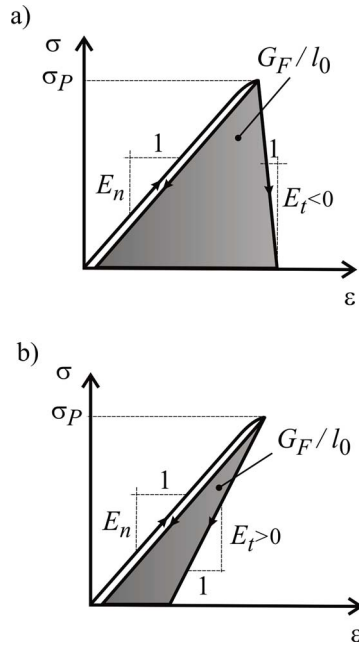


Fig. 4 Mean slope E_t of the post-peak softening curve

parameters in Eq. (45) have meaning similar to the case of mode II.

The formulation presented so far is based on the assumption that the strain modes are substantially independent. As it will be shown in the companion paper [45], even if it is possible to closely reproduce with this assumption the uniaxial behavior of the composite, the simulation of shear versus normal stress strength envelopes is not accurate. A mild interaction between modes significantly improves the fitting of these envelopes and enhances the overall predictive capability of the model. A detailed discussion of this interaction will appear in the subsequent companion paper.

Fracture Mechanics Aspects. Based on the data from the literature, it is impossible to identify the equilibrium post-peak softening curve because in all the tests either the machine stiffness was too low or the specimen size too large to keep the response stable (Sec. 13.2 in [5]). Nevertheless, the mean slope of the post-peak softening curve, or the area under it, can be identified from the value of fracture energy G_F and the material characteristic length l_0 representing roughly the FPZ width, which in turn is approximately equal to the minimum possible spacing h of parallel cracks or the crack band width in the crack band model.

The crack band width can be roughly estimated from visual observations of fracture and typically equals 1.5 to 2 times the material inhomogeneity size (such as spacing of the weave of reinforcing fabric, spacing of fiber strands, or of lamina, etc.). G_F can be most easily determined from size effect tests [16,46], as shown in [9,11]. Since $\sigma_p/E_n + \sigma_p/E_t = G_F/l_0 = \text{area under the post-peak softening curve and the elastic unloading curve of slope } E_t$ emanating from the peak stress point (Fig. 4). If this curve is assumed to be linear, its slope can be calculated as

$$E_t = - \left(\frac{G_F}{l_0 \sigma_p} - \frac{1}{E_n} \right)^{-1} \quad (46)$$

provided that both G_F and l_0 are known; here σ_p = peak stress across the fracture plane, and E_n pertains to stress across the expected crack plane; E_t can be positive or negative, and for a vertical drop it is ∞ . An equation analogous to (46) is used in the crack band model (Sec. 13.2 in [5]).

A more detailed discussion of fracture simulation with the present model is beyond the scope of this paper and is planned for subsequent study.

Closing Comment. The spectral stiffness microplane model developed here has several attractive features that need to be translated into a computational algorithm and verified by comparison. This is pursued in the second part of this study, which follows.

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Spectral Stiffness Microplane Model for Quasibrittle Composite Laminates—Part II: Calibration and Validation

The spectral stiffness microplane (SSM) model developed in the preceding Part I of this study is verified by comparisons with experimental data for uniaxial and biaxial tests of unidirectional and multidirectional laminates. The model is calibrated by simulating the experimental data on failure stress envelopes analyzed in the recent so-called “World Wide Failure Exercise,” in which various existing theories were compared. The present theory fits the experiments as well as the theories that were best in the exercise. In addition, it can simulate the post-peak softening behavior and fracture, which is important for evaluating the energy-dissipation capability of composite laminate structures. The post-peak softening behavior and fracture are simulated by means of the crack band approach which involves a material characteristic length. [DOI: 10.1115/1.2744037]

Keywords: fiber composites, laminates, spectral methods, microplane model, fracture energy, crack band model, damage, failure criteria

Introduction

In Part I of this study [1], a new three-dimensional constitutive model for transversely isotropic (unidirectional) composite laminates has been proposed in the framework of a kinematically constrained spectral stiffness microplane (SSM) model [2]. The formulation exploits the spectral decomposition of the orthotropic stiffness matrix to decompose the strain tensor into four orthogonal strain modes. Strain Modes I and III govern the transverse behavior (which is matrix dominated), Mode II governs the longitudinal behavior (which is fiber dominated), and Mode IV governs the behavior in pure shear. This mode decomposition is a generalization of the deviatoric-volumetric split already adopted in the formulation of microplane models for isotropic materials [3,2]. As it will be seen, in the elastic regime, the four strain modes can be considered to be uncoupled, whereas in the inelastic regime a certain degree of coupling is necessary to achieve good fits of the experimental results.

In the present Part II, the model will be calibrated and validated by comparing numerical results to published experimental data found for typical uniaxial and biaxial tests. Simulations of the behavior of unidirectional and multidirectional carbon–epoxy laminates (AS4 3501-6) will be compared with the experimental data reported in Refs. [4,5] (and references therein). This collection of data was used in the “World Wide Failure Exercise” [6–8], for the purpose of comparing the accuracy of several failure criteria for laminates. Among them, the Tsai–Wu criterion [9] performed the best.

The literature, unfortunately, does not include, for one and the same laminate, a complete set of test data for both the multiaxial failure envelopes and the stress-strain behavior, not even the uniaxial behavior. Therefore, the failure envelopes from other

kinds of laminates in Ref. [5] will be scaled, in a proper way, to deduce the likely form of the missing experimental curves for the carbon–epoxy laminates studied here.

The calibrated model will subsequently be used to predict the behavior of multidirectional laminates. Only short-time loading, for which creep is unimportant, and loading rates low enough for dynamic wave propagation effects to be absent, will be considered. All the definitions and notations from Part I will be retained.

Elastic Behavior of Unidirectional Laminates

The elastic behavior of unidirectional carbon–epoxy laminates depends on the elastic properties of the epoxy matrix, and the elastic properties of the fibers. Micromechanical models might be used to get approximate values of the elastic moduli of the laminates based on the experimental data for their components. However, it is preferable to deduce these moduli from the experimental data measured for the given laminate. The experimental elastic properties for carbon/epoxy AS4 3501-6 reported in Ref. [4] are: $E_{33}=126$ GPa, $E_{11}=11$ GPa, $\nu_{12}=0.4$, $\nu_{31}=0.28$, and $G_{13}=6.6$ GPa, where E_{33} =longitudinal modulus (in the fiber direction); E_{11} =transverse modulus (in the direction orthogonal to the fibers); $\nu_{12}=\nu_{21}$ =transverse Poisson ratio; ν_{31} ($=\nu_{32}$)=major Poisson ratio; and G_{13} ($=G_{23}$)=in-plane shear modulus. Note that the longitudinal modulus E_{33} is actually a secant modulus because the elastic uniaxial stress-strain curve shows (Fig.2) a stiffening behavior due to straightening of the fibers in tension. In absence of more precise experimental data, the same elastic modulus is also assumed to govern the elastic behavior for compression in the longitudinal direction.

The elastic moduli at the microplane level are easy to identify because they equal the eigenvalues of the stiffness matrix [1] associated with the elastic parameters introduced previously, which are: $\lambda_1=7.9$ GPa, $\lambda_2=145.4$ GPa, $\lambda_3=18.2$ GPa, and $\lambda_4=13.2$ GPa. The microplane elastic stiffness relevant to strain Mode II (λ_2) is significantly larger than the others (λ_1 , λ_3 , and λ_4). This confirms that strain Mode II is the mode governing the laminate behavior for loading in fiber direction.

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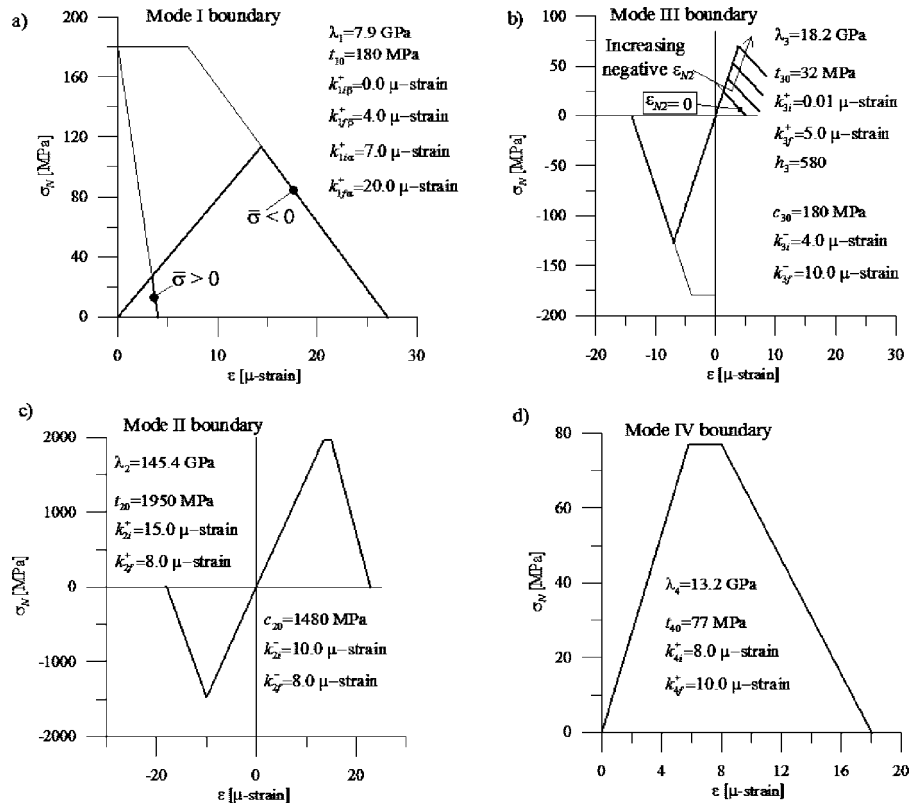


Fig. 1 Stress-strain boundary for the microplane normal component of: (a) Mode I; (b) Mode III; (c) Mode II; and (d) Mode IV

Uniaxial Stress-Strain Curves of Unidirectional Laminates

For the uniaxial case, the calibration procedure consists of two steps. First the experimental stress-strain curves are matched in the prepeak region. The post-peak softening parameters are adjusted to match the fracture energy, for which the width of the localization band h [10], representing a material characteristic length, is also needed. In theory, the value of h must be obtained from the fracture energy measured a priori by fracture tests. However, since no data are available, an educated guess based on the size of the inhomogeneities is the only option left. We consider the localization band width of $h=2$ mm (for specimens of length $L=100$ mm) and we assume the same band width for fracture, compression failure, and shear failure. Doubtless, this assumption is crude and, in general, different band widths may be expected to be associated with different phenomena. Nevertheless, by experience with other materials [11], this assumption seems adequate for finite element simulations, and having only one band width is convenient since the characteristic finite element size is usually made to coincide with this width.

Let us now analyze the laminate behavior under transverse compression and tension. The experimental data available for these loading conditions are the peak stress in tension and compression ($F_{1t}=48$ MPa, $F_{1c}=-200$ MPa), peak strains in tension and compression ($\epsilon_{1t}=4.36$ μstrain, $\epsilon_{1c}=-20$ μstrain), stress-strain curve in compression up to the peak, and the interlaminar fracture energy ($G_f^t=220$ J/m²) identified from a double cantilever beam specimen [4]. The interlaminar fracture energy can be assumed to be a good approximation of the transverse fracture energy. This assumption is supported by Daniel and Lee's [12] measurements of the progressive damage in $[0_n/90_m]_s$ coupons of graphite-epoxy, in which the 90 deg layers fractured and the 0 deg layers remained elastic. From these experiments, it is pos-

sible to deduce the fracture energy in transverse direction, and its value turns out to be very close to the aforementioned value of the interlaminar fracture energy.

The transverse behavior is governed by Mode I and Mode III [1]. The identified microplane parameters that fit the aforementioned macroscopic properties (for $h=2$ mm) are $t_{10}=c_{10}=180.0$ MPa, $k_{1i\alpha}^+=k_{1i\alpha}^-=7.0 \times 10^{-6}$, $k_{1f\alpha}^+=k_{1f\alpha}^-=20.0 \times 10^{-6}$, $k_{1i\beta}^+=k_{1i\beta}^-=0.0$, $k_{1f\beta}^+=k_{1f\beta}^-=4.0 \times 10^{-6}$, $t_{30}=32.0$ MPa, $k_{3i}^+=0.01 \times 10^{-6}$, $k_{3f}^+=5.0 \times 10^{-6}$, $c_{30}=180.0$ MPa, $k_{3i}^-=4.0 \times 10^{-6}$, $k_{3f}^-=10.0 \times 10^{-6}$, $s_{10}=77$ MPa, and $s_{30}=90$ MPa.

Figures 1(a) and 1(b) show the stress-strain boundaries of the microplane normal component for Mode I (tension) and Mode III (tension and compression) according to the foregoing parameters.

The parameters of Mode I compressive boundary are assumed to be equal to those of the tensile boundary. This assumption is necessary because strain Mode I is a quasideviatoric mode in the plane of isotropy exhibiting both tension and compression at the microplane level, regardless of the sign of the applied macroscopic transverse stress. The macroscopic behavior is, of course, not symmetric and the tensile strength is much smaller than the compressive strength. To take into account this asymmetry, the Mode I boundary has been made to depend upon the mean stress $\bar{\sigma}$ in the plane of isotropy (Fig. 1(a)); $\bar{\sigma}$ provides a measure of confinement of the fiber and permits distinguishing between Mode I strains associated with the macroscopic transverse tension ($\bar{\sigma} > 0$) and with the macroscopic transverse compression ($\bar{\sigma} < 0$).

Because of the lack of data in the post-peak portion of the compressive stress-strain curve, parameters $k_{1i\alpha}^-$, $k_{1i\beta}^-$, $k_{1f\alpha}^-$, $k_{1f\beta}^-$, k_{3i}^- , and k_{3f}^- cannot be properly calibrated. The reported values are obtained by matching only the pre-peak nonlinearity (Fig. 2(d)). The identification of these parameters is, of course, less than satisfactory and further studies are needed.

Figures 2(c) and 2(d) compares the experimental data (points)

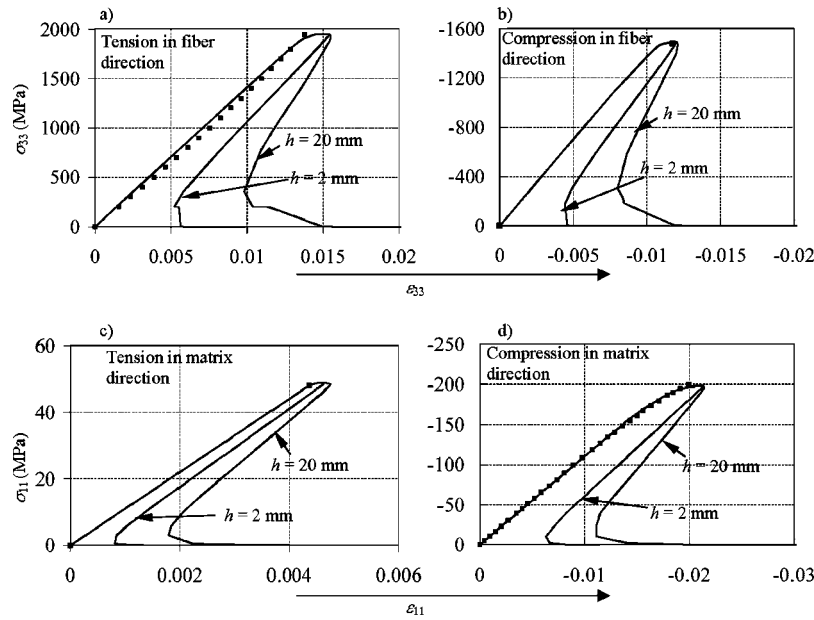


Fig. 2 Uniaxial stress-strain curves. Comparison between numerical simulations (solid line) and experimental results (points) from Soden et al. [4,5] for: (a) tension in fiber direction; (b) compression in fiber direction; (c) tension in transverse direction; and (d) compression in transverse direction

with the calculated stress-strain curves (solid lines) for $h=2$ mm. The curves clearly exhibit snapback [13], which explains why, in experiments, the failure of laminates is dynamic and sudden, making the post-peak equilibrium curve unobservable.

For comparison, Figs. 2(c) and 2(d) also show the stress-strain curves obtained with the aforementioned parameters and $h=20$ mm. Of course in this case the fracture energy turns out to be ten times larger than the experimental value.

As far as the uniaxial behavior in the fiber direction is concerned, the available experimental data consist of the peak stress in tension and compression ($F_{3t}=1950$ MPa, $F_{3c}=1480$ MPa), peak strain in tension and compression ($\varepsilon_{3t}=13.8 \times 10^{-6}$, $\varepsilon_{3c}=-11.75 \times 10^{-6}$), and the stress-strain curve in tension up to the peak. Data are available on the post-peak for neither tension nor compression. The peak stress in tension is matched by setting $t_{20}=1950$ MPa and $k_{2t}^+=15.0 \times 10^{-6}$.

The fitting of the compressive peak stress is slightly more complicated because high compressive stress in the direction of the fibers generates Mode III tensile stresses on those microplanes whose normal orientation is close to the transverse direction. These stresses must quickly lead to a splitting-like failure because of the low tensile strength associated with Mode III. The experimental results, however, do not indicate this kind of failure. On the contrary, they show the failure to be triggered by microbuckling of the fibers. Since microbuckling must be associated with Mode II, and not with Mode III, the splitting-like failure must be somehow prevented in these simulations. To this end, the peak of Mode III normal boundary is radially scaled up as a function of the negative Mode II strain (Fig. 1(b)). The radial scaling is formulated through functions h_3^+ and k_{3i}^+ presented in Part I [1]. The identified microplane parameters are $c_{20}=1480$ MPa, $k_{2i}^-=10.0 \times 10^{-6}$, $s_{20}=77$ MPa, and $h_3=580$ (h_3 governs the radial scaling of the boundary associated with Mode III as a function of the negative normal strain of Mode II).

In absence of any information about the post-peak part of the stress-strain curves, it is impossible to identify from such curves parameters k_{2f}^+ and k_{2f}^- that define the post-peak slope of the normal boundary, associated with Mode II for tension and compression, respectively. We assume $k_{2f}^+=k_{2f}^-=8.0 \times 10^{-6}$. Figure 1(c)

shows the compressive and tensile boundaries for Mode II used in the numerical simulations, and Figs. 2(a) and 2(b) show the comparison between the experimental data (points) and the numerical simulations (solid lines).

To conclude the analysis of uniaxial behavior of the unidirectional laminate, we need to analyze the response under macroscopic in-plane shear stress. This response is governed solely by the strain Mode IV [1]. The optimization of the relevant microplane model parameters can be done by fitting the stress-strain curve, which, however, was available only up to the peak. The best fits, shown in Fig. 3, have been obtained by setting $t_{40}=c_{40}=77.0$ MPa, $k_{4i}^+=k_{4i}^-=8.9 \times 10^{-6}$. We also assumed that $k_{4f0}^+=k_{4f0}^-=10.0 \times 10^{-6}$. These last two parameters govern the undocumented post-peak segment of the curve.

Biaxial Failure Envelope of Unidirectional Laminates Without Mode Interaction

Let us now analyze failure of the carbon-epoxy unidirectional laminates under biaxial loading. Following the "World Wide Failure Exercise," we consider three different biaxial loading conditions: (1) transverse and in-plane shear loadings; (2) longitudinal

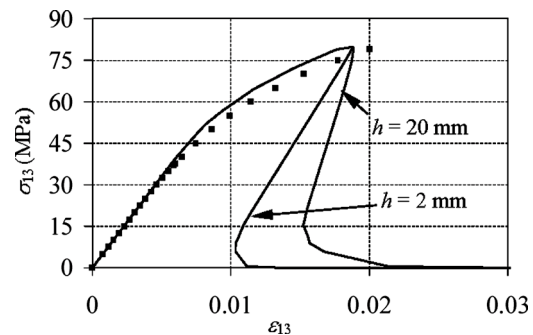


Fig. 3 Comparison between numerical simulations (solid line) and experimental results (points) from Soden et al. [4,5] for shear loading

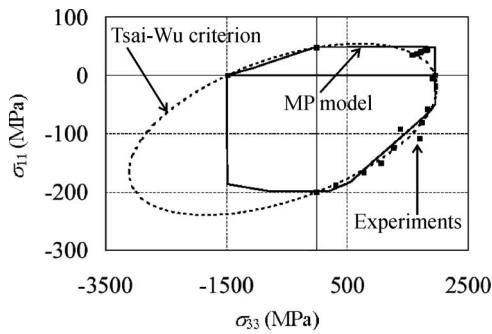


Fig. 4 Comparison between numerical simulations (solid line), Tsai-Wu criterion (dashed line), and experimental results (points) from Soden et al. [4,5] for multiaxial failure envelope with: (a) no interaction of modes; and (b) with interaction

and in-plane shear loadings; and (3) longitudinal and transverse loadings. Unfortunately, the associated failure envelopes available in the literature and reported in Ref. [5] were measured on three different kinds of laminates. To check the ability of the model to predict, with the same set of parameters, the response of unidirectional carbon-epoxy laminates under all the three aforementioned loading conditions, the biaxial failure envelopes are assumed to be scaled radially according to the uniaxial strengths.

Figure 4 compares the experimental data (points), the microplane model prediction (solid line), and the prediction of the Tsai-Wu criterion (dashed line), which performed the best during the “World Wide Failure Exercise.”

In the first quadrant (tension-tension), the microplane model, and the Tsai-Wu criterion agree quite well for matrix-dominated failures (low and moderate longitudinal stresses). For high longitudinal stresses, the Tsai-Wu criterion is more conservative than the microplane model. Unfortunately, the experimental results do not help to validate one theory more than the other because they are quite scattered and lie in the middle between the two predicted curves. In the second quadrant (longitudinal tension—transverse compression) the agreement between the two theories and the ex-

perimental results is excellent. The same good agreement can be seen in the fourth quadrant (transverse tension—longitudinal compression).

An important discrepancy between the microplane model and the Tsai-Wu criterion is in the third quadrant (compression—compression), in which the Tsai-Wu criterion predicts a much larger strength than the microplane model. Unfortunately, there are no experimental data available to check which theory is correct.

Figure 5(a) shows the failure envelopes for transverse and in-plane shear loading. In this case, the microplane model prediction resembles the maximum stress criterion and is poor. The prediction of the Tsai-Wu criterion is somewhat better but it does not predict the skew character of the envelopes, which shows that the shear strength in the presence of transverse compression is higher, as expected. A similar picture arises from analyzing the failure envelope for longitudinal and in-plane shear loading. Without mode interaction (Fig. 5(c)), the prediction of the microplane model cannot be fully accurate, but still is reasonable.

Formulation With Mode Interaction

Performance of the microplane model can be significantly enhanced even if only a limited interaction between the strain modes is introduced. In Fig. 5(a), the vertical portions of the envelope are associated with failures in the matrix and fiber-matrix interface. Consequently they are governed by Modes I and III. On the contrary, the horizontal portion of the envelope is associated with pure shear failure and is governed by Mode IV. Therefore, to match the experimental data, one needs to introduce interaction between Modes I, III and Mode IV. This can be achieved by formulating the following interaction functions

$$f_{14}^+(\epsilon_p) = f_{14}^-(\epsilon_p) = g_{14}(\epsilon_p) = f_{34}^+(\epsilon_p) = f_{34}^-(\epsilon_p) = g_{34}(\epsilon_p) = 1 + \tau_4 \epsilon_4 \quad (1)$$

$$f_{41}^+(\epsilon_p) = f_{41}^-(\epsilon_p) = g_{41}(\epsilon_p) = \langle 1 - \text{sign}(\epsilon_{N3}) |\epsilon_{N1}| / \tau_1 \rangle \quad (2)$$

where $\tau_4 = \tau_4^-$ for $\bar{\sigma} < 0$, and $\tau_4 = \tau_4^+$ for $\bar{\sigma} > 0$.

The best fit of the failure envelope in Fig. 5(b) can be obtained upon setting $\tau_4^+ = 0.1$, $\tau_4^- = 0.02$, and $\tau_1^+ = 0.002$. The microplane

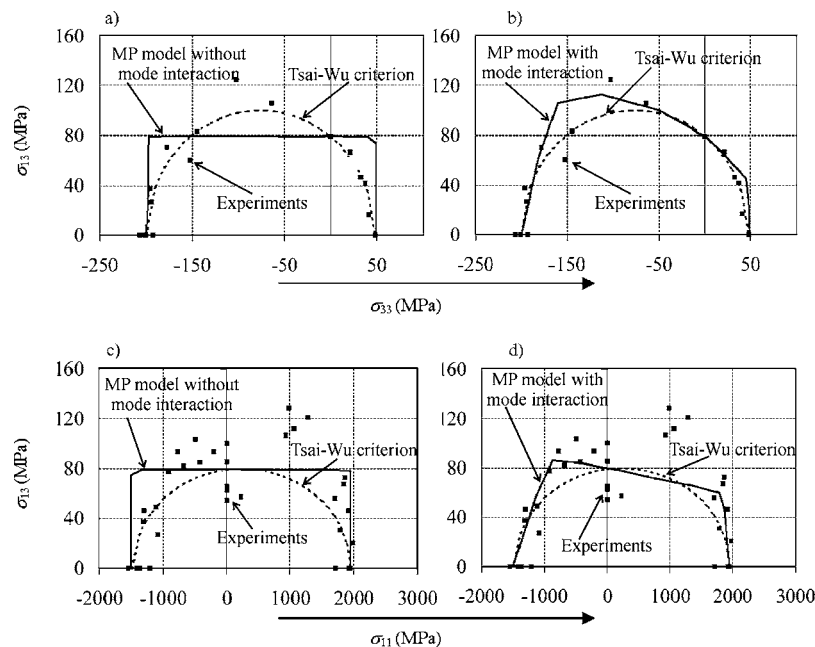


Fig. 5 Comparison between numerical simulations (solid line) and experimental results (points) from Soden et al. [4,5] for uniaxial loading in fiber direction

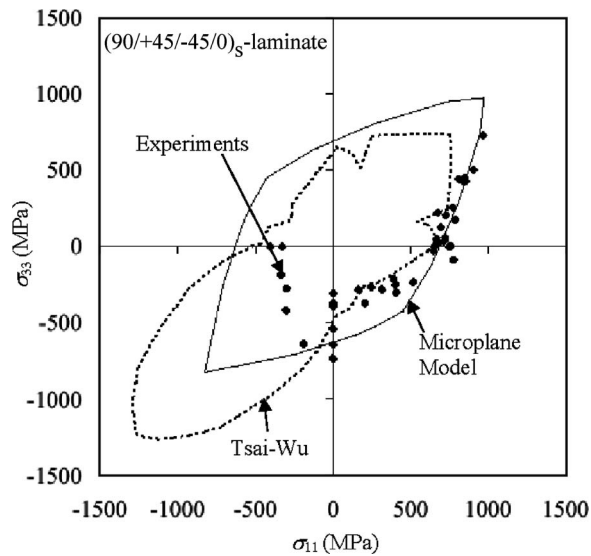


Fig. 6 Comparison between numerical simulations (solid line), Tsai-Wu criterion (dashed line), and experimental results (points) from Soden et al. [4,5] for multiaxial failure envelope with: (a) no interaction of modes; and (b) with interaction

model prediction is now in good agreement with the experimental data and is more accurate than the prediction of Tsai-Wu criterion.

The microplane prediction of the failure envelope in Fig. 5(c) can also be improved through mode interaction. In this case, the relevant modes are Mode II (behavior in fiber direction) and Mode IV (pure shear), and so it is these two that should be linked. The effect of Mode IV on Mode II is formulated in the same manner as the effect of Mode IV on Modes I and III

$$f_{24}^+(\epsilon_P) = \tilde{f}_{24}(\epsilon_P) = g_{24}(\epsilon_P) = 1 + \tau_4 \epsilon_4 \quad (3)$$

In addition, the effect of Mode II on Mode IV can be formulated as follows

$$f_{42}^+(\epsilon_P) = \tilde{f}_{42}(\epsilon_P) = g_{42}(\epsilon_P) = \langle 1 - \tau_2 \epsilon_{N2} \rangle \quad (4)$$

where $\langle x \rangle = \max(x, 0)$. The best fit of the failure envelope, shown in Fig. 5(d), is obtained by setting $\tau_2 = 50$.

Analysis of Multidirectional Laminates

A widely used laminate layup is $(90/+45/-45/0)_s$, which is quasi-isotropic. Several authors studied the AS4/3501-6 carbon-epoxy quasi-isotropic laminate. The relevant data and reference to the original publications can be found in Ref. [5]. The tests were carried out by subjecting tubular specimens to pressure and axial loads. To avoid spurious ruptures at the connections with the loading platens, the specimens had been reinforced at the ends. Because of this end constraint, it is reasonable to assume that the specimens could not have experienced any torsional rotation. This means that the plies were subjected only to longitudinal and circumferential (hoop) strains, and that the shear strains vanished in all the plies. This, in turn, implies the appearance of shear stresses in the +45 and -45 plies. These stresses are important and, if neglected, lead to a significant underestimation of the laminate strength.

The behavior of this multidirectional laminate is here simulated assuming each ply to be governed by the microplane model for unidirectional laminates (see Part I, [1]) and calibrated in the preceding sections. At this stage, no adjustment to the model parameters is allowed, so that the numerical solution would truly be a prediction. An iterative Newton-Rapson procedure is used to converge to a vanishing out-of-plane normal stress for all plies.

Figure 6 shows a comparison between the experiments (data points), the microplane model prediction (solid line), and the pre-

diction of the Tsai-Wu criterion (dashed line). The microplane model theory agrees very well with the experimental data in the tension-tension quadrant of the envelope. For the tension-compression quadrant, the prediction is less accurate but still satisfactory. However, marked disagreement is found in the compression-compression quadrant, in which both the microplane model and the Tsai-Wu criterion severely overestimate the laminate strength. This shortcoming is, most probably, due to the fact that the microplane calibration for the unidirectional laminate could not rely on any data for the compression-compression quadrant.

Conclusions

1. The spectral decomposition theorem is a powerful tool to analyze generally anisotropic materials. This is the *only* known exact and rigorous approach for the anisotropic generalization of the microplane model.
2. The mechanical behavior of laminate composites can be subdivided into characteristic loading conditions representing the longitudinal tension or compression, transverse tension or compression, and shear, most of which are dominated by one spectral mode, and none by more than two spectral modes.
3. Interaction of modes, which it has been possible to avoid, with minor exceptions, in the previous microplane formulations, helps to achieve better fit of some experimental multiaxial failure envelopes of laminates. Even if this interaction is not dominant, its neglect always causes a single mode to reach its strength limit before the others, which impairs the fits of some multiaxial data.
4. The present SSM model describes well the experimentally observed behavior of fiber composites, not only for uniaxial stress-strain curves, but also for multiaxial failure envelopes. The fitting capabilities for multiaxial failure envelopes are superior to the Tsai-Wu criterion, which has so far been considered as overall the best model for laminates.
5. The main advantage of the SSM model is that one and the same model can simulate the orthotropic stiffness, failure envelopes, and the post-peak behavior, which include strain-softening damage and fracture mechanics aspects. This further implies that the SSM model must be able to automatically predict the energetic size effect.
6. The material parameters of the SSM model can be identified from experiments by a sequential procedure. Post-peak data are needed to identify the parameters governing the softening, and an estimate can be based on the width of the localization band taken equal to the inhomogeneity size.
7. The experimental failure envelopes can be represented by the present model as well or better than with the existing failure criteria, including Tsai-Wu.
8. Although, due to stability limitations of current testing, post-peak measurements are missing, the post-peak softening curve can be approximately inferred from the notched fracture energy test and an estimate of the fracture process zone width.
9. The SSM model can be implemented as a material subroutine in finite element codes, either implicit or explicit. From experience with microplane models for concrete (used in the commercial code ATENA) the kinematically constrained formulation is known to be very stable in finite element analysis.

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Appendix: Calibration Procedure

The procedure of sequential identification of microplane model parameters may be summarized as follows:

- (1) Identify elastic parameters λ_1 , λ_2 , λ_3 , and λ_4 (eigenvalues of the stiffness matrix) on the basis of the measured elastic properties (see Eqs. (28)–(31) in Ref. [1]).
- (2) Identify parameters t_{20} , c_{20} , and s_{20} on the basis of the uniaxial tensile, compressive, and shear strength in the fiber direction.
- (3) Identify parameters k_{2i}^+ , k_{2f}^+ , k_{2i}^- , k_{2f}^- according to the post-peak softening behavior of the unidirectional lamina tested in fiber direction.
- (4) Identify parameters t_{10} , c_{10} , s_{10} , t_{30} , c_{30} , and s_{30} on the basis of the uniaxial tensile, compressive, and shear strength in the direction transverse to the fiber.
- (5) Identify parameters k_{3i}^+ , k_{3f}^+ , k_{3i}^- , k_{3f}^- , $k_{1i\alpha}^+$, $k_{1i\alpha}^-$, $k_{1f\alpha}^+$, $k_{1f\alpha}^-$, $k_{1i\beta}^+$, $k_{1i\beta}^-$, $k_{1f\beta}^+$, $k_{1f\beta}^-$ according to the post-peak softening behavior of the unidirectional lamina tested in direction transverse to the fiber.
- (6) Identify parameters t_{40} , c_{40} , s_{40} , k_{4i}^+ , k_{4i}^- , k_{4f}^+ , k_{4f}^- according to the in-plane shear strength and post-peak softening characteristics of the unidirectional laminate.

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Synchronization of Multiple Chaotic Gyroscopes Using the Fundamental Equation of Mechanics

This paper provides a simple, novel approach for synchronizing the motions of multiple “slave” nonlinear mechanical systems by actively controlling them so that they follow the motion of an independent “master” mechanical system. The multiple slave systems need not be identical to one another. The method is inspired by recent results in analytical dynamics, and it leads to the determination of the set of control forces to create such synchronization between highly nonlinear dynamical systems. No linearizations or approximations are involved, and the exact control forces needed to synchronize the nonlinear systems are obtained in closed form. The method is applied to the synchronization of multiple, yet different, chaotic gyroscopes that are required to replicate the motion of a master gyro, which may have a chaotic or a regular motion. The efficacy of the method and its simplicity in synchronizing these mechanical systems are illustrated by two numerical examples, the first dealing with a system of three different gyros, the second with five different ones. [DOI: 10.1115/1.2793132]

1 Introduction

Gyrodynamics is an area of mechanics that has been of significant interest for more than a century to both the scientific and the engineering communities. Gyroscopes, from a purely scientific viewpoint, show many strange and interesting properties, and from an engineering viewpoint, they have great utility in the navigation of aircraft, rockets, and spacecraft and in the control of complex mechanical systems. It has been known for some time now [1–6] that symmetric gyros, when subjected to harmonic vertical base excitations, exhibit a variety of interesting dynamic behaviors that can span the range all the way from regular to chaotic motions. Various investigators have looked at gyro models that involve different types of damping, the most common type being linear plus cubic [3–5]. Depending on the parameters that describe these gyrosystems, they can exhibit fixed points, periodic behavior, period doubling behavior, quasiperiodic behavior, and chaotic motions.

Synchronization of two chaotic systems is an important problem in nonlinear science, and it has received considerable attention in recent years since it was first carried out by Pecora and Carroll [7] and Lakshmanan and Murali [8]. When one has more than one gyro operating in a mechanical system, synchronizing these gyros so that a master gyro drives a bunch of slave gyros in such a manner that the slaves “exactly” replicate the motion of the master is a problem of considerable interest both in navigation and in the transmission of encrypted messages [9]. While many researchers have considered the synchronization of two coupled chaotic systems whose motions may or may not synchronize depending on the coupling between them, in this paper we consider the synchronization of a set of “slave” mechanical systems that may or may not be coupled, each synchronized to the motions of an independent “master” mechanical system.

The way the synchronization of the motion of two chaotic systems has been usually achieved—the systems are usually, it appears, taken to be *identical*, but starting with different initial

conditions—is through the application of a control signal (a coupling) to one of them (the slave system), which is often some linear or nonlinear function of the difference in the motion between the master and the slave. The methodology is perhaps best described as belonging to a kind of generalized feedback control philosophy. For example, Chen [4] considered two identical chaotic gyros, used a variety of such control laws, and showed that when the feedback gain exceeds a certain value, the slave gyro synchronizes with the master gyro. The value of this feedback gain, above which such synchronization occurs, is typically obtained through numerical experimentation [4]. Modern nonlinear control theory has also been used to look at the gyro synchronization problem. Here, the system is conceived as an autonomous set of first order nonlinear differential equations, and the difference in the response between the master and the slave gyro is taken to be an error signal. A suitable time-varying control is then applied to the slave gyro to drive this error signal to zero. Often, this is done by using feedback linearization; the nonlinear terms in the equation governing the error signal are eliminated, and then standard linear feedback control theory is applied [10]. Such strategies, which may be commonly found in the literature, become difficult, if not impossible, to use when we have many slaves that may be coupled to one another (not just one) and that need to be driven to yield the same motions as a single independent master, and especially so when the dynamical characteristics of these slaves are *not* identical with one another and/or with those of the master gyro. Considering that it is very difficult to exactly replicate the properties of multiple mechanical systems even when they “seem” identical, it is interesting that the problem of driving *nonidentical* slaves using a master that may also be different from each of the slaves has only recently begun to be broached in the nonlinear science literature [11,12].

In this paper, we explore a new and different strategy for synchronizing the response of n nonlinear mechanical systems that is inspired by some recent advances in analytical dynamics [13]. We consider a system of n gyros—not necessarily identical—some, or all, of which may exhibit a chaotic behavior, and we pose the problem of synchronizing the motion of all the others with, say, that of the i th gyro (the master). We frame this in the context of a tracking control problem, in which the $n-1$ slave gyros are re-

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quired to exactly track the motion of the master gyro. We then further reformulate the tracking problem as a problem of constrained motion, where we want the control (constraint) forces to be such that all the gyros, which are highly nonlinear systems, are constrained to have the same motion. We use the explicit closed form analytical control given by the fundamental equation [13] to then yield the control force that will cause, in a theoretical sense, *exact* synchronization of these gyros. We show that this approach to the synchronization of such gyroscopic systems—and, indeed, general nonidentical, nonlinear mechanical systems—which is based on these deeper results from analytical mechanics, has several advantages, most important of which are that the control forces obtained are continuous functions of time and that they can be found in closed form and hence can be determined simply and efficaciously. Furthermore, in a sense, the minimum forces that need to be exerted to synchronize these nonlinear systems are obtained, and they yield, theoretically speaking, exact synchronization. As we shall show, of some importance is the fact that the manner in which synchronization is achieved can be controlled easily and with little difficulty.

The paper is organized as follows. In Sec. 2, we provide a brief description of the equation of motion of a symmetric gyro subjected to a vertical periodic base motion. We use the Lagrangian approach and obtain the requisite equations of motion. In Sec. 3, we present the fundamental equation that provides the explicit equation of motion for general nonlinear mechanical systems that are constrained. In Sec. 4 (and in Appendix B), we apply the fundamental equation to the problem of synchronizing n gyros, providing a closed form solution to the determination of the control forces required to be applied to each of these nonlinear systems that yields exact synchronization of their motions. In Sec. 5, we present several numerical results to illustrate the behavior of the proposed control, and its simplicity and efficacy. In the last section, we present our conclusions.

2 Equation of Motion for the Symmetric Gyro

Consider the symmetric gyro, whose point of support, o , undergoes a vertical harmonic motion of frequency ω and amplitude d_0 , as shown in Fig. 1. Using the Euler angles θ (nutation), φ (precession), and ψ (spin) [14], the Lagrangian for the system is given by (see Appendix A¹)

$$L = \frac{1}{2}I(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\varphi} \cos \theta)^2 - mgr \sin \theta - mgr \cos \theta \quad (1)$$

where m is the mass of the gyro, $I := I_1 + mr^2$, $I_1 = I_2$ is the principal equatorial moment of inertia through the center of mass (c.m.) of the gyro, and I_3 is the polar moment of inertia about the symmetry axis. In Fig. 1, the point of support of the gyro is denoted by o , so that the moments of inertia about the axes ox and oy are each equal to I . The dots in Eq. (1) refer to differentiation with respect to time t . The quantity r denotes the distance along the polar axis of the c.m. of the gyro from its point of support, and $d(t) = d_0 \sin \omega t$ is the time-varying amplitude of the vertical support motion that has frequency ω .

Since φ and ψ are cyclic coordinates, the corresponding angular momenta $p_\psi = I_3(\dot{\psi} + \dot{\varphi} \cos \theta)$ and $p_\varphi = I\dot{\varphi} \sin^2 \theta + p_\psi \cos \theta$ are conserved. The angular velocities $\dot{\varphi}$ and $\dot{\psi}$ can be eliminated by using the Routhian [14],

¹We provide the Lagrangian in Appendix A. This is specifically because the Lagrangian given in Ref. [2] is incorrect and, consequently, the equation of motion obtained from it is also invalid. Unfortunately, this error has found its way into the current literature dealing with this topic, as in Refs. [1–5] and Ref. [10].

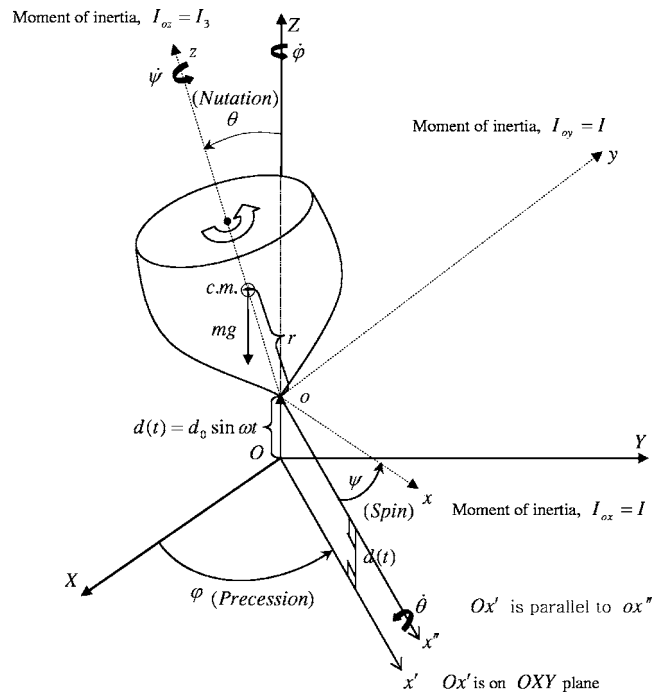


Fig. 1 Symmetric gyro with vertical support excitation $d(t) = d_0 \sin(\omega t)$

$$R(\theta, \dot{\theta}, t) = L - p_\varphi \dot{\varphi}(p_\varphi, p_\psi, \theta) - p_\psi \dot{\psi}(p_\varphi, p_\psi, \theta) \quad (2)$$

The equation of motion, which is given by $(d/dt)(\partial R / \partial \dot{\theta}) - \partial R / \partial \theta = F_d$, then reduces to

$$I\ddot{\theta} + \frac{(p_\varphi - p_\psi \cos \theta)(p_\psi - p_\varphi \cos \theta)}{I \sin^3 \theta} - mgr \sin \theta - mr \sin \theta \ddot{d}(t) = F_d \quad (3)$$

where F_d is the nonconservative force of damping, which we take here to be of linear-plus-cubic type [3], so that $F_d = -\hat{c}\dot{\theta} - \hat{e}\dot{\theta}^3$. Along with previous researchers [2–5], for simplicity, we only consider damping related to the θ coordinate.

Were we to further assume that $p_\varphi = p_\psi = \bar{p}$ (which permits the gyro to be in the so-called “sleeping” position, removing the singularity in Eq. (2)), Eq. (3) can be further simplified to

$$\ddot{\theta} + \alpha^2 \frac{(1 - \cos \theta)^2}{\sin^3 \theta} + c\dot{\theta} + e\dot{\theta}^3 - \beta \sin \theta = -\gamma \sin \theta \sin \omega t \quad (4)$$

Under this assumption, Eq. (4) then is the differential equation that describes the motion of the symmetric gyro, where we have denoted $\alpha = \bar{p}/I$, $c = \hat{c}/I$, $e = \hat{e}/I$, $\beta = mgr/I$, and $\gamma = \omega^2 mrd_0/I$. The parameter set $P = \{\alpha, \beta, c, e, \gamma, \omega\}$ specifies the physical characteristics of the gyro and the harmonic vertical motion of the base on which it is supported. It may be pointed out that no assumption on the magnitude of the vertical displacement d_0 of the base has been made in arriving at this equation. We note in passing that no singularity arises in Eq. (4) due to the $\sin \theta$ term in the denominator in Eq. (4).

3 Fundamental Equation

This equation deals with the explicit equation of motion for a mechanical system when the system is constrained to satisfy a set of consistent constraints. Consider an unconstrained discrete mechanical system whose equation of motion is described by the equations

$$\mathbf{M}(t, \mathbf{q}) \ddot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t) \quad \mathbf{q}(0) = \mathbf{q}_0 \quad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0 \quad (5)$$

where $\mathbf{M}$ is an $n \times n$ symmetric, positive definite matrix, the n vector $\mathbf{q}$ represents the generalized coordinates used to describe the configuration of the system, and the right hand side is a known function of $\mathbf{q}$, $\dot{\mathbf{q}}$, and t . The dots refer to differentiation with respect to time. By unconstrained we mean here that the components of the initial velocity $\dot{\mathbf{q}}_0$ can be arbitrarily specified. Equation (5) results from the application of Lagrange's equations to a mechanical system, or from Newtonian mechanics.

Let this system be subjected to a set of s constraints of the form

$$\mathbf{h}(\mathbf{q}(t)) = 0 \quad (6)$$

that are satisfied by the initial conditions so that

$$\mathbf{h}(\mathbf{q}_0) = 0 \quad \text{and} \quad \dot{\mathbf{h}}(\mathbf{q}_0, \dot{\mathbf{q}}_0) = 0 \quad (7)$$

Here, $\mathbf{h}$, is an s vector. Differentiating Eq. (6) twice with respect to time, we obtain the set of matrix equations

$$\mathbf{A}(\mathbf{q}, \dot{\mathbf{q}}) \ddot{\mathbf{q}} = \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) \quad (8)$$

where $\mathbf{A}$ is an $s \times n$ matrix. The equation of motion of the constrained system that satisfies these constraints exactly is then explicitly given by [13]

$$\mathbf{M} \ddot{\mathbf{q}} = \mathbf{f}(\mathbf{q}, \dot{\mathbf{q}}, t) + \mathbf{F}^c(\mathbf{q}, \dot{\mathbf{q}}, t) \quad (9)$$

where

$$\mathbf{F}^c(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathbf{M}^{1/2}(\mathbf{A}\mathbf{M}^{-1/2})^+(b - \mathbf{A}\mathbf{M}^{-1}\mathbf{f}) \quad (10)$$

Here, $\mathbf{X}^+$ denotes the Moore–Penrose (MP) inverse of the matrix $\mathbf{X}$ (see Ref. 13). We shall denote the n components of the n vector $\mathbf{F}^c$ by f_i^c , $i=1, 2, \dots, n$. We notice that the constraint (Eq. (6)) is actually implemented as Eq. (8). In what follows, we shall suppress the arguments of the various quantities unless needed for clarity.

When relations (7) are not satisfied by the initial conditions, one could replace the equation set (Eq. (8)) by any other system of constraint equations [15] whose solution asymptotically tends to $\mathbf{h}=0$, as $t \rightarrow \infty$. For example, the system of equations

$$\ddot{\mathbf{h}} + \Delta \dot{\mathbf{h}} + \Sigma \mathbf{h} = 0 \quad (11)$$

where Δ and Σ are diagonal matrices with positive entries, would lead to $\mathbf{h} \rightarrow 0$ exponentially, as $t \rightarrow \infty$, and could be used by placing it in the form given in Eq. (8). It should be pointed out that the force $\mathbf{F}^c$ given by Eq. (10) minimizes, at each instant of time, the quantity $(\mathbf{F}^c)^T \mathbf{M}^{-1} \mathbf{F}^c$ —the weighted norm of the active control force $\mathbf{F}^c$ [13].

The general results obtained in analytical mechanics (see Ref. [13] for more details) are far more extensive than those presented above; here, we have particularized them to only cover the present problem of interest—synchronization of n nonidentical gyroscopes (see Ref. [15] for a more extensive treatment).

4 Synchronization of n Different Gyros

Consider n different, independent gyros described by the non-autonomous nonlinear equations,

$$\ddot{\theta}_i = -\alpha_i^2 \frac{(1 - \cos \theta_i)^2}{\sin^3 \theta_i} - c_i \dot{\theta}_i - e_i \dot{\theta}_i^3 + \beta_i \sin \theta_i - (\gamma_i \sin \theta_i) \sin \omega_i t \quad (12a)$$

$$i = 1, 2, \dots, n$$

$$:= f_i(\theta_i, \dot{\theta}_i, t; P_i) \quad i = 1, 2, \dots, n \quad (12b)$$

with

$$\theta_i(t=0) = \theta_i^0 \quad \text{and} \quad \dot{\theta}_i(t=0) = \dot{\theta}_i^0 \quad i = 1, 2, \dots, n \quad (13)$$

We have explicitly included the parameter set $P_i = \{\alpha_i, \beta_i, c_i, e_i, \gamma_i, \omega_i\}$ on the right hand side of Eq. (12b), indicating that each of the n symmetric gyros could have different physi-

cal characteristics and may be mounted on surfaces that harmonically vibrate vertically at different frequencies and with different amplitudes of vibration.

Our aim is to synchronize the motion of all n gyros so that $n-1$ of them “follow” the motion of the master gyro. Without any loss of generality, from here on we shall take the master gyro to be the first gyro in our set of n gyros and refer to it (the master gyro) by the subscript 1. Hence, we require

$$\theta_i(t) = \theta_1(t) \quad i = 2, \dots, n \quad (14)$$

where $\theta_1(t)$ is the solution of the nonlinear, nonautonomous differential equation given in Eq. (12a) with $i=1$. We note that the equation set (Eq. (14)) constitutes a set of $n-1$ independent conditions. The problem of synchronization can be interpreted as one of ensuring that the tracking conditions (Eq. (14)) are satisfied by the gyros whose equations of motion are given by Eqs. (12a) and (12b). Alternatively, we think of this problem as one in which Eqs. (12a), (12b), and (13) represent an unconstrained, n degree of freedom, mechanical system on which the $n-1$ independent constraints (14) are required to be imposed. In fact, we can modify this set of constraints to include all the $s := n(n-1)/2$ constraints,

$$h_{ij}(t) = (\theta_i(t) - \theta_j(t)) = 0 \quad \forall i < j \quad i, j \in (1, n) \quad (15)$$

of which $(n-1)(n-2)/2$ are redundant, though all of them are consistent [11,16]. Enforcing these constraints would make the motion of all the gyros identical. As mentioned before, among these s constraints, only $(n-1)$ are independent. Noting that in general the initial conditions (Eq. (13)) may not satisfy the constraints (Eq. (14)) (or, alternatively, Eq. (15)), we further modify the constraints (Eq. (15)) to

$$\ddot{h}_{ij} + \delta \dot{h}_{ij} + k h_{ij} = 0 \quad \forall i < j \quad i, j \in (1, n) \quad (16)$$

where δ and k are positive constants [15]. Since the solution of the set of s equations given by Eq. (16) satisfies the condition that $h_{ij} \rightarrow 0$ as $t \rightarrow \infty$, we have asymptotic (and exponential) convergence toward the satisfaction of the constraints (Eq. (15)) and hence obtain synchronization of the n different gyros.

It is important to point out that by altering the parameters δ and k in Eq. (16), one can describe different “paths” taken by the system of gyros toward their eventual synchronization. For simplicity, we have chosen the same constants δ and k for each equation of the set (16). In general, we could have used different values of δ and k for the different equations in this set (provided all the equations in the set are consistent with one another), signifying our intent to synchronize some of the gyros earlier (in time) than others since the values of δ and k for each of the equations in the set (16) control the rate and nature of convergence of $h_{ij}(t)$ to zero. Even more generally than is shown in the Eq. (16), we could have chosen the paths toward synchronization to be described by any set of consistent second order nonlinear differential equations that would be globally asymptotic to the solution $h_{ij}=0$, $i < j$, $i, j \in (1, n)$, so that the paths taken by the different gyros toward synchronization can be controlled pretty much at will.

Equations (16) can be put in the form of Eq. (8) where the n vector $\mathbf{q} = [\theta_1, \theta_2, \dots, \theta_n]^T$, so that

$$\mathbf{A} \ddot{\mathbf{q}} = -\delta \mathbf{A} \dot{\mathbf{q}} - k \mathbf{A} \mathbf{q} := \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) \quad (17)$$

where matrix $\mathbf{A}$ is an $s \times n$ matrix, containing 0's, 1's, and -1 's. For example, when we have four gyros so $n=4$ and $s=6$, the 6×4 matrix $\mathbf{A}$ takes the form

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad (18)$$

We note the form of matrix $\mathbf{A}$, which we will use to our advantage in our subsequent derivations: Each row of $\mathbf{A}$ has all its elements zero, except for two elements, which are 1 and -1 . As expected, only $(n-1)$ rows of matrix $\mathbf{A}$ are linearly independent. Comparing Eq. (5) with Eq. (12b), we see that the matrix $\mathbf{M}$ that describes the unconstrained motion of the mechanical system consisting of n gyros is given by $\mathbf{M} = \mathbf{I}_n$. Also, the n components of the n vector $\mathbf{f}$ in Eq. (5) are given by the f_i 's, $i=1, 2, \dots, n$ defined in Eq. (12b). From Eq. (10), the explicit generalized control force n vector, $\mathbf{F}^c$, required to enforce the constraint set (Eq. (17)) is given by

$$\mathbf{F}^c = \mathbf{A}^+(\mathbf{b} - \mathbf{A}\mathbf{f}) \quad (19)$$

where $\mathbf{A}^+$ is the MP inverse of matrix $\mathbf{A}$, the s vector $\mathbf{b}$ is given in Eq. (17), and the f_i given in Eq. (12b) form the n components of the n vector $\mathbf{f}$. For $n=4$ and matrix $\mathbf{A}$ given in Eq. (18), we easily determine (this can be done using MATLAB or MAPLE)

$$\mathbf{A}^+ = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} \quad (20)$$

which when substituted in relation (19) will yield the explicit control forces to exactly satisfy the s constraint equations (Eq. (17)) or, alternatively, (Eq. (16)).

Noting Eq. (9), we then see that the synchronized motion of the n gyros is obtained by providing the generalized control force f_i^c to the i th gyro, where f_i^c is the i th component of the n vector $\mathbf{F}^c$ obtained explicitly in Eq. (19). The equations of motion for the (asymptotically) synchronized gyros will then be

$$\ddot{\theta}_i = f_i(\theta_i, \dot{\theta}_i, t; P_i) + f_i^c \quad i = 1, 2, \dots, n \quad (21)$$

From Eq. (21), we observe that, in general, $f_1^c(t) \neq 0$. Hence, though the motion of all the gyros is fully synchronized (asymptotically) by subjecting the i th gyro to the control force f_i^c , the synchronized motion will, in general, not be that of the master gyro, unless $f_1^c = 0$. In order to synchronize the motion of the $(n-1)$ slave gyros with the motion of the first (master, $i=1$) gyro, we then need to simply subtract the force f_1^c from each component of the control force n vector $\mathbf{F}^c$ determined from Eq. (19). (The proof of this statement is somewhat long, and in order not to disturb the flow of thought, we present it in Appendix B.) The active control force needed to be applied to synchronize the remaining $n-1$ gyros with the motion of the first (master) gyro is then given by

$$\mathbf{F}^{\text{syn}} = \mathbf{F}^c - [\mathbf{1}]f_1^c = [0, f_2^c - f_1^c, f_3^c - f_1^c, \dots, f_n^c - f_1^c]^T \quad (22)$$

where $[\mathbf{1}]$ denotes the $n \times 1$ column vector each of whose elements is unity.

We thus obtain the equations of motion of the system of n gyros as

$$\ddot{\theta}_i = f_i(\theta_i, \dot{\theta}_i, t; P_i) + f_i^{\text{syn}} \quad i = 1, 2, \dots, n \quad (23)$$

where f_i^{syn} is the i th component of the control force n vector $\mathbf{F}^{\text{syn}}$ (explicitly given in Eq. (22)), which causes the slave gyros to exactly follow the motion of the master. Note that the first component of the n vector $\mathbf{F}^{\text{syn}}$ is zero since the first gyro ($i=1$) is the master gyro, so that from Eq. (23), we have $\ddot{\theta}_1 = f_1(\theta_1, \dot{\theta}_1, t; P_1)$. The nonidentical slave gyros ($i=2, 3, \dots, n$) are subjected to the

last $(n-1)$ components of the generalized control force n vector $\mathbf{F}^{\text{syn}}$, which thus enforces exact synchronization of the slave gyros with the master gyro's motion.

5 Numerical Examples

In this section, we consider two examples. The first example deals with the synchronization of three nonidentical gyros, each with its own physical characteristics. For the parameters chosen to describe these gyros, each gyro exhibits chaotic dynamics, and the two slave gyros are required to follow the master's chaotic motions. The second example deals with five different gyros, whose motion is required to be synchronized. One of the four slave gyros in this set has properties that show regular motion, the others have properties that show chaotic motions. They are synchronized with the motion of the master gyro, which in this example is periodic, though complex.

Example 1. Consider three gyros each described by Eqs. (12a) and (12b) that need to be synchronized so that they each follow the motion of the first (master) gyro. Each uncontrolled gyro exhibits a chaotic motion. We shall take these three dynamical systems to be different from each other, described by the parameter sets $P_i = \{\alpha_i, \beta_i, c_i, e_i, \gamma_i, \omega_i\}$, $i=1, 2, 3$, and their dynamics will be investigated for the initial condition sets $\text{IC}_i = \{\theta_i^0, \dot{\theta}_i^0\}$, $i=1, 2, 3$, given by

$$P_1 = \{10, 1, 0.5, 0.03, 35.8, 2.05\} \quad \text{IC}_1 = \{\theta_1^0 = -0.5, \dot{\theta}_1^0 = 1\} \quad (24)$$

$$P_2 = \{10, 1, 0.5, 0.05, 35.5, 2\} \quad \text{IC}_2 = \{\theta_2^0 = 0.5, \dot{\theta}_2^0 = 1\} \quad (25)$$

and

$$P_3 = \{10.5, 1, 0.5, 0.04, 38.5, 2.1\} \quad \text{IC}_3 = \{\theta_3^0 = 1, \dot{\theta}_3^0 = -0.5\} \quad (26)$$

The equation of motion (Eq. (12a)) for the i th gyro can be expressed as a set of three first order autonomous equations given by

$$\dot{\theta}_i = v_i$$

$$\dot{v}_i = -\alpha_i^2 \frac{(1 - \cos \theta_i)^2}{\sin^3 \theta_i} - c_i v_i - e_i v_i^3 + \beta_i \sin \theta_i - (\gamma_i \sin \theta_i) \sin \tau_i$$

$$\dot{\tau}_i = \omega_i \quad (27)$$

Each of the gyro systems described by the parameter sets P_i , $i=1, 2, 3$, given by Eqs. (24)–(26) is chaotic and has a different chaotic attractor.

The Lyapunov exponents for each of the dynamical systems are computed over a time span of 1000 s using the method described in Ref. [17]. The integration for determining these exponents is performed using MATLAB ODE45 using a relative error tolerance of 10^{-9} and an absolute error tolerance of 10^{-13} . The Lyapunov exponent sets, l_i , of the three different dynamical systems are computed to be $l_1 \approx \{0.211, -0.896, 0\}$, $l_2 \approx \{0.216, -1.001, 0\}$, and $l_3 \approx \{0.208, -0.936, 0\}$. The positive value of the largest Lyapunov exponent in each set indicates that the motions are chaotic for each of these gyros. Furthermore, the chaotic attractors for each system are different.

Figure 2 shows plots of $(\theta_i, \dot{\theta}_i)$, $i=1, 2, 3$, for $50 \leq t \leq 100$ for the three uncoupled gyros along with a figure (lower right corner) in which all three plots are superposed. The integration of the equations of motion throughout this study is carried out using MATLAB ODE45 with a relative error tolerance of 10^{-9} and an absolute error tolerance of 10^{-12} . The differences in the responses between the three gyros, $h_{ij}(t) = \theta_i(t) - \theta_j(t)$, are shown in Fig. 3.

We shall now use the scheme described in Sec. 4 to couple these gyros and synchronize them, the first gyro being the master. In this demonstration, the synchronization is done using equation

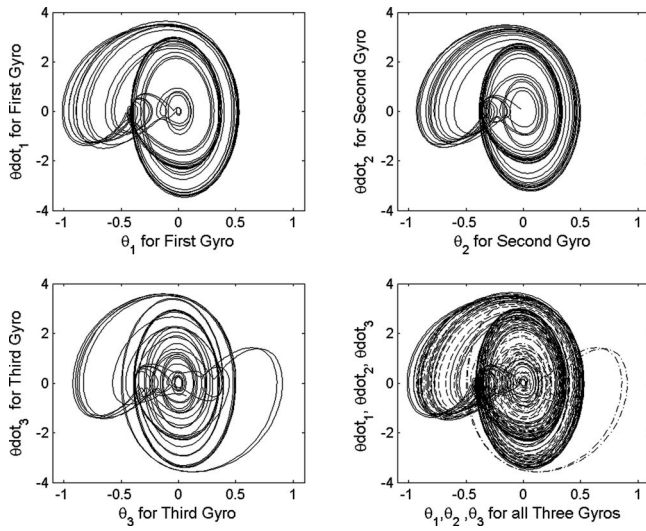


Fig. 2 $(\theta_i, \dot{\theta}_i)$ plots showing the dynamics of the three uncoupled gyros for $50 \leq t \leq 100$. The lower right corner shows these plots superposed on one another; the first gyro is shown with a solid line, the second with a dashed line, and the third with a dashed-dotted line.

set (16) using $\delta=1$ and $k=2$. Since we have three dynamical systems, the number of constraints for synchronization are given by $s=3$. Matrix **A** becomes

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \quad (28)$$

so that

$$\mathbf{A}^+ = \frac{1}{3} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \quad (29)$$

We note that only two rows of matrix **A** given in relation (28) are independent, signifying that we have two constraints that are in-

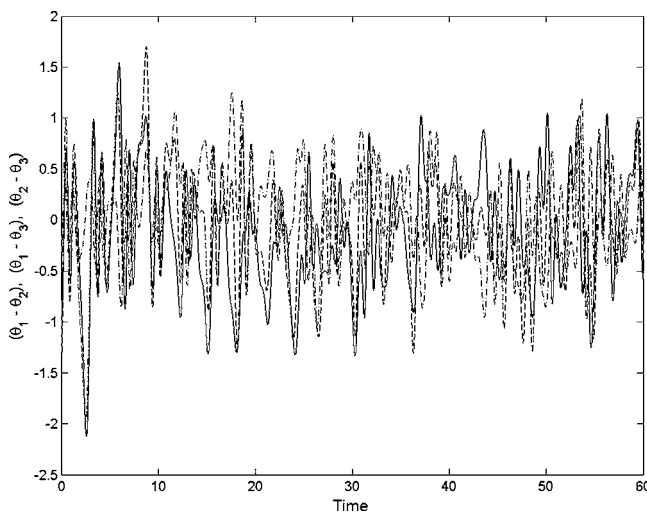


Fig. 3 The differences in the responses between the three uncoupled, unsynchronized gyros shown for a duration of 60 s. $h_{12}(t) = \theta_1(t) - \theta_2(t)$ is shown by the solid line, $h_{13}(t) = \theta_1(t) - \theta_3(t)$ is shown by the dashed line, and $h_{23}(t) = \theta_2(t) - \theta_3(t)$ is shown by the dashed-dotted line.

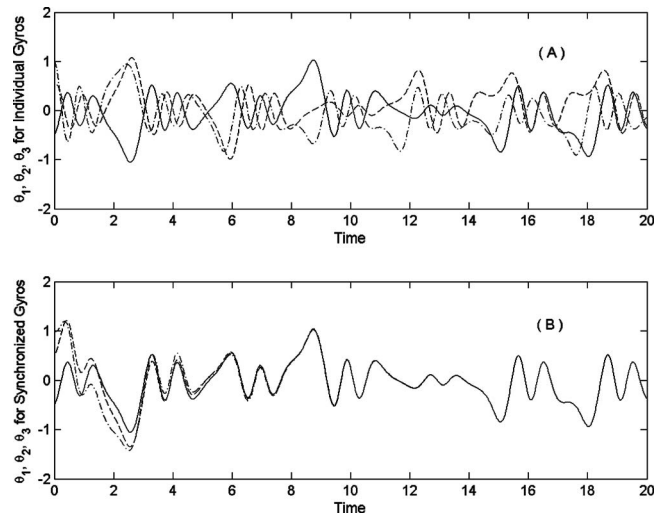


Fig. 4 (A) First 20 s of the response of the uncoupled gyros with the master gyro shown with a solid line, the second gyro shown with a dashed line, and the third gyro shown with a dashed-dotted line. (B) Synchronization of the gyros showing the slave gyros following the master (solid line), as required by the constraint set (16) with $\delta=1$ and $k=2$.

dependent. The explicit, generalized control forces f_i^{syn} required to be applied to the slave gyros ($i=2,3$) are obtained using relations (17)–(22). Figure 4(a) shows the time responses for the first 20 s. of the three uncoupled gyros, and Fig. 4(b) shows their synchronized response, where the latter two gyros ($i=2,3$) are now slaved to the first gyro. We observe that the error between the responses gradually reduces to zero, as required by Eq. (17).

The plots in the $(\theta_i, \dot{\theta}_i)$ plane, $i=1,2,3$, superposed on one another for all three gyros are shown in Fig. 5, indicating synchronization of the two slave gyros with the chaotic motion of the master gyro. The plots are made using the response of each of the gyros over a 50 s interval of time starting at 50 s. We note that in this figure, there are three plots that are superimposed on top of one another.

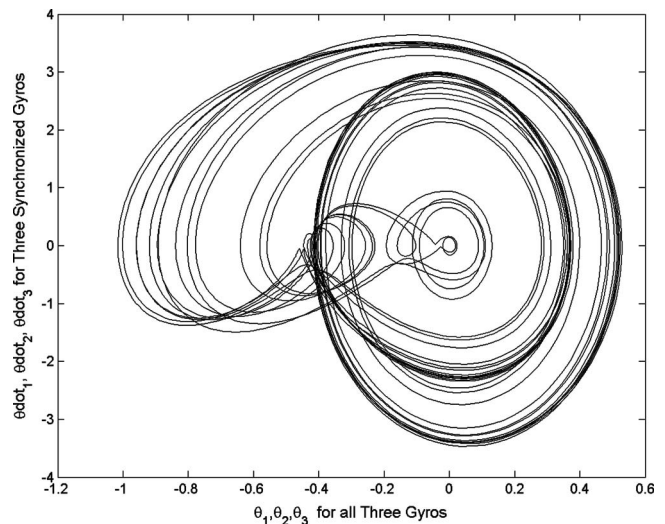


Fig. 5 Superimposed plots of $(\theta_i, \dot{\theta}_i)$, $i=1,2,3$, of the three synchronized gyros for $50 \leq t \leq 100$. The master gyro is a chaotic system and its Lyapunov exponents [17] are $\lambda_1 \approx \{0.211, -0.896, 0\}$. Each of the gyros execute the entire motion shown in the plot.

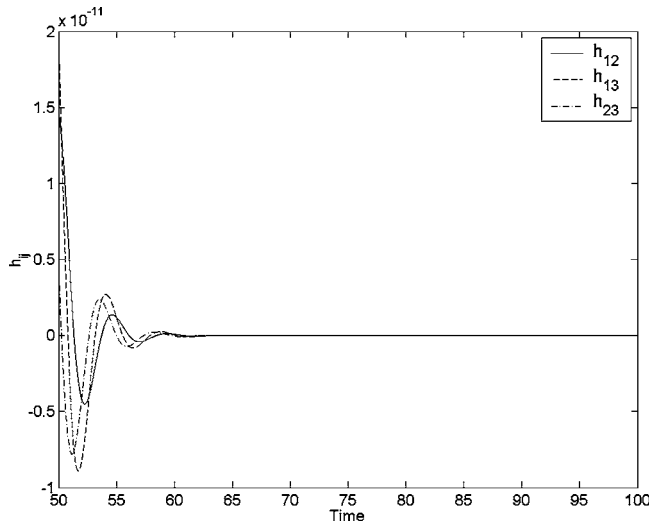


Fig. 6 $h_{12}(t)=\theta_1(t)-\theta_2(t)$ (solid line), $h_{13}(t)=\theta_1(t)-\theta_3(t)$ (dashed line), and $h_{23}(t)=\theta_2(t)-\theta_3(t)$ (dashed-dotted line) for $50 \leq t \leq 100$. Note the exponential convergence of the h_{ij} 's, as demanded by Eq. (16), and also the vertical scale, which indicates that the error in synchronization is of the order of the numerical integration error tolerance, 10^{-12} .

The differences in the responses, $h_{ij}(t)=\theta_i(t)-\theta_j(t)$, $50 \leq t \leq 100$, between the motions of the three synchronized gyros are shown in Fig. 6. We notice that this error soon becomes of the same order of magnitude as the numerical integration error tolerance (10^{-12}). The exponential convergence of $h_{ij}(t)$ toward zero, as demanded by relation (16), is obvious. Lastly, we show the generalized control forces that need to be applied to the slave gyros ($i=2,3$) to synchronize their motions with that of the master. This is shown in Fig. 7 for the entire time segment $0 \leq t \leq 100$.

Example 2. We consider here five different gyro systems, and our aim is to track the motion of the first gyro (master, with parameter set P_1), which in this case is a periodic motion, though considerably complex in nature (see Fig. 9). The four slave gyros exhibit both regular and chaotic motions when uncontrolled. The

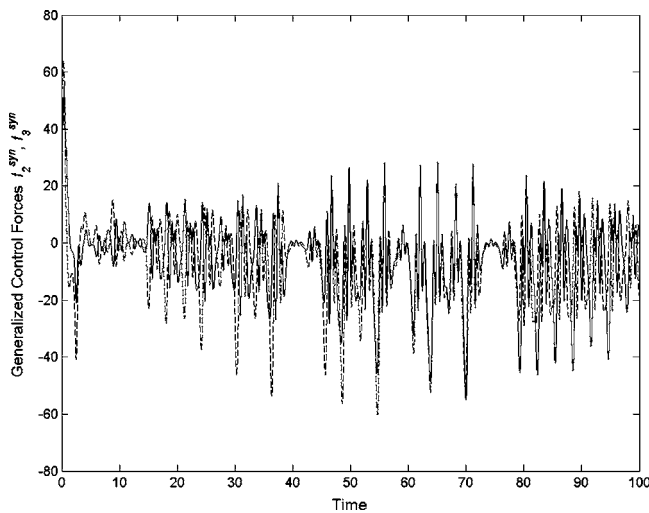


Fig. 7 The solid line shows the generalized force f_2^{yn} required to be applied to second gyro ($i=2$) to achieve synchronization with the motion of the master gyro ($i=1$). The dashed line shows the generalized force f_3^{yn} required to be applied to the third gyro ($i=3$).

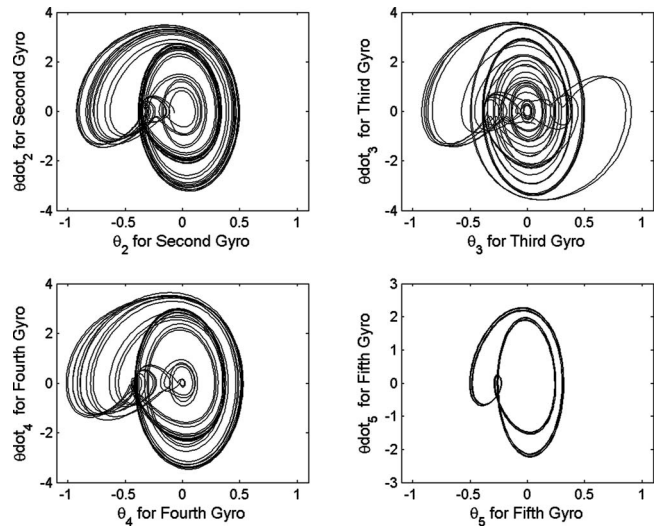


Fig. 8 $(\theta_i, \dot{\theta}_i)$, $i=2,3,4,5$ plot for $50 \leq t \leq 100$ of the four uncoupled slave gyro systems showing different dynamical behaviors for each gyro. The lower right figure shows the transient motions of this ($i=5$) dynamical system, which has not yet attained its regular periodic behavior. The other three dynamical systems ($i=2,3,4$) exhibit chaotic motions, as indicated by the computed Lyapunov exponents.

parameter sets $P_i=\{\alpha_i, \beta_i, c_i, e_i, \gamma_i, \omega_i\}$, $i=1,2,\dots,5$, and the initial condition sets for the dynamical systems are taken to be

$$P_1=\{10.5, 1, 0.5, 0.02, 38.7, 2.2\} \quad IC_1=\{\theta_1^0=-1, \dot{\theta}_1^0=0.5\} \quad (30)$$

$$P_2=\{10, 1, 0.5, 0.05, 35.5, 2\} \quad IC_2=\{\theta_2^0=0.5, \dot{\theta}_2^0=1\} \quad (31)$$

$$P_3=\{10.5, 1, 0.5, 0.04, 38.5, 2.1\} \quad IC_3=\{\theta_3^0=1, \dot{\theta}_3^0=-0.5\} \quad (32)$$

$$P_4=\{10, 1, 0.5, 0.03, 35.8, 2.05\} \quad IC_4=\{\theta_4^0=-0.5, \dot{\theta}_4^0=1\} \quad (33)$$

and

$$P_5=\{10.5, 1, 0.45, 0.045, 36, 2.05\} \quad IC_5=\{\theta_5^0=0.5, \dot{\theta}_5^0=0.5\} \quad (34)$$

The Lyapunov exponent sets, l_i , for these five different gyros—three of which have the same properties as those in Example 1—computed over a time interval of 1000 s, are found to be [17]

$$l_1 \approx \{-0.180, -0.50, 0\} \quad l_2 \approx \{0.216, -1.001, 0\}$$

$$l_3 \approx \{0.208, -0.936, 0\}$$

$$l_4 \approx \{0.211, -0.896, 0\} \quad l_5 \approx \{-0.017, -0.606, 0\} \quad (35)$$

The numerical integration error tolerances for computing the Lyapunov exponents are identical to those used in the previous example. From the values of set l_1 , we see that the master gyro has a periodic motion, while the slave gyros ($i=2,3,4,5$) show a variety of both chaotic and regular motions. From the largest Lyapunov exponent, we see that three of the slaves exhibit chaotic motions, while one shows a periodic motion.

Figure 8 shows the $(\theta_i, \dot{\theta}_i)$, $i=2,3,4,5$ plots for the four slave gyro systems for $50 \leq t \leq 100$. Except for the dynamical system

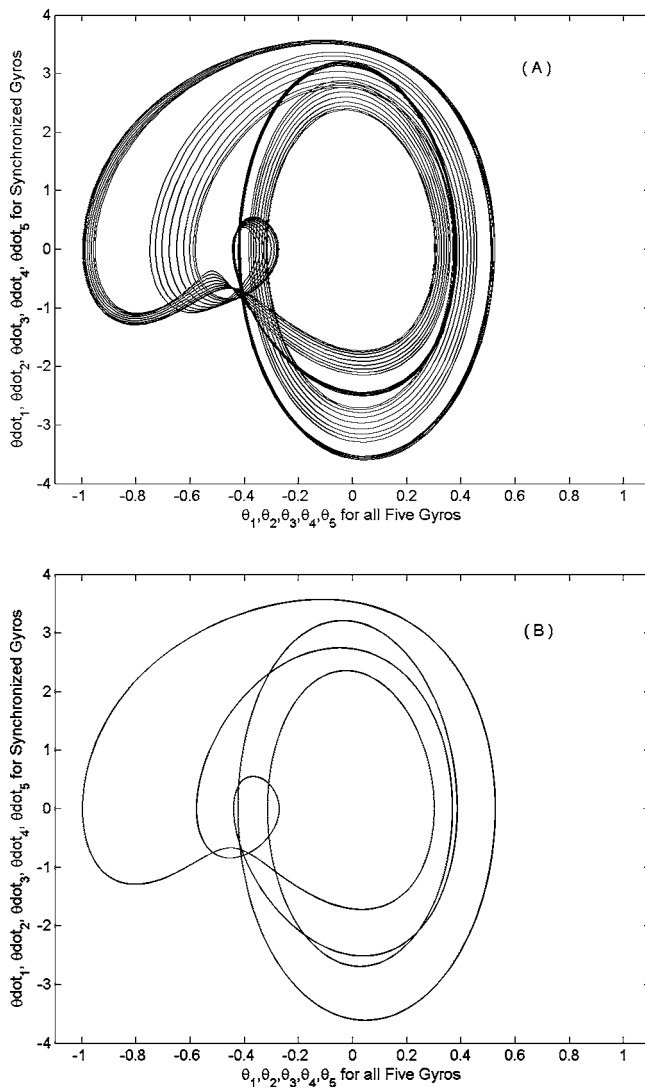


Fig. 9 (A) $(\theta_i, \dot{\theta}_i)$, $i=1,2,3,4,5$, plot for $50 \leq t \leq 100$ of the five gyro systems superimposed on each other showing that the four slaves follow the master gyro. As is seen, the motion of the master is a complex transient motion, which has not yet reached its stable periodic orbit, which is characterized by the Lyapunov exponents $I_1 \approx \{-0.180, -0.50, 0\}$. (B) $(\theta_i, \dot{\theta}_i)$, $i=1,2,3,4,5$, plot for $150 \leq t \leq 200$ of the five gyro systems superimposed on each other showing that the four slaves follow the master gyro. The master gyro has reached a periodic orbit, and the four slaves synchronize with the master's motion. The motion of the five gyros is shown superposed on each other.

($i=5$) shown in the lower right, the other three slaves exhibit a chaotic behavior, as indicated from the computed Lyapunov numbers shown in Eq. (35).

The synchronized motion—we again choose $\delta=1$ and $k=2$ —of the five systems with the four slaves following the master is shown in Fig. 9(a). We see that the tracking during the transient period when the orbit of the master gyro is being attracted to its stable periodic orbit is very well executed by the control. Here, the uncontrolled motion of the first (master) gyro is first plotted, and superimposed on it are plots of the motions of the four slaves for $50 \leq t \leq 100$. The results of the synchronization procedure when the integration is extended to 200 s are shown in Fig. 9(b), where we have plotted the motions of the five different systems for $150 \leq t \leq 200$. The plots fall exactly on top of each other, indicating synchronization. We notice that the master gyro's mo-

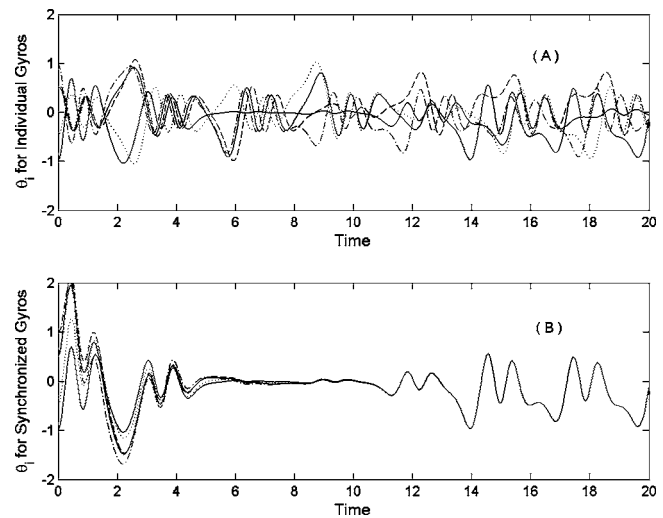


Fig. 10 The upper figure shows the motion of the five uncoupled gyros over the first 20 s. of response. The lower figure shows the manner in which the synchronization occurs over time, the five gyros following the motions of the master gyro, which in turn is asymptotically attracted to a stable periodic orbit, as shown in Fig. 9(b).

tion has now settled down to being periodic, and the four slaves follow this periodic, though complex, motion. Note that the figure shows the motion of all five gyros superposed on one another.

The manner in which the synchronization occurs over time is illustrated in Fig. 10, where we show the first 20 s. of the motion of both the uncoupled system and the synchronized system. The solid line in the two panels denotes the master gyro; the dashed line, the second gyro; the dashed-dotted line the third gyro; the dotted line, the fourth gyro; and another solid line, the fifth. From the lower panel, which shows synchronization with the master gyro, we can identify the motion of the master in the upper panel.

Figure 11 shows the control forces needed to be applied to the four slave gyros for synchronization for $0 \leq t \leq 100$. The errors in synchronization, $h_{ij}(t) = \theta_i(t) - \theta_j(t)$, for the time intervals $50 \leq t$

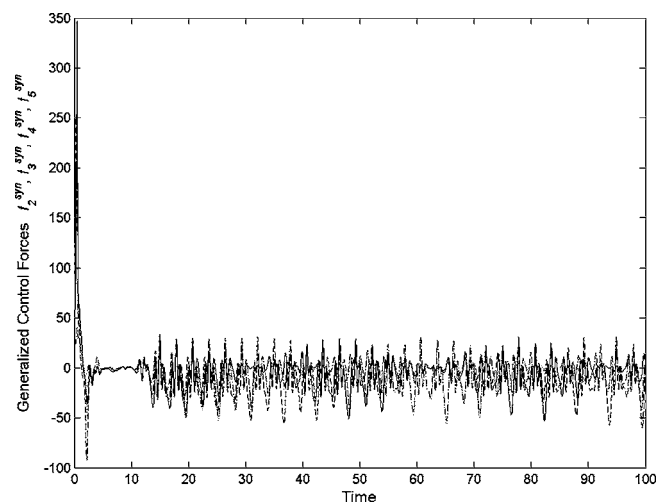


Fig. 11 Control forces required to be applied to the four slave gyros. The solid line shows the generalized control force on the second gyro, the dashed line that on the third gyro, the dashed-dotted line that on the fourth gyro, and the dotted line that on the fifth gyro.

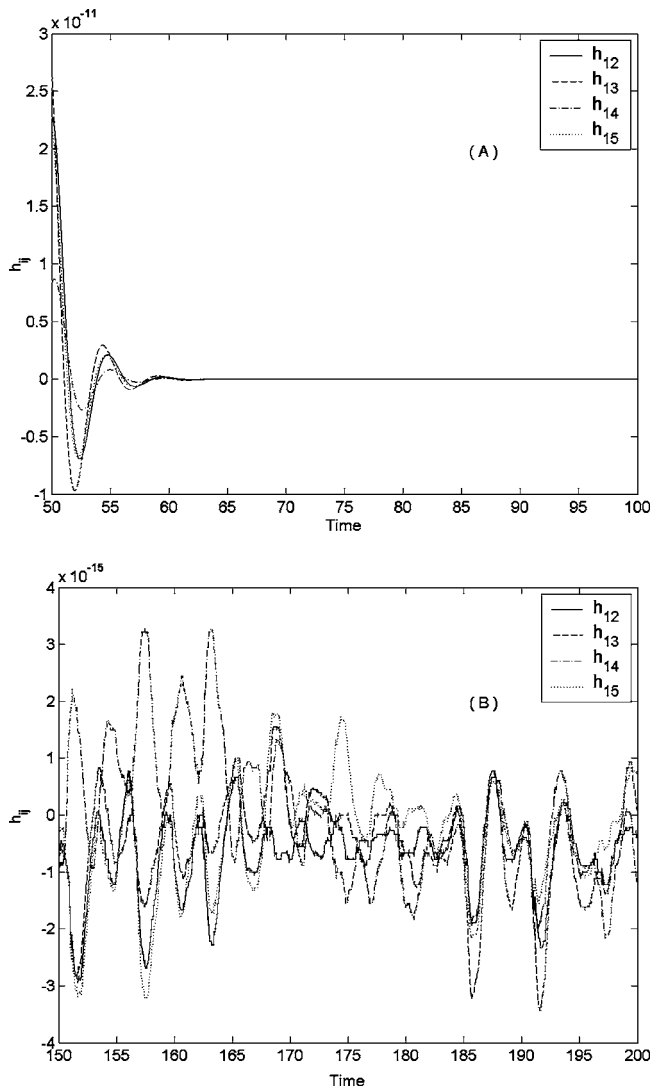


Fig. 12 (A) The errors $h_{ij}(t)$ as functions of time for $50 \leq t \leq 100$, showing that they exponentially reduce. (B) The errors $h_{ij}(t)$ as functions of time for $150 \leq t \leq 200$. Note the vertical scales. Errors in synchronization of the motion are less than the integration error tolerance used.

≤ 100 and $150 \leq t \leq 200$ are shown in Fig. 12, which shows the same sort of characteristics, including exponential convergence, that were observed earlier in Fig. 6.

6 Conclusions

In this paper, we have described an analytical dynamics based approach to the synchronization of highly nonlinear mechanical systems that yields the explicit generalized active control forces so that a set of slave systems can follow an independent master mechanical system. This paper focuses on gyroscopic systems—by way of demonstration—due to their importance in the guidance and control of airships and spacecraft and in the accurate control of complex mechanical systems, such as robotic and autonomous systems. While for simplicity, the slave systems have been considered to be independent of each other in this paper, the same methodology is applicable to slave systems that may be linearly or nonlinearly coupled to one another. The main contributions of this paper are the following.

- 1 The novel strategy used here is to formulate the problem of synchronization of highly nonlinear mechanical systems first

as a tracking control problem, and then further recast this tracking control problem as a problem of constrained motion of nonlinear dynamical systems. We accordingly constrain the motion of the slave systems to exactly follow the master system and thereby obtain the exact control forces required to be applied to the slaves for synchronization with the master. The constraint (control) forces that need to be applied for exact synchronization are determined *explicitly* and in *closed form* using the newly developed general theory of constrained motion of nonlinear mechanical systems. The theory [11,15] that underlies the approach is much broader than what is required for the specific problem at hand of synchronizing chaotic/regular gyroscopic systems since it is applicable to general nonlinear mechanical systems. This makes the approach presented here applicable to the synchronization of general nonlinear systems.

- 2 In Sec. 4 and Appendix B, we prove a general result that hereto appears to be not known, and we use it to develop a simple, yet powerful, methodology for the synchronization of complex nonlinear mechanical systems.
- 3 The method yields control forces for the synchronization of nonlinear mechanical systems that have the following salient and beneficial characteristics. The control forces (1) are continuous in time, (2) are obtained explicitly in closed form so that they are simple and efficacious to determine, (3) lead, theoretically speaking, to exact synchronization of the nonlinear mechanical systems, (4) provide, in a sense, the *minimum forces* that need to be exerted for such synchronization [18], and (5) are not found by methods using any approximations of the nonlinear system.
- 4 Whereas most such synchronization studies are done with dynamical systems that are identical, we show that the method developed here can be used with equal ease and facility to couple *different* slave systems—each displaying varying kinds of regular and chaotic motions. This is important because, unlike many electrical systems, multiple copies of mechanical systems can seldom be built to have identical dynamical characteristics.
- 5 We show the efficacy of the methodology by illustrating two examples. In the first example, two slave gyros with different dynamical characteristics are synchronized with the motions of yet another master gyro whose dynamical characteristics differ from those of both the slaves; the master's motion is chaotic. In the second example, we consider five different gyro systems, some of which have chaotic motions, and we synchronize them with the stable periodic motions of the master gyro. While the dynamics of the slave gyros have been taken for simplicity to be independent of one another in this paper, the same general methodology works with coupled slave gyros as well.
- 6 We observe that while most methods (e.g., Ref. [10]) of synchronization deal with applying control signals to each of the first order differential equations that describe a mechanical system's dynamics (each gyro here can be represented by three, first order autonomous, nonlinear equations), the method proposed here deals directly, and simply, with the second order nonautonomous Lagrange equations of motion and obtains in explicit form the generalized control forces required to synchronize the different mechanical systems. The control we obtain is continuous in time, unlike what might be obtained using methods such as sliding mode control [20]; yet, theoretically speaking, it leads to exact synchronization.
- 7 Lastly, the approach allows the paths in phase space along which the synchronization occurs to be easily and accurately controlled, so that different slaves can be brought into synchronization with the master with varying levels of rapidity, as desired.

Appendix A

The Lagrangian in Eq. (1) can be obtained as follows:

- 1 The kinetic energy (KE) of the symmetrical gyro (Fig. 1) with respect to the inertial frame of reference $OXYZ$ $= (1/2)m\bar{u}_{c.m.} \cdot \bar{u}_{c.m.} + \text{KE of rotation about the c.m. of the gyro}$. Here, $\bar{u}_{c.m.}$ is the velocity of the c.m. of the gyro with respect to the inertial frame $OXYZ$. Denoting by $\bar{x}_{c.m.}$ the position vector of the c.m. of the gyro, we have

$$\bar{x}_{c.m.} = (r \sin \varphi \sin \theta) \bar{\mathbf{I}} - (r \sin \theta \cos \varphi) \bar{\mathbf{J}} + (r \cos \theta + d) \bar{\mathbf{K}} \quad (\text{A1})$$

where $\bar{\mathbf{I}}$, $\bar{\mathbf{J}}$, and $\bar{\mathbf{K}}$ are the unit vectors along the inertial coordinate directions OX , OY , and OZ , respectively. Differentiating Eq. (A1) with respect to time and noting that the vertical support excitation $d(t) = d_0 \sin \omega t$, we obtain the velocity of the c.m. of the gyro to be

$$\begin{aligned} \bar{u}_{c.m.} = & (r \dot{\theta} \sin \varphi \cos \theta + r \dot{\varphi} \cos \varphi \sin \theta) \bar{\mathbf{I}} + (r \dot{\varphi} \sin \varphi \sin \theta \\ & - r \dot{\theta} \cos \varphi \cos \theta) \bar{\mathbf{J}} + (\dot{d} - r \dot{\theta} \sin \theta) \bar{\mathbf{K}} \end{aligned} \quad (\text{A2})$$

$$\text{Hence, } \bar{u}_{c.m.} \cdot \bar{u}_{c.m.} = r^2(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) + \dot{d}^2 - 2r\dot{d}\dot{\theta} \sin \theta \quad (\text{A3})$$

The total KE of the gyro [14] is then given by

$$\begin{aligned} \text{KE} = & \frac{1}{2}m[r^2(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) + \dot{d}^2 - 2r\dot{d}\dot{\theta} \sin \theta] \\ & + \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\varphi} \cos \theta)^2 \end{aligned} \quad (\text{A4})$$

Here, I_1 and I_3 refer to the moments of inertia about the equatorial and polar directions through the c.m. of the symmetric gyro. This expression simplifies to

$$\begin{aligned} \text{KE} = & \frac{1}{2}I(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\varphi} \cos \theta)^2 - mr\dot{d}\dot{\theta} \sin \theta \\ & + \frac{1}{2}m\dot{d}^2 \end{aligned} \quad (\text{A5})$$

where $I = (mr^2 + I_1)$ is the moment of inertia of the gyro about an axis through the point of support o , which is parallel to the principal axis direction that goes through the c.m.

- 2 The potential energy (PE) of the gyro with respect to the inertial frame $OXYZ$ is

$$\text{PE} = mgd + mgr \cos \theta \quad (\text{A6})$$

- 3 Therefore, the effective Lagrangian—we ignore terms that are purely functions of time— $L = \text{KE} - \text{PE}$, is then

$$\begin{aligned} L = & \frac{1}{2}I(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\varphi} \cos \theta)^2 - mr\dot{d}\dot{\theta} \sin \theta \\ & - mgr \cos \theta \end{aligned} \quad (\text{A7})$$

Appendix B

We obtain here the explicit control force n vector $\mathbf{F}^{\text{syn}}$, as given in Eq. (22), which is required to be applied to the set of n nonlinear mechanical systems so that the slave systems, $i=2, 3, \dots, n$, follow the master system, $i=1$.

We begin with two lemmas.

LEMMA 1. Consider the $s \times n$ matrix $\mathbf{A}$ of Eq. (17), an instantiation of which is provided for $n=4$ in Eq. (18). Augment matrix $\mathbf{A}$ by the n -component row vector

$$\mathbf{g} = [1, 0, 0, \dots, 0] \quad (\text{B1})$$

to form the $(s+1) \times n$ matrix

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A} \\ \mathbf{g} \end{bmatrix} \quad (\text{B2})$$

Then, the row vector

$$\mathbf{h} := \mathbf{g}[\mathbf{I}_n - \mathbf{A}^+ \mathbf{A}] \quad (\text{B3})$$

is simply the n -component row vector $(1/n)[1, 1, \dots, 1]$. Here, $\mathbf{X}^+$ denotes the MP inverse of the matrix $\mathbf{X}$.

Proof. We notice that only $(n-1)$ rows of matrix $\mathbf{A}$ are linearly independent. Hence, $\mathbf{A}$ is rank deficient. As shown in Ref. [13], the column space of the $n \times n$ matrix $[\mathbf{I}_n - \mathbf{A}^+ \mathbf{A}]$ is the same as the null space of matrix $\mathbf{A}$. However, the null space of matrix $\mathbf{A}$ has dimension 1 and consists of n -component column vectors, each of the form $\lambda[1, 1, 1, \dots, 1]^T$, where we disallow the value $\lambda=0$ since it leads to a trivial vector. Thus, the n columns of the $n \times n$ matrix $[\mathbf{I}_n - \mathbf{A}^+ \mathbf{A}]$ must be of the form $\lambda_i[1, 1, 1, \dots, 1]^T$, $i=1, 2, \dots, n$, where the constants $\lambda_i \neq 0$, $i=1, 2, 3, \dots, n$, remain yet to be determined.

However, the matrix $[\mathbf{I}_n - \mathbf{A}^+ \mathbf{A}]$ is symmetric since $[\mathbf{I}_n - \mathbf{A}^+ \mathbf{A}]^T = \mathbf{I}_n - (\mathbf{A}^+ \mathbf{A})^T = \mathbf{I}_n - (\mathbf{A}^+ \mathbf{A})$ (see Ref. [13]). Hence, $\lambda_1 = \lambda_2 = \dots = \lambda_n = \lambda$. Furthermore, $[\mathbf{I}_n - \mathbf{A}^+ \mathbf{A}]$ is idempotent; hence, $n\lambda^2 = \lambda$, which implies that $\lambda = 1/n$. The matrix $[\mathbf{I}_n - \mathbf{A}^+ \mathbf{A}]$ therefore has identical columns, and every entry in the matrix is $1/n$. Noting Eq. (B1), the result now follows.

From this proof, it follows that the result of this lemma is true even when our matrix $\mathbf{A}$ has any row dimension r , $(n-1) \leq r \leq s = n(n-1)/2$, provided it always has $(n-1)$ linearly independent rows. $\square$

LEMMA 2. The MP generalized inverse of matrix $\tilde{\mathbf{A}}$ defined in Eq. (B2) is given by

$$\tilde{\mathbf{A}}^+ = [\mathbf{V} \mathbf{A}^+ \quad | \quad [1]] \quad (\text{B4})$$

where $[1]$ is the n -component column vector each of whose components is unity, and the $n \times n$ matrix

$$\mathbf{V} = \begin{bmatrix} 0 & | & 0 & \dots & \dots & \dots & 0 \\ -1 & | & \dots & \dots & \dots & \dots & \dots \\ -1 & | & \dots & \dots & \dots & \dots & \dots \\ \vdots & | & \dots & \dots & \dots & \dots & \dots \\ \vdots & | & \dots & \dots & \dots & \dots & \dots \\ -1 & | & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \quad (\text{B5})$$

where $\mathbf{I}_{n-1}$ is the $(n-1) \times (n-1)$ identity matrix.

Proof. Greville [19] gives the MP inverse of a matrix $\tilde{\mathbf{A}}$, which is obtained by augmenting any matrix $\mathbf{A}$ with the row $\mathbf{g}$, as

$$\tilde{\mathbf{A}}^+ = \begin{bmatrix} \mathbf{A} \\ \mathbf{g} \end{bmatrix}^+ = [(\mathbf{I}_n - \mathbf{h}^+ \mathbf{g}) \mathbf{A}^+ \quad | \quad \mathbf{h}^+] \quad \text{for } \mathbf{h} = \mathbf{g}(\mathbf{I}_n - \mathbf{A}^+ \mathbf{A}) \neq 0 \quad (\text{B6})$$

For our specific matrix $\mathbf{A}$ and row vector $\mathbf{g}$, the row vector $\mathbf{h}$ is given by Eq. (B3). The MP inverse of $\mathbf{h}$, namely, $\mathbf{h}^+ = [1, 1, \dots, 1]^T = [1]$ (see Ref. [13]). Noting that $\mathbf{g} = [1, 0, 0, \dots, 0]$, we have $(\mathbf{I}_n - \mathbf{h}^+ \mathbf{g}) = \mathbf{V}$, and the result follows equation (B6). $\square$

Main Result

The control force that synchronizes the $(n-1)$ slave gyro systems to the motion of the first (master, $i=1$) gyro is given by the n vector

$$\mathbf{F}^{\text{syn}} = \mathbf{F}^c - [1]f_1^c = [0, f_2^c - f_1^c, f_3^c - f_1^c, \dots, f_n^c - f_1^c]^T \quad (\text{B7})$$

where the f_i^c 's are defined as in Eqs. (19) and (21).

Proof. We add to the s constraints given by Eq. (17) the addi-

tional constraint $\ddot{q}_1 := \ddot{\theta}_1 = f_1(\theta_1, \dot{\theta}_1, t) = f_1(q_1, \dot{q}_1, t)$, so that our set of constraints now becomes

$$\tilde{\mathbf{A}}\ddot{\mathbf{q}} = \begin{bmatrix} \mathbf{A} \\ \mathbf{g} \end{bmatrix} \ddot{\mathbf{q}} = \begin{bmatrix} \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) \\ f_1(q_1, \dot{q}_1, t) \end{bmatrix} := \tilde{\mathbf{b}}(\mathbf{q}, \dot{\mathbf{q}}, t) \quad (\text{B8})$$

instead, where the column vector $\mathbf{b}$ is the same as that in Eq. (17), $\mathbf{g}$ is the row vector defined in Eq. (B1), and $\tilde{\mathbf{A}}$ is now an $(s+1) \times n$ matrix. The last constraint simply enforces the condition that the motion of the master gyro is not to be disturbed through the addition of any control force applied to it.

The control force that causes these constraints (Eq. (B8)) to be satisfied is then simply given, like before, by [13]

$$\mathbf{F}^{\text{syn}} = \tilde{\mathbf{A}}^+(\tilde{\mathbf{b}} - \tilde{\mathbf{A}}\mathbf{f}) \quad (\text{B9})$$

where $\mathbf{f} = [f_1, f_2, \dots, f_n]^T$. Using Lemma 2 and Eq. (B8), this can be rewritten as

$$\begin{aligned} \mathbf{F}^{\text{syn}} &= \tilde{\mathbf{A}}^+ \left\{ \begin{bmatrix} \mathbf{b}(\mathbf{q}, \dot{\mathbf{q}}) \\ f_1(q_1, \dot{q}_1, t) \end{bmatrix} - \begin{bmatrix} \mathbf{A} \\ \mathbf{g} \end{bmatrix} \mathbf{f} \right\} \\ &= \tilde{\mathbf{A}}^+ \begin{bmatrix} \mathbf{b} - \mathbf{A}\mathbf{f} \\ 0 \end{bmatrix} = [\mathbf{V}\mathbf{A}^+ \quad | \quad [\mathbf{1}]] \begin{bmatrix} \mathbf{b} - \mathbf{A}\mathbf{f} \\ 0 \end{bmatrix} \end{aligned} \quad (\text{B10})$$

where matrix $\mathbf{V}$ is defined in Eq. (B5).

Since $\mathbf{F}^c := [f_1^c, f_2^c, f_3^c, \dots, f_n^c]^T = \mathbf{A}^+(\mathbf{b} - \mathbf{A}\mathbf{f})$, as given in Eq. (19), relation (B10) becomes

$$\mathbf{F}^{\text{syn}} = [\mathbf{V}\mathbf{A}^+ \quad | \quad [\mathbf{1}]] \begin{bmatrix} \mathbf{b} - \mathbf{A}\mathbf{f} \\ 0 \end{bmatrix} = \mathbf{V}\mathbf{F}^c \quad (\text{B11})$$

Noting the form of $\mathbf{V}$ in Lemma 2, equation (B11) thus reduces to

$$\mathbf{F}^{\text{syn}} = \mathbf{V}\mathbf{F}^c = [0, f_2^c - f_1^c, f_3^c - f_1^c, \dots, f_n^c - f_1^c]^T \quad (\text{B12})$$

which is the required result. As expected, there is no control force required to be applied to the master gyro because this is the motion that we are requiring the slave gyros to follow.

It is important to note that from all the control forces $\hat{\mathbf{F}}^{\text{syn}}(t)$ that can be applied to the system to cause synchronization, the control force $\mathbf{F}^{\text{syn}}(t)$, which is given explicitly in equation (B12), minimizes at each instant of time the quantity $[\hat{\mathbf{F}}^{\text{syn}}(t)]^T \hat{\mathbf{F}}^{\text{syn}}(t)$ (see Ref. [18]). That is, of all the control forces that will cause synchronization, $\mathbf{F}^{\text{syn}}(t)$ has, at each instant of time, the smallest Euclidean norm. $\square$

COROLLARY. The result above is valid when we use any r appropriate and consistent equations, $(n-1) \leq r \leq s = n(n-1)/2$ for synchronization, of the form

$$\theta_i(t) = \theta_j(t) \quad i < j \quad i, j \in (1, n) \quad (\text{B13})$$

to synchronize the $(n-1)$ nonlinear mechanical systems with the master system ($i=1$), as long as $(n-1)$ of these equations are linearly independent.

Proof. If the conditions of the corollary are satisfied, the rank of the $r \times n$ matrix $\mathbf{A}$ is $(n-1)$, and the null space of $\mathbf{A}$ will have dimension 1. Noting the form of $\mathbf{A}$, the columns of the $n \times n$ matrix $[\mathbf{I}_n - \mathbf{A}^+\mathbf{A}]$ will then each be of the form $\lambda[1, 1, \dots, 1]^T$. According to Lemma 1 then,

$$\mathbf{h} := \mathbf{g}[\mathbf{I}_n - \mathbf{A}^+\mathbf{A}] = (1/n)[1, 1, \dots, 1] \quad (\text{B14})$$

so that, again,

$$\mathbf{h}^+ = [1, 1, \dots, 1]^T := [\mathbf{1}] \quad (\text{B15})$$

and the entire argument goes through. $\square$

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Stability of SDOF Linear Viscoelastic System Under the Excitation of Wideband Noise

The moment Lyapunov exponents of a single degree-of-freedom viscoelastic system under the excitation of a wideband noise are studied in this paper. A realistic example of such a system is the transverse vibration of a viscoelastic column under the excitation of stochastic axial compressive load. The method of averaging, both first order and second order, is applied. The averaged Itô differential equation governing the pth norm is established and the pth moment Lyapunov exponent is then obtained. White noise and real noise are considered as models of wideband noises. The variations of the moment Lyapunov exponents with the change of different parameters are discussed.

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1 Introduction

1.1 Wideband Noises and Stochastic Stability. Typical examples of engineering systems subjected to parametric and external random excitations include buildings under earthquake and wind loads, off-shore platforms subjected to ocean waves, vehicles running on rough roads, airplanes in turbulent streams, etc. The common property of the loads is their uncertainty, or the stochastic effect. In order to perform analytical analysis to these systems, mathematical models are required under the framework of stochastic dynamics and stability.

Mathematically, random excitations can be described as stochastic processes. With the theory of stochastic processes, analytical models of different noises can be established. Gaussian white noise process $\xi(t)$ is formally the derivative of the Wiener process given by

$$\xi(t) = \sigma \dot{W}(t) \quad (1.1)$$

with constant power spectral density $S(\omega) = \sigma^2$. Such a noise does not exist in reality since its power will be infinity. However, it provides a very simple and useful mathematical idealization for theoretical analysis. A real noise, or Ornstein–Uhlenbeck process, is defined by

$$d\xi(t) = -\alpha \xi(t)dt + \sigma dW(t) \quad (1.2)$$

with power spectral density

$$S(\omega) = \frac{\sigma^2}{\alpha^2 + \omega^2} \quad (1.3)$$

For large values of α , its power will spread over a wide frequency band; thus, by suitably selecting the parameter α , a real noise may be used as the mathematical model of a wideband noise.

The moment Lyapunov exponents, which are defined by

$$\Lambda(p) = \lim_{t \rightarrow \infty} \frac{1}{t} \log E[\|\mathbf{X}(t)\|^p] \quad (1.4)$$

characterize the moment stability of a stochastic dynamical system with state vector $\mathbf{X}(t)$, where $E[\cdot]$ denotes the expected value and $\|\cdot\|$ denotes a suitable vector norm. The p th moment of the response of the system is asymptotically stable if $\Lambda(p) < 0$. More-

over, $\Lambda(p)$ is a convex function of p and $\Lambda'(0)$ is equal to the largest Lyapunov exponent λ , which is defined by

$$\lambda = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|\mathbf{X}(t)\| \quad (1.5)$$

and describes the almost-sure or sample stability of the system.

Generally speaking, Lyapunov exponent is easier to obtain. However, in general, the almost-sure stability cannot assure the moment stability. Therefore, it is important to obtain the moment Lyapunov exponents of stochastic systems so that the complete properties of dynamic stability can be described.

1.2 Single Degree-of-Freedom Linear Viscoelastic System Under Dynamical Load.

Viscoelasticity has been observed in a number of materials such as polymers, composite materials, metals, and alloys at high temperatures. Generally speaking, elastic materials have the capacity to store mechanical energy without dissipation because they are capable of recovering to the original states after they are unloaded. However, viscoelastic materials both store and dissipate mechanical energy. This property is useful in engineering applications, such as for suppressing the vibration of structures. It is of practical importance to investigate the behavior of viscoelastic materials under random dynamic loads.

In classical elasticity, there are no strain-rate effects, i.e., the strain at time t depends only on the stress at time t and vice versa. However, for viscoelastic materials, stress is not a function of instantaneous strain but depends on the past time history of strain; this dependency relationship also holds for strain. It had been observed that the strain of a viscoelastic material under constant stress typically increases with time in creep test, while the stress decreases with time under constant strain in relaxation test. For linear nonaging materials, the relation between uniaxial strain $\varepsilon(t)$ and stress $\sigma(t)$ can be expressed as [1]

$$\begin{aligned} \varepsilon(t) &= \int_0^t F(t-\tau) d\sigma(\tau) = F(0)\sigma(t) + \int_0^t \dot{F}(t-\tau)\sigma(\tau) d\tau \\ \sigma(t) &= \int_0^t G(t-\tau) d\varepsilon(\tau) = G(0)\varepsilon(t) + \int_0^t \dot{G}(t-\tau)\varepsilon(\tau) d\tau \end{aligned} \quad (1.6)$$

where $F(t)$ is the creep function and $G(t)$ is the relaxation function. Obviously, $G(0)$ corresponds to the instantaneous elastic modulus. It is assumed that the following conditions:

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$$\int_0^\infty |\dot{G}(t)|dt < \infty \quad \text{and} \quad \int_0^\infty t|\dot{G}(t)|dt < \infty \quad (1.7)$$

are satisfied such that the equilibrium elastic modulus $G(\infty)$ exists. One usual choice of relaxation and creep functions under uniaxial strain status is

$$G(t) = Ee^{-t/\lambda} \quad F(t) = \frac{1}{E} \left(1 + \frac{t}{\lambda} \right) \quad (1.8)$$

where E is the general elastic modulus and λ is known as the relaxation time. Equation (1.8) describes the stress relaxation and creep phenomena and is associated with the well-known differential Maxwell model [1]. Furthermore, the generalized Maxwell model, which consists of a sequence of differential Maxwell units coupled in parallel, can be used as an approximation to most linear viscoelastic behavior as close as possible [1,2]. Obviously, the relaxation function for the generalized Maxwell model will be given by

$$G(t) = \sum_{j=1}^M E_j e^{-t/\lambda_j} \quad (1.9)$$

where M is the number of Maxwell units in parallel chain.

It is known that the equation of motion of an elastic beam under dynamical axial compressive load $P(t)$ is given by

$$\rho A \frac{\partial^2 v}{\partial t^2} + \beta_0 \frac{\partial v}{\partial t} + EI \frac{\partial^4 v}{\partial x^4} + P(t) \frac{\partial^2 v}{\partial x^2} = 0$$

where $v(x, t)$ is the transverse deflection of the beam, x the axial coordinate, ρA the mass per unit length of the beam, β_0 the damping constant, and EI the flexural rigidity of the beam. Since the term including EI is associated with the constitutive relation, the equation of motion for the viscoelastic case becomes, using Eq. (1.6),

$$\rho A \frac{\partial^2 v}{\partial t^2} + \beta_0 \frac{\partial v}{\partial t} + EI(1 - \mathcal{H}) \frac{\partial^4 v}{\partial x^4} + P(t) \frac{\partial^2 v}{\partial x^2} = 0 \quad (1.10)$$

where the material relaxation operator $\mathcal{H}$ is taken as

$$\mathcal{H}[u(t)] = \int_0^t h(t-s)u(s)ds \quad (1.11)$$

in which $h(t)$ is the relaxation function describing the property of viscoelasticity.

If the beam is simply supported, the transverse deflection can be expressed as

$$v(x, t) = \sum_{n=1}^{\infty} q_n(t) \sin \frac{n\pi x}{L} \quad (1.12)$$

Substituting Eq. (1.12) into Eq. (1.10) leads to the equations of motion,

$$\ddot{q}_n(t) + 2\beta_n \dot{q}_n(t) + \omega_n^2 \left[1 - \frac{P(t)}{P_n} - \mathcal{H} \right] q_n(t) = 0 \quad n = 1, 2, \dots \quad (1.13)$$

where

$$\beta_n = \frac{\beta_0}{2\rho A} \quad \omega_n^2 = \frac{EI}{\rho A} \left(\frac{n\pi}{L} \right)^4 \quad P_n = EI \left(\frac{n\pi}{L} \right)^2$$

If only the n th mode is considered and the damping, viscoelastic effect, and the amplitude of load are all small, by introducing a small parameter ε , the equation of motion can be written as

$$\ddot{q}(t) + 2\varepsilon\beta\dot{q}(t) + \omega^2 \left\{ [1 + \varepsilon^{1/2}\xi(t)]q(t) - \varepsilon \int_0^t h(t-s)q(s)ds \right\} = 0 \quad (1.14)$$

The dynamic stability of viscoelastic systems has been investigated by some authors. Ariaratnam [3] studied the almost-sure stability of a single degree-of-freedom (SDOF) linear viscoelastic system subjected to random fluctuation in the stiffness parameter by evaluating the largest Lyapunov exponent using the method of stochastic averaging for integrodifferential equations due to Lari-onov [4]. Potapov [5] studied the almost-sure stability of a viscoelastic column under the excitation of a random wideband stationary process using Lyapunov's direct method. Potapov [6] described the behavior of stochastic viscoelastic systems by numerical evaluation of Lyapunov exponents of linear integrodifferential equations.

As what has been indicated before, because almost-sure stability cannot assure moment stability, it is important to study the moment stability of SDOF linear viscoelastic system (1.14) in terms of moment Lyapunov exponents.

For small ε such that $\omega^2 - \varepsilon^2\beta^2 > 0$, letting

$$q(t) = x(t)e^{-\varepsilon\beta t} \quad (1.15)$$

the damping term in Eq. (1.14) can be removed to yield

$$\ddot{x}(t) + \tilde{\omega}^2 \left\{ [1 + \varepsilon^{1/2}\tilde{\xi}(t)]x(t) - \varepsilon \int_0^t \tilde{h}(t-s)x(s)ds \right\} = 0 \quad (1.16)$$

where

$$\tilde{\omega}^2 = \omega^2 - \varepsilon^2\beta^2 \quad \tilde{\xi}(t) = \frac{\omega^2}{\omega^2 - \varepsilon^2\beta^2} \xi(t) \quad \tilde{h}(t) = \frac{\omega^2}{\omega^2 - \varepsilon^2\beta^2} e^{-\varepsilon\beta t} h(t)$$

It is easy to verify that the moment Lyapunov exponents of systems (1.14) and (1.16) are related as

$$\Lambda_{q(t)}(p) = -\varepsilon p\beta + \Lambda_{x(t)}(p) \quad (1.17)$$

Therefore, without loss of generality, the stochastic stability of the SDOF viscoelastic system

$$\ddot{q}(t) + \omega^2 \left\{ [1 + \varepsilon^{1/2}\xi(t)]q(t) - \varepsilon \int_0^t h(t-s)q(s)ds \right\} = 0 \quad (1.18)$$

will be considered in the remaining of this paper by determining its p th moment Lyapunov exponent, where $\xi(t)$ is a wideband stationary noise with zero mean.

Generally speaking, it is difficult to obtain the exact moment Lyapunov exponents of Eq. (1.18). Therefore, some approximate methods have to be applied to study the properties of Eq. (1.18). In this paper, the method of averaging, both first order and second order, will be used to obtain the differential equations governing the p th moment. The moment stability of viscoelastic system (1.18) can then be determined by solving the averaged equations.

2 First-Order Stochastic Averaging

The method of averaging for stochastic dynamical systems was proposed by Stratonovich [7,8], and developed by Khasminskii [9,10]. The purpose is to approximate the solution of a stochastic dynamical system by a Markov diffusion process which satisfies the Itô stochastic differential equation when the correlation function of random excitation decays to zero fast enough or the excitation is a wideband process. After this approximation, it may be easier to obtain the solution or its dynamical properties of the approximated or averaged system.

In order to use the method of stochastic averaging to investigate system (1.18), the following transformation is applied:

$$\begin{aligned} q(t) &= a(t) \cos \Phi(t) & \dot{q}(t) &= -\omega a(t) \sin \Phi(t) & \Phi(t) &= \omega t + \varphi(t) \end{aligned} \quad (2.1)$$

From the first two equations of Eq. (2.1), one has

$$\dot{a}(t) \cos \Phi(t) - a(t) \dot{\varphi}(t) \sin \Phi(t) = 0 \quad (2.2)$$

Substituting Eq. (2.1), into system (1.18) yields

$$\begin{aligned} &\dot{a}(t) \sin \Phi(t) + a(t) \dot{\varphi}(t) \cos \Phi(t) \\ &= -\varepsilon^{1/2} \omega \xi(t) a(t) \cos \Phi(t) - \varepsilon \omega \int_0^t h(t-s) a(s) \cos \Phi(s) ds \end{aligned} \quad (2.3)$$

Letting $P=a^p$, it is easy to see that P is the p th norm of system (1.18). Thus, from Eqs. (2.2) and (2.3), $P(t)$ and $\varphi(t)$ can be solved as

$$\begin{Bmatrix} \dot{P}(t) \\ \dot{\varphi}(t) \end{Bmatrix} = \varepsilon \mathbf{F}^{(1)}(P, \varphi, t) + \varepsilon^{1/2} \mathbf{F}^{(0)}(P, \varphi, \xi, t) \quad (2.4)$$

where

$$\begin{aligned} \mathbf{F}^{(1)}(P, \varphi, t) &= \begin{Bmatrix} -\omega P I_h P(t) \sin \Phi(t) \\ -\omega I_h \cos \Phi(t) \end{Bmatrix} = \begin{Bmatrix} F_1^{(1)}(P, \varphi, t) \\ F_2^{(1)}(P, \varphi, t) \end{Bmatrix} \\ \mathbf{F}^{(0)}(P, \varphi, \xi, t) &= \begin{Bmatrix} (1/2) \omega P P(t) \xi(t) \sin 2\Phi(t) \\ (1/2) \omega \xi(t) [1 + \cos 2\Phi(t)] \end{Bmatrix} = \begin{Bmatrix} F_1^{(0)}(P, \varphi, \xi, t) \\ F_2^{(0)}(P, \varphi, \xi, t) \end{Bmatrix} \\ I_h &= \int_0^t h(t-s) \left[\frac{P(s)}{P(t)} \right]^{1/p} \cos \Phi(s) ds \end{aligned}$$

Then, system (2.4) can be approximated by the following averaged equations:

$$d \begin{Bmatrix} \bar{P}(t) \\ \bar{\varphi}(t) \end{Bmatrix} = \varepsilon \begin{Bmatrix} \bar{m}_P \\ \bar{m}_\varphi \end{Bmatrix} dt + \varepsilon^{1/2} \bar{\sigma} d\mathbf{W}(t) \quad (2.5)$$

where

$$\begin{aligned} \bar{m}_P &= \mathcal{M}_t \left\{ F_1^{(1)}(P, \varphi, t) + \int_{-\infty}^0 \mathbb{E} \left[\frac{\partial F_1^{(0)}}{\partial P} F_{1\tau}^{(0)} + \frac{\partial F_1^{(0)}}{\partial \varphi} F_{2\tau}^{(0)} \right] d\tau \right\} \\ \bar{m}_\varphi &= \mathcal{M}_t \left\{ F_2^{(1)}(P, \varphi, t) + \int_{-\infty}^0 \mathbb{E} \left[\frac{\partial F_2^{(0)}}{\partial P} F_{1\tau}^{(0)} + \frac{\partial F_2^{(0)}}{\partial \varphi} F_{2\tau}^{(0)} \right] d\tau \right\} \\ [\bar{\sigma} \bar{\sigma}^T]_{ij} &= \mathcal{M}_t \left\{ \int_{-\infty}^{\infty} \mathbb{E} [F_i^{(0)} F_{j\tau}^{(0)}] d\tau \right\} \quad i, j = 1, 2 \\ \mathcal{M}\{\cdot\} &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \{\cdot\} dt \\ F_{j\tau}^{(0)} &= F_j^{(0)}[P, \varphi, \xi(t+\tau), t+\tau] \quad j = 1, 2 \end{aligned}$$

Noting that Eq. (2.4) is an integrodifferential equation, the averaging method for integrodifferential equations [4] should also be applied. That is to say, in order to simplify the system, the method of stochastic averaging due to Khasminskii [9,10] is used to obtain the approximate Itô stochastic differential equations, and the averaging method for integrodifferential equations due to Lari-onov [4] is used to obtain the approximate drift terms in the Itô equations in which the viscoelastic terms are involved.

When applying the averaging operation, $P(t)$ and $\varphi(t)$ are treated as unchanged, i.e., they are replaced by $\bar{P}$ and $\bar{\varphi}$ directly. Now, consider $\bar{m}_P$ first. By substituting in the corresponding terms, one has

$$\bar{m}_P = -\omega p \bar{P} \mathcal{M}_t \{I^{sc}\} + \frac{1}{4} \omega^2 p \bar{P} \mathcal{M}_t \{J_1\}$$

where

$$I^{sc} = \sin \Phi(t) \int_0^t h(t-s) \cos \Phi(s) ds$$

$$\begin{aligned} J_1 &= \int_{-\infty}^0 R(\tau) \{p \sin 2\Phi(t) \sin 2\Phi(t+\tau) \\ &\quad + 2 \cos 2\Phi(t) [1 + \cos 2\Phi(t+\tau)]\} d\tau \end{aligned}$$

and $R(\tau) = \mathbb{E}[\xi(t)\xi(t+\tau)]$ is the correlation function of the wide-band noise $\xi(t)$.

Observing that conditions (1.7) are required, i.e., $h(t)$ and $t \cdot h(t)$ are integrable over $[0, \infty)$, then applying the transformation $s = t - \tau$ and changing the order of integration lead to

$$\begin{aligned} \mathcal{M}_t \{I^{sc}\} &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t=0}^T \int_{s=0}^t h(t-s) \cos \Phi(s) \sin \Phi(t) ds dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t=0}^T \int_{\tau=0}^t h(\tau) \sin \Phi(t) \cos \Phi(t-\tau) d\tau dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{\tau=0}^T \int_{t=\tau}^T h(\tau) [\sin(2\omega t - \omega\tau + 2\bar{\varphi}) + \sin \omega\tau] dt d\tau \\ &= \frac{1}{2} \int_0^\infty h(\tau) \sin \omega\tau d\tau = \frac{1}{2} H^s(\omega) \end{aligned} \quad (2.6)$$

Similarly, it can be shown that

$$\mathcal{M}_t \{I^{cc}\} = \mathcal{M}_t \left[\cos \Phi(t) \int_0^t h(t-s) \cos \Phi(s) ds \right] = \frac{1}{2} H^c(\omega)$$

where

$$H^s(\omega) = \int_0^\infty h(\tau) \sin \omega\tau d\tau \quad \text{and} \quad H^c(\omega) = \int_0^\infty h(\tau) \cos \omega\tau d\tau \quad (2.7)$$

are the sine and cosine transformations of the viscoelastic kernel function $h(t)$.

On the other hand,

$$\begin{aligned} \mathcal{M}_t \{J_1\} &= \lim_{T \rightarrow \infty} \frac{p+2}{2T} \int_{t=0}^T \int_{\tau=-\infty}^0 R(\tau) \cos 2\omega\tau d\tau dt \\ &\quad + \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t=0}^T \int_{\tau=-\infty}^0 R(\tau) \left[\frac{2-p}{2} \cos(4\omega t + 2\omega\tau + 4\bar{\varphi}) \right. \\ &\quad \left. + 2 \cos(2\omega t + 2\bar{\varphi}) \right] d\tau dt \\ &= \frac{p+2}{2} \int_{\tau=-\infty}^0 R(\tau) \cos 2\omega\tau d\tau = \frac{p+2}{4} S(2\omega) \end{aligned}$$

where the cosine and sine power spectral density functions of noise $\xi(t)$ are given by

$$\begin{aligned} S(\omega) &= \int_{-\infty}^\infty R(\tau) \cos \omega\tau d\tau = 2 \int_0^\infty R(\tau) \cos \omega\tau d\tau \\ &= 2 \int_{-\infty}^0 R(\tau) \cos \omega\tau d\tau \end{aligned}$$

$$\Psi(\omega) = 2 \int_0^\infty R(\tau) \sin \omega \tau d\tau = -2 \int_{-\infty}^0 R(\tau) \sin \omega \tau d\tau \quad (2.8)$$

Similarly, $\bar{m}_\varphi$ and $[\bar{\sigma}\bar{\sigma}^T]_{ij}$ can be evaluated to yield

$$\begin{aligned} \bar{m}_p &= \omega p \bar{P} \left[-\frac{1}{2} H^s(\omega) + \frac{p+2}{16} \omega S(2\omega) \right] \\ \bar{m}_\varphi &= -\omega \left[\frac{1}{2} H^c(\omega) + \frac{1}{8} \omega \Psi(2\omega) \right] \end{aligned} \quad (2.9)$$

$$[\bar{\sigma}\bar{\sigma}^T]_{11} = b_{11} = \frac{1}{8} \omega^2 p^2 \bar{P}^2 S(2\omega) \quad [\bar{\sigma}\bar{\sigma}^T]_{12} = [\bar{\sigma}\bar{\sigma}^T]_{21} = 0$$

$$[\bar{\sigma}\bar{\sigma}^T]_{22} = b_{22} = \frac{1}{8} \omega^2 [2S(0) + S(2\omega)]$$

Noting that the transition probability density function for the solution of the averaged equation is the solution of the Fokker-Planck equation, which depends on the diffusion matrix $\bar{\sigma}\bar{\sigma}^T$ but not every single element $\bar{\sigma}_{ij}$, thus one can take

$$\bar{\sigma}_{12} = \bar{\sigma}_{21} = 0 \quad \bar{\sigma}_{11} = \sqrt{b_{11}} = \omega p \bar{P} \sqrt{\frac{S(2\omega)}{8}} \quad (2.10)$$

$$\bar{\sigma}_{22} = \sqrt{b_{22}} = \omega \sqrt{\frac{2S(0) + S(2\omega)}{8}}$$

Finally, the averaged Itô differential equations are

$$\begin{aligned} d\bar{P} &= \varepsilon \omega p \bar{P} \left[-\frac{1}{2} H^s(\omega) + \frac{p+2}{16} \omega S(2\omega) \right] dt \\ &+ \varepsilon^{1/2} \omega p \bar{P} \sqrt{\frac{S(2\omega)}{8}} dW_1(t) \end{aligned} \quad (2.11)$$

$$\begin{aligned} d\bar{\varphi} &= -\varepsilon \omega \left[\frac{1}{2} H^c(\omega) + \frac{1}{8} \omega \Psi(2\omega) \right] dt \\ &+ \varepsilon^{1/2} \omega \sqrt{\frac{2S(0) + S(2\omega)}{8}} dW_2(t) \end{aligned} \quad (2.12)$$

It can be seen that $\bar{P}(t)$ does not depend on $\bar{\varphi}(t)$; therefore, it can be solved independently. The property of independent increment for the Wiener process indicates that the expectation of the second term in Eq. (2.11) is zero. Therefore, taking the expected value on both sides of Eq. (2.11) yields

$$dE[\bar{P}] = \varepsilon \omega p \left[-\frac{1}{2} H^s(\omega) + \frac{p+2}{16} \omega S(2\omega) \right] E[\bar{P}] dt \quad (2.13)$$

From Eq. (2.13), it is easy to obtain the moment Lyapunov exponents for the averaged system by

$$\Lambda(p) = \lim_{t \rightarrow \infty} \frac{\log E[\bar{P}]}{t} = \varepsilon \omega p \left[-\frac{1}{2} H^s(\omega) + \frac{p+2}{16} \omega S(2\omega) \right] \quad (2.14)$$

and the Lyapunov exponent is given by

$$\lambda = \Lambda'(0) = \varepsilon \omega \left[-\frac{1}{2} H^s(\omega) + \frac{1}{8} \omega S(2\omega) \right] \quad (2.15)$$

From Eqs. (2.14) and (2.15), it is clear that the viscoelasticity helps to stabilize the system, whereas noises destabilize the system. The stronger the noise, the more unstable the system. The boundaries for the almost-sure stability and the p th moment stability are determined by $\lambda=0$ and $\Lambda(p)=0$, respectively.

Equations (2.14) and (2.15) show that the stability of averaged system (2.5) is determined by the power spectral density of the wideband random excitation at 2ω . Using Eqs. (1.1) and (1.3), the p th moment Lyapunov exponent for the *Gaussian white noise* model is reduced by

$$\Lambda(p) = \varepsilon \omega p \left[-\frac{1}{2} H^s(\omega) + \frac{p+2}{16} \omega \sigma^2 \right] \quad (2.16)$$

and that for *real noise* model becomes

$$\Lambda(p) = \varepsilon \omega p \left[-\frac{1}{2} H^s(\omega) + \frac{p+2}{16} \frac{\omega \sigma^2}{\alpha^2 + 4\omega^2} \right] \quad (2.17)$$

A general choice of the viscoelastic kernel function is, according to Eq. (1.9) of the generalized Maxwell model,

$$h(t) = \sum_{j=1}^M \gamma_j e^{-\kappa_j t} \quad (2.18)$$

Its sine and cosine transformations are given by

$$H^s(\omega) = \sum_{j=1}^M \frac{\omega \gamma_j}{\kappa_j^2 + \omega^2} \quad H^c(\omega) = \sum_{j=1}^M \frac{\gamma_j \kappa_j}{\kappa_j^2 + \omega^2} \quad (2.19)$$

Thus, from Eqs. (2.14) and (2.15), when

$$S(2\omega) < \sum_{j=1}^M \frac{4\gamma_j}{\kappa_j^2 + \omega^2}$$

system (2.5) is asymptotically stable almost surely. When

$$S(2\omega) < \sum_{j=1}^M \frac{8\gamma_j}{(p+2)(\kappa_j^2 + \omega^2)}$$

the p th moment of system (2.5) is asymptotically stable. These results indicate that the stronger the viscoelastic effect (i.e., larger γ), the wider the stability region; the larger the relaxation time (i.e., smaller κ), the wider the stability region.

3 Second-Order Stochastic Averaging

The first-order stochastic averaging may not be adequate in some applications. Similar to deterministic systems, higher-order averaging may be applied to obtain better approximations. Hijawi et al. [11] studied the dynamic response of nonlinear elastic structure under random load using both the first-order and the second-order stochastic averaging methods. Lin and Cai [12] also presented several examples where some terms in equations may not be small enough. In this section, a second-order averaging method is applied and the results are compared with those obtained using the first-order averaging.

Rewrite Eq. (2.4) as

$$\begin{Bmatrix} \dot{P}(t) \\ \dot{\varphi}(t) \end{Bmatrix} = \varepsilon \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} + \varepsilon^{1/2} \begin{Bmatrix} g_1 \\ g_2 \end{Bmatrix} \quad (3.1)$$

where

$$f_1 = -\omega p P \int_0^t h(t-s) \left[\frac{P(s)}{P(t)} \right]^{1/p} \sin(\omega t + \varphi) \cos(\omega s + \varphi) ds$$

$$f_2 = -\omega \int_0^t h(t-s) \left[\frac{P(s)}{P(t)} \right]^{1/p} \cos(\omega t + \varphi) \cos(\omega s + \varphi) ds$$

$$g_1 = \frac{1}{2} \omega p P \xi(t) \sin(2\omega t + 2\varphi)$$

$$g_2 = \frac{1}{2} \omega \xi(t) [1 + \cos(2\omega t + 2\varphi)]$$

Let

$$P(t) = \bar{P}(t) + \varepsilon P_1(\bar{P}, \bar{\varphi}, t) \quad \varphi(t) = \bar{\varphi}(t) + \varepsilon \varphi_1(\bar{P}, \bar{\varphi}, t) \quad (3.2)$$

where $\bar{P}(t)$ and $\bar{\varphi}(t)$, as will be shown, are the results of the first-order averaging and will be determined later. Differentiating Eqs. (3.2) with respect to time t yields

$$\begin{Bmatrix} \dot{P} \\ \dot{\varphi} \end{Bmatrix} = \mathbf{A} \begin{Bmatrix} \dot{\bar{P}} \\ \dot{\bar{\varphi}} \end{Bmatrix} + \varepsilon \begin{Bmatrix} \partial P_1 / \partial t \\ \partial \varphi_1 / \partial t \end{Bmatrix} \quad (3.3)$$

where

$$\mathbf{A} = \begin{bmatrix} 1 + \varepsilon(\partial P_1 / \partial \bar{P}) & \varepsilon(\partial P_1 / \partial \bar{\varphi}) \\ \varepsilon(\partial \varphi_1 / \partial \bar{P}) & 1 + \varepsilon(\partial \varphi_1 / \partial \bar{\varphi}) \end{bmatrix}$$

It is easy to check that

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 - \varepsilon(\partial P_1 / \partial \bar{P}) & -\varepsilon(\partial P_1 / \partial \bar{\varphi}) \\ -\varepsilon(\partial \varphi_1 / \partial \bar{P}) & 1 - \varepsilon(\partial \varphi_1 / \partial \bar{\varphi}) \end{bmatrix} + o(\varepsilon)$$

Substituting Eq. (3.3) into Eqs. (3.1) yields

$$\begin{aligned} \begin{Bmatrix} \dot{\bar{P}} \\ \dot{\bar{\varphi}} \end{Bmatrix} &= \varepsilon \mathbf{A}^{-1} \begin{Bmatrix} f_1 - \partial P_1 / \partial t \\ f_2 - \partial \varphi_1 / \partial t \end{Bmatrix} + \varepsilon^{1/2} \mathbf{A}^{-1} \begin{Bmatrix} g_1 \\ g_2 \end{Bmatrix} = \varepsilon \begin{Bmatrix} f_1 - \partial P_1 / \partial t \\ f_2 - \partial \varphi_1 / \partial t \end{Bmatrix} \\ &+ \varepsilon^2 \begin{Bmatrix} -(\partial P_1 / \partial \bar{P})(f_1 - \partial P_1 / \partial t) - (\partial P_1 / \partial \bar{\varphi})(f_2 - \partial \varphi_1 / \partial t) \\ -(\partial \varphi_1 / \partial \bar{P})(f_1 - \partial P_1 / \partial t) - (\partial \varphi_1 / \partial \bar{\varphi})(f_2 - \partial \varphi_1 / \partial t) \end{Bmatrix} \\ &+ \varepsilon^{1/2} \begin{Bmatrix} g_1 \\ g_2 \end{Bmatrix} + \varepsilon^{3/2} \begin{Bmatrix} -(\partial P_1 / \partial \bar{P})g_1 - (\partial P_1 / \partial \bar{\varphi})g_2 \\ -(\partial \varphi_1 / \partial \bar{P})g_1 - (\partial \varphi_1 / \partial \bar{\varphi})g_2 \end{Bmatrix} + o(\varepsilon^2) \end{aligned} \quad (3.4)$$

Expanding f_1 , f_2 , g_1 , and g_2 at $\bar{P}$ and $\bar{\varphi}$ leads to

$$f_1 = -\omega p [\bar{P} I^{\text{sc}} + \varepsilon P_1 I^{\text{sc}} + \varepsilon \bar{P} \varphi_1 (I^{\text{cs}} - I^{\text{ss}})] - \varepsilon \omega \bar{P} J^{\text{sc}} + o(\varepsilon)$$

$$f_2 = -\omega [I^{\text{cc}} - \varepsilon \varphi_1 (I^{\text{cs}} + I^{\text{sc}})] - \varepsilon \frac{\omega}{p} J^{\text{cc}} + o(\varepsilon)$$

$$\begin{aligned} g_1 &= \frac{1}{2} \omega p \bar{P} \xi(t) \sin(2\omega t + 2\bar{\varphi}) + \varepsilon \omega p \xi(t) \left[\frac{P_1}{2} \sin(2\omega t + 2\bar{\varphi}) \right. \\ &\quad \left. + \bar{P} \varphi_1 \cos(2\omega t + 2\bar{\varphi}) \right] + o(\varepsilon) \end{aligned}$$

$$g_2 = \frac{1}{2} \omega \xi(t) [1 + \cos(2\omega t + 2\bar{\varphi})] - \varepsilon \omega \varphi_1 \xi(t) \sin(2\omega t + 2\bar{\varphi}) + o(\varepsilon)$$

where

$$I^{\text{cc}} = \int_0^t h(t-s) \left[\frac{\bar{P}(s)}{\bar{P}(t)} \right]^{1/p} \cos(\omega t + \bar{\varphi}) \cos(\omega s + \bar{\varphi}) ds$$

$$I^{\text{ss}} = \int_0^t h(t-s) \left[\frac{\bar{P}(s)}{\bar{P}(t)} \right]^{1/p} \sin(\omega t + \bar{\varphi}) \sin(\omega s + \bar{\varphi}) ds$$

$$I^{\text{cs}} = \int_0^t h(t-s) \left[\frac{\bar{P}(s)}{\bar{P}(t)} \right]^{1/p} \cos(\omega t + \bar{\varphi}) \sin(\omega s + \bar{\varphi}) ds$$

$$I^{\text{sc}} = \int_0^t h(t-s) \left[\frac{\bar{P}(s)}{\bar{P}(t)} \right]^{1/p} \sin(\omega t + \bar{\varphi}) \cos(\omega s + \bar{\varphi}) ds$$

$$J^{\text{cc}} = \int_0^t h(t-s) \left[\frac{\bar{P}(s)}{\bar{P}(t)} \right]^{1/p} \left[\frac{P_1}{\bar{P}(s)} - \frac{P_1}{\bar{P}(t)} \right] \cos(\omega t + \bar{\varphi}) \cos(\omega s + \bar{\varphi}) ds$$

$$J^{\text{sc}} = \int_0^t h(t-s) \left[\frac{\bar{P}(s)}{\bar{P}(t)} \right]^{1/p} \left[\frac{P_1}{\bar{P}(s)} - \frac{P_1}{\bar{P}(t)} \right] \sin(\omega t + \bar{\varphi}) \cos(\omega s + \bar{\varphi}) ds$$

Thus, Eq. (3.3) can be written as

$$\begin{Bmatrix} \dot{\bar{P}} \\ \dot{\bar{\varphi}} \end{Bmatrix} = \varepsilon \begin{Bmatrix} f_1^* \\ f_2^* \end{Bmatrix} + \varepsilon^2 \begin{Bmatrix} f_1^{**} \\ f_2^{**} \end{Bmatrix} + \varepsilon^{1/2} \begin{Bmatrix} g_1^* \\ g_2^* \end{Bmatrix} + \varepsilon^{3/2} \begin{Bmatrix} g_1^{**} \\ g_2^{**} \end{Bmatrix} + o(\varepsilon^2) \quad (3.5)$$

where

$$f_1^* = -\omega p \bar{P} I^{\text{sc}} - \frac{\partial P_1}{\partial t} \quad f_2^* = -\omega I^{\text{cc}} - \frac{\partial \varphi_1}{\partial t}$$

$$g_1^* = \frac{1}{2} \omega p \bar{P} \xi(t) \sin(2\omega t + 2\bar{\varphi}) \quad g_2^* = \frac{1}{2} \omega \xi(t) [1 + \cos(2\omega t + 2\bar{\varphi})]$$

$$f_1^{**} = -\frac{\partial P_1}{\partial \bar{P}} f_1^* - \frac{\partial P_1}{\partial \bar{\varphi}} f_2^* - \omega [p P_1 I^{\text{sc}} + p \bar{P} \varphi_1 (I^{\text{cc}} - I^{\text{ss}}) + \bar{P} J^{\text{sc}}]$$

$$f_2^{**} = -\frac{\partial \varphi_1}{\partial \bar{P}} f_1^* - \frac{\partial \varphi_1}{\partial \bar{\varphi}} f_2^* + \omega \left[\varphi_1 (I^{\text{cs}} + I^{\text{sc}}) - \frac{1}{p} J^{\text{cc}} \right]$$

$$\begin{aligned} g_1^{**} &= -\frac{\partial P_1}{\partial \bar{P}} g_1^* - \frac{\partial P_1}{\partial \bar{\varphi}} g_2^* + \omega p \xi(t) \left[\frac{P_1}{2} \sin(2\omega t + 2\bar{\varphi}) \right. \\ &\quad \left. + \bar{P} \varphi_1 \cos(2\omega t + 2\bar{\varphi}) \right] \end{aligned}$$

$$g_2^{**} = -\frac{\partial \varphi_1}{\partial \bar{P}} g_1^* - \frac{\partial \varphi_1}{\partial \bar{\varphi}} g_2^* - \omega \varphi_1 \xi(t) \sin(2\omega t + 2\bar{\varphi}) \quad (3.6)$$

From Sec. 2, it is known that

$$\mathcal{M}_t \{I^{\text{sc}}\} = \frac{1}{2} \mathcal{H}^f(\omega)$$

in which $\bar{P}$ is treated as a constant under the averaging operation.

The first-order term in the $\bar{P}$ of Eq. (3.5) is given by f_1^* , which, after averaging, should be the same as the result of the first-order averaging. Setting f_1^* to the averaged result of the deterministic term in the P of Eq. (2.4), i.e., $\mathcal{M}_t \{F_1^{(1)}\}$ in $\bar{m}_P$ of Eq. (2.5), one obtains

$$\frac{\partial P_1}{\partial t} = -\omega p \bar{P} \left[I^{\text{sc}} - \frac{1}{2} \mathcal{H}^f(\omega) \right]$$

When $\bar{P}$ is treated as a constant in I^{sc} , it can be seen that

$$\begin{aligned}
I^{sc} - \frac{1}{2}H^s(\omega) &= \int_0^t h(t-s)\sin(\omega t + \bar{\varphi})\cos(\omega s + \bar{\varphi})ds \\
&\quad - \frac{1}{2}\int_0^\infty h(\tau)\sin\omega\tau d\tau \\
&= \int_0^t h(\tau)\sin(\omega t + \bar{\varphi})\cos(\omega t - \omega\tau + \bar{\varphi})d\tau \\
&\quad - \frac{1}{2}\int_0^\infty h(\tau)\sin\omega\tau d\tau \\
&= \frac{1}{2}\int_0^\infty h(\tau)\sin(2\omega t - \omega\tau + 2\bar{\varphi})d\tau - \int_t^\infty h(\tau)\sin(\omega t \\
&\quad + \bar{\varphi})\cos(\omega t - \omega\tau + \bar{\varphi})d\tau
\end{aligned} \quad (3.7)$$

and

$$\left| \int_t^\infty h(\tau)\sin(\omega t + \bar{\varphi})\cos(\omega t - \omega\tau + \bar{\varphi})d\tau \right| \leq \int_t^\infty |h(\tau)|d\tau$$

From conditions (1.7), $h(t)$ is absolutely integrable over $[0, \infty)$, this means that the second integral in Eq. (3.7) tends to zero as t tends to infinity. Therefore, one can approximately choose

$$\begin{aligned}
\frac{\partial P_1}{\partial t} &= -\frac{1}{2}\omega p \bar{P} \int_0^\infty h(\tau)\sin(2\omega t - \omega\tau + 2\bar{\varphi})d\tau \\
&= -\frac{1}{2}\omega p \bar{P} [H^c(\omega)\sin(2\omega t + 2\bar{\varphi}) - H^s(\omega)\cos(2\omega t + 2\bar{\varphi})]
\end{aligned}$$

i.e.,

$$P_1 = \frac{1}{4}p \bar{P} [H^c(\omega)\cos(2\omega t + 2\bar{\varphi}) + H^s(\omega)\sin(2\omega t + 2\bar{\varphi})] \quad (3.8)$$

Similarly, noticing that

$$\frac{\partial \varphi_1}{\partial t} = -\omega \left[I^{cc} - \frac{1}{2}H^c(\omega) \right]$$

it can be approximately set

$$\frac{\partial \varphi_1}{\partial t} = -\frac{1}{2}\omega [H^c(\omega)\cos(2\omega t + 2\bar{\varphi}) + H^s(\omega)\sin(2\omega t + 2\bar{\varphi})]$$

or

$$\varphi_1 = \frac{1}{4} [H^s(\omega)\cos(2\omega t + 2\bar{\varphi}) - H^c(\omega)\sin(2\omega t + 2\bar{\varphi})] \quad (3.9)$$

Since P_1 and φ_1 have been determined, Eqs. (3.5) can be simplified by substituting Eqs. (3.8) and (3.9) into Eqs. (3.6), and then the stochastic averaging method presented in Sec. 2 can be performed for Eqs. (3.5). Thus, following the same procedure as the first-order averaging, after some tedious deduction, the averaged version of Eqs. (3.5) is given by, still denoted by $\bar{P}$ and $\bar{\varphi}$,

$$d\bar{P} = \bar{m}_p^* dt + \bar{\sigma}_{11}^* dW_1 + \bar{\sigma}_{12}^* dW_2$$

$$d\bar{\varphi} = \bar{m}_\varphi^* dt + \bar{\sigma}_{21}^* dW_1 + \bar{\sigma}_{22}^* dW_2$$

where higher-order terms have been neglected, and

$$\begin{aligned}
\bar{m}_p^* &= \varepsilon \omega p \bar{P} \left[-\frac{1}{2}H^s(\omega) + \frac{p+2}{16}\omega S(2\omega) \right] \\
&\quad + \varepsilon^2 \frac{p(p+2)}{16} \omega^2 \bar{P} H^c(\omega) S(2\omega)
\end{aligned} \quad (3.10)$$

$$\begin{aligned}
\bar{m}_\varphi^* &= \varepsilon \omega \left[-\frac{1}{2}H^c(\omega) - \frac{1}{8}\omega \Psi(2\omega) \right] + \varepsilon^2 \frac{1}{8} \omega \{ -[H^c(\omega)]^2 - [H^s(\omega)]^2 \\
&\quad - \omega H^c(\omega) \Psi(2\omega) \}
\end{aligned}$$

$$b_{11}^* = [\bar{\sigma}^* \bar{\sigma}^{*T}]_{11} = \frac{1}{8} \omega^2 p^2 \bar{P}^2 S(2\omega) [\varepsilon + \varepsilon^2 H^c(\omega)]$$

$$b_{12}^* = [\bar{\sigma}^* \bar{\sigma}^{*T}]_{12} = b_{21}^* = [\bar{\sigma}^* \bar{\sigma}^{*T}]_{21} = 0$$

$$b_{22}^* = [\bar{\sigma}^* \bar{\sigma}^{*T}]_{22} = \frac{1}{8} \omega^2 [S(2\omega) + 2S(0)] [\varepsilon + \varepsilon^2 H^c(\omega)]$$

Similar to the first-order averaging, it can be seen that $\bar{\sigma}_{12}^* = \bar{\sigma}_{21}^* = 0$ and thus

$$\begin{aligned}
\bar{\sigma}_{11}^* &= \sqrt{b_{11}^*} = \omega p \bar{P} \sqrt{\frac{1}{8} S(2\omega) [\varepsilon + \varepsilon^2 H^c(\omega)]} \\
\bar{\sigma}_{22}^* &= \sqrt{b_{22}^*} = \omega \sqrt{\frac{1}{8} [S(2\omega) + 2S(0)] [\varepsilon + \varepsilon^2 H^c(\omega)]}
\end{aligned} \quad (3.11)$$

Therefore, by taking the expectation on both sides of the Itô differential equation for $\bar{P}$, one has

$$\begin{aligned}
dE[\bar{P}] &= \left\{ \varepsilon \omega p \left[-\frac{1}{2}H^s(\omega) + \frac{p+2}{16}\omega S(2\omega) \right] \right. \\
&\quad \left. + \varepsilon^2 \frac{p(p+2)}{16} \omega^2 H^c(\omega) S(2\omega) \right\} E[\bar{P}] dt
\end{aligned}$$

and the p th moment Lyapunov exponents, including the second-order term, are

$$\begin{aligned}
\Lambda(p) &= \varepsilon \omega p \left[-\frac{1}{2}H^s(\omega) + \frac{p+2}{16}\omega S(2\omega) \right] \\
&\quad + \varepsilon^2 \frac{p(p+2)}{16} \omega^2 H^c(\omega) S(2\omega)
\end{aligned} \quad (3.12)$$

Obviously, the Lyapunov exponent for the second-order averaging is given by

$$\lambda = \varepsilon \omega \left[-\frac{1}{2}H^s(\omega) + \frac{1}{8}\omega S(2\omega) \right] + \varepsilon^2 \frac{1}{8} \omega^2 H^c(\omega) S(2\omega) \quad (3.13)$$

4 Numerical Results and Discussion

In order to check the accuracy of the approximate results obtained by the method of stochastic averaging, Monte Carlo simulation is applied to compute the moment Lyapunov exponents.

In the Monte Carlo simulation, the viscoelastic kernel function takes the form of Eq. (2.18), i.e.,

$$h(t) = \sum_{j=1}^M \gamma_j e^{-\kappa_j t}$$

where M is selected to be 2. Two different models of wideband noise approximation, i.e., Gaussian white noise and real noise, will be discussed separately.

Case I. The wideband noise is taken as the simplest model, i.e., Gaussian white noise (1.1).

Letting

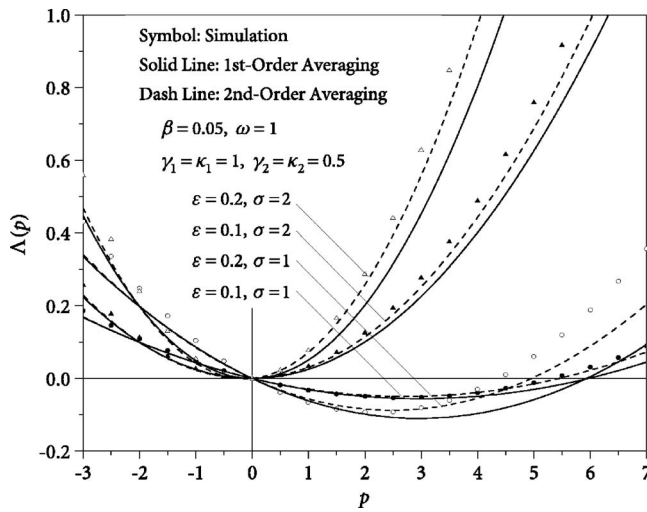


Fig. 1 Moment Lyapunov exponents under white noise excitation for different ε and σ

$$x_1(t) = q(t) \quad x_2(t) = \dot{q}(t) \quad (4.1)$$

$$x_{j+2}(t) = \int_0^t \gamma_j e^{-\kappa_j(t-s)} q(s) ds \quad j = 1, \dots, M$$

Eq. (1.14) can be written as an $(M+2)$ degree-of-freedom system of Itô differential equations

$$d \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{M+2} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\omega^2 & -2\varepsilon\beta & \varepsilon\omega^2 & \cdots & \varepsilon\omega^2 \\ \gamma_1 & 0 & -\kappa_1 & 0 & 0 \\ \vdots & 0 & 0 & \ddots & 0 \\ \gamma_M & 0 & 0 & 0 & -\kappa_M \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{M+2} \end{Bmatrix} dt + \begin{Bmatrix} 0 \\ -\varepsilon^{1/2}\sigma\omega^2 x_1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} dW(t) \quad (4.2)$$

Equation (4.2) is linear homogeneous, and the algorithm introduced in [13] can be applied to simulate the moment Lyapunov exponents. The norm for calculating the moment Lyapunov exponents is $\|\mathbf{x}\| = (x_1^2 + x_2^2)^{1/2}$. The iteration equations are given by, using the explicit Euler scheme [14],

$$\begin{aligned} x_1^{k+1} &= x_1^k + x_2^k \Delta t \\ x_2^{k+1} &= x_2^k + \left(-\omega^2 x_1^k - 2\varepsilon\beta x_2^k + \varepsilon\omega^2 \sum_{j=1}^M x_{j+2}^k \right) \Delta t - \varepsilon^{1/2}\sigma\omega^2 x_1^k \Delta W^k \\ x_j^{k+1} &= x_j^k + (\gamma_j x_1^k - \kappa_j x_{j+2}^k) \Delta t \quad j = 1, \dots, M \end{aligned} \quad (4.3)$$

with Δt being the time step and k denoting the k th iteration.

Figure 1 shows typical results of the moment Lyapunov exponents for different values of ε and σ , where the parameters are taken as $\beta=0.05$, $\gamma_1=\kappa_1=1$, $\gamma_2=\kappa_2=0.5$, and $\omega=1$. The analytical results from the first-order and the second-order averaging are also included in the figure. In Monte Carlo simulation, the sample size for estimating the expected value is $S=5000$, time step is $\Delta t=0.0001$, and the total length of time for simulation is $T=5000$, i.e., the number of iteration is $N=5 \times 10^7$.

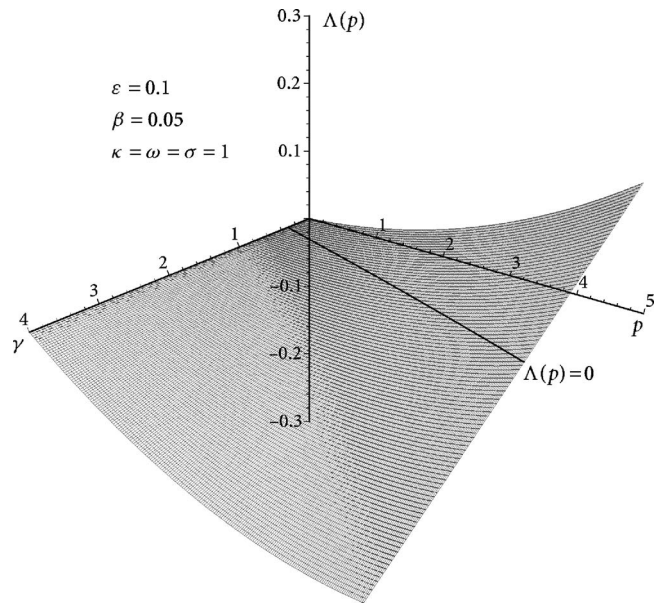


Fig. 2 Moment Lyapunov exponents under white noise excitation

It can be seen that the first-order averaging results agree with the simulation results very well when ε and σ are small, i.e., the intensity of noise is weak. The second-order averaging does give better approximation. As shown in Eqs. (2.14) and (2.15), when the intensity of noise σ increases, the system becomes more and more unstable in the sense that the Lyapunov exponents and moment Lyapunov exponents (for $p > 0$) increase.

Figures 2 and 3 illustrate the variation of moment Lyapunov exponents from second-order averaging with respect to the viscoelastic characteristic parameters γ and κ . The curves $\Delta(p)=0$ give the boundaries of the moment stability. The p th moments for $p > 0$ are of interest in application. As the figures show, with the increase of viscoelastic intensity γ , the stability region for $p > 0$ becomes wider, which indicates that the viscoelasticity helps to stabilize the system. Moreover, when κ increases, i.e., when the relaxation time of viscoelasticity decreases, the stability region for $p > 0$ becomes narrower, implying that larger relaxation time helps to stabilize the system.

Case II. The wideband noise is approximated by a real noise, the Ornstein–Uhlenbeck process, given by Eq. (1.2).

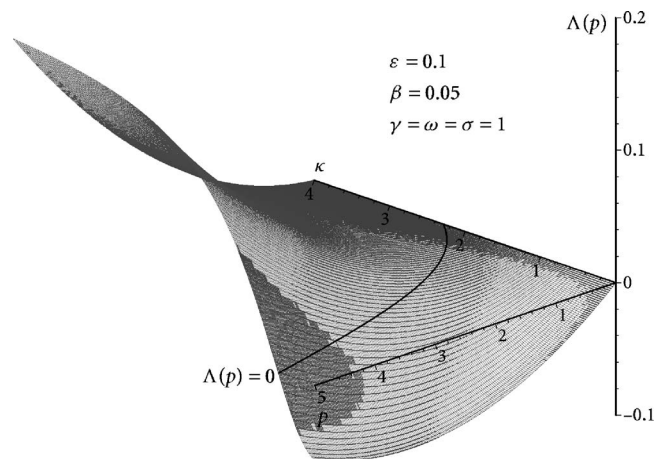


Fig. 3 Moment Lyapunov exponents under white noise excitation

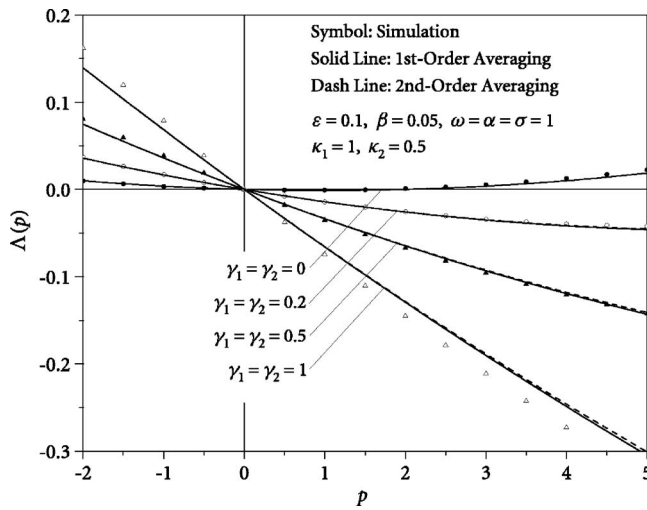


Fig. 4 Moment Lyapunov exponents under real noise excitation for different γ

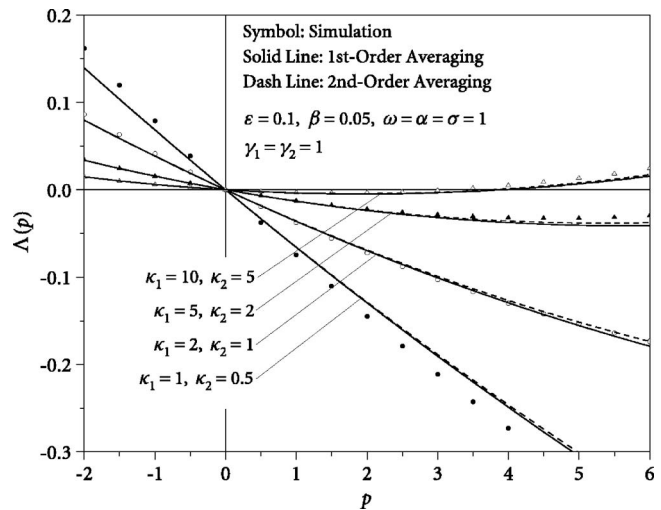


Fig. 5 Moment Lyapunov exponents under real noise excitation for different κ

Denoting

$$x_1(t) = q(t) \quad x_2(t) = \dot{q}(t) \quad (4.4)$$

$$x_{j+2}(t) = \int_0^t \gamma_j e^{-\kappa_j(t-s)} q(s) ds \quad j = 1, \dots, M \quad x_{M+3}(t) = \xi(t)$$

Eq. (1.14) is converted to the Itô differential equations

$$d \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{M+2} \\ x_{M+3} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\omega^2 & -2\varepsilon\beta & \varepsilon\omega^2 & \cdots & \varepsilon\omega^2 & -\varepsilon^{1/2}\omega^2 x_1 \\ \gamma_1 & 0 & -\kappa_1 & 0 & 0 & 0 \\ \vdots & 0 & 0 & \ddots & 0 & 0 \\ \gamma_M & 0 & 0 & 0 & -\kappa_M & 0 \\ 0 & 0 & 0 & 0 & 0 & -\alpha \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{M+2} \\ x_{M+3} \end{pmatrix} dt + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \sigma \end{pmatrix} dW(t) \quad (4.5)$$

Thus, the discretized equations using the explicit Euler scheme are

$$\begin{aligned} x_1^{k+1} &= x_1^k + x_2^k \Delta t \\ x_2^{k+1} &= x_2^k + \left(-\omega^2 x_1^k - 2\varepsilon\beta x_2^k + \varepsilon\omega^2 \sum_{j=1}^M x_{j+2}^k - \varepsilon^{1/2}\omega^2 x_{M+3}^k \right) \Delta t \\ x_{j+2}^{k+1} &= x_{j+2}^k + (\gamma_j x_1^k - \kappa_j x_{j+2}^k) \Delta t \quad j = 1, \dots, M \\ x_{M+3}^{k+1} &= x_{M+3}^k + (-\alpha x_{M+3}^k) \Delta t + \sigma \Delta W^k \end{aligned} \quad (4.6)$$

The norm for calculating the moment Lyapunov exponents is taken as $\|x\| = (x_1^2 + x_2^2)^{1/2}$. The moment Lyapunov exponents for different values of viscoelastic parameters are illustrated in Figs. 4 and 5, with $\varepsilon=0.1$, $\beta=0.05$, and $\omega=\alpha=\sigma=1$. Other parameters for numerical iterations are the same as for the case of Gaussian white noise. Figure 4 shows that the stronger the viscoelasticity (i.e., larger γ_j), the more stable the system, and larger relaxation times (i.e., smaller κ_j) make the system more stable, as shown in Fig. 5. These conclusions are indicated by Eq. (2.17) and are the same as the case of Gaussian white noise excitation.

Figures 4 and 5 also indicate that the second-order averaging method does not improve the accuracy of approximation significantly. Therefore, the approximate results from the first-order averaging are acceptable in engineering applications.

Figure 6 illustrates the comparison of moment Lyapunov exponents for different values of α , with $\varepsilon=0.1$, $\beta=0.05$, $\gamma_1=1$, $\kappa_1=2$, $\gamma_2=\kappa_2=\omega=1$, and $\sigma=2$. As discussed in Sec. 1, the power spectral density of real noise is flatter for larger values of α and thus a real noise can be considered as a wideband noise. The simulation results show that, when α is large but still in the same order as σ , this approximation is acceptable.

It should also be mentioned that, for the Euler scheme in the Monte Carlo simulation of stochastic differential equations, the error of discrete approximation is of the order of Δt in the weak sense, i.e., convergence in probability. Since the numerical estimation of moment Lyapunov exponents requires large time T for simulation, the iteration time step must be small enough; this increases the computation resource significantly. Therefore, the discrepancy between the simulation results and averaging results is partly contributed by the error in discretization.

Figures 7 and 8 show the variation of moment Lyapunov exponents from second-order averaging with respect to the parameters

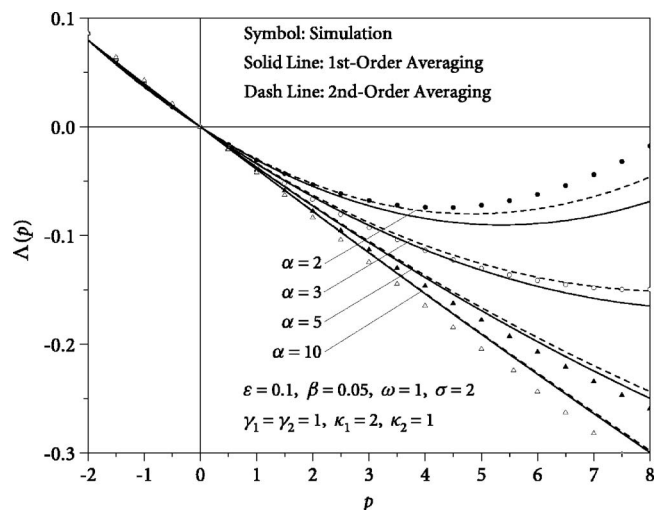


Fig. 6 Moment Lyapunov exponents under real noise excitation for different α

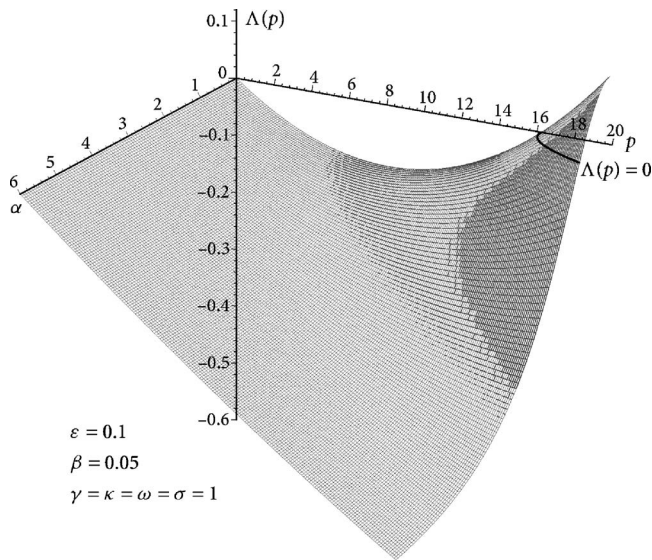


Fig. 7 Moment Lyapunov exponents under real noise excitation

of real noise. From Fig. 7, it can be seen that the larger the noise parameter α , the wider the stability region for $p > 0$, i.e., the more stable the system. According to Eq. (1.3), larger α means that the power of noise spreads over a wider frequency band, which reduces the power of the noise in the neighborhood of resonance. Figure 8 shows that with the increase of noise intensity σ , the stability region of the p th moment (for $p > 0$) dwindles away as expected.

5 Conclusion

To investigate the stochastic stability of a SDOF viscoelastic system under the excitation of wideband noise, the method of

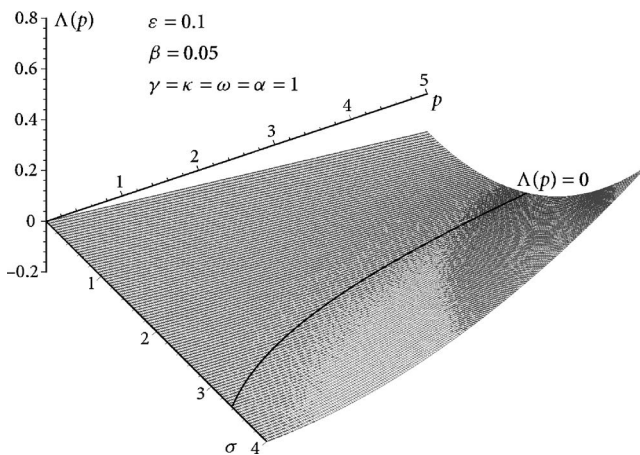


Fig. 8 Moment Lyapunov exponents under real noise excitation

averaging, both first order and second order, is applied to determine the moment Lyapunov exponents. Since the viscoelastic term can be expressed as an integral, the method of stochastic averaging for general stochastic systems by Khasminskii and the method of averaging for integrodifferential equations by Larionov are combined to obtain the approximate moment Lyapunov exponents analytically. Approximate results of the p th moment Lyapunov exponent are obtained for general kernel functions satisfying the boundedness conditions. In Monte Carlo simulation, kernels in the form of combination of Maxwell models are used to illustrate the results. Simulation results show that the averaging methods provide good approximations for SDOF viscoelastic systems under the excitation of weak wideband noises.

It can be concluded, from the approximate analytical results and the Monte Carlo simulation results of the moment Lyapunov exponents, that the increase of the intensity of material relaxation γ , relaxation time $1/\kappa$, and the decrease of noise intensity σ help to stabilize the viscoelastic system. This result is useful in engineering applications.

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Vibrations of Double-Walled Carbon Nanotubes With Different Boundary Conditions Between Inner and Outer Tubes

Free vibrations of a double-walled carbon nanotube (DWNT) are studied. The inner and outer carbon nanotubes are modeled as two individual elastic beams interacting each other by van der Waals forces. An original method is proposed to calculate the first seven order resonant frequencies and relative vibrational modes. Detailed results are demonstrated for DWNTs according to the different boundary conditions between inner and outer tubes, such as fixed-free, cantilever-free, fixed-simple and fixed-fixed (reduced form) supported ends. Our results indicate that there is a special invariable frequency for a DWNT that is not affected by different combinations of boundary conditions. All vibrational modes of the DWNT must be coaxial when the resonant frequency is smaller than this frequency. Some noncoaxial vibrations will occur when their resonant frequencies exceed the frequency. Especially, the first noncoaxial resonant frequency is still invariable for all different boundary conditions. A change of resonant frequency for various lengths of DWNTs is discussed in detail. In addition, our model predicts a new coaxial-noncoaxial vibrational mode in fixed-simple supports for inner and outer tubes of a DWNT. [DOI: 10.1115/1.2793133]

Keywords: carbon nanotube, vibration, resonant frequency, vibrational mode, boundary condition

1 Introduction

Carbon nanotubes (CNTs) have become the most promising materials for nanoelectronics, nanodevices, and nanocomposites because of their unusual electronic properties and superior mechanical strength [1–4]. The mechanical behavior of CNTs, including vibrational behavior, has been the subject of numerous recent studies. Since controlled experiments at nanoscale are difficult and molecular dynamics simulations remain formidable for large scale systems, continuum elastic models have been widely used to study the mechanical behavior of CNTs. The classical Euler and Timoshenko elastic-beam models have been effectively used to study overall mechanical deformation of CNTs, including mechanical properties, static deflection, column buckling, resonant frequencies and modes, and sound wave propagation in CNTs [5–11]. In most previous works, multiwall nanotubes (MWNTs) have been modeled as Euler or Timoshenko elastic beams with the same boundary conditions for all tubes. In particular, the single elastic-beam model has been used to study static and dynamic behaviors of MWNTs [12,13]. As shown in Ref. [14] for the vibration of MWNTs, such a simplified model is adequate for a MWNT of aspect (length-to-diameter) ratio no less than 10. Many proposed applications and designs of CNTs are involved with aspect ratio about 10, or periodically supported CNTs with finite spans. Such examples include suspended crossing CNTs with spans about 20 nm [15], CNTs as single-electron transistors of length down to 20 nm [16], MWNTs of aspect ratio around 20 (about 300 nm long and 10–20 nm in diameter) as electrometers [17] or building blocks in nanoelectronics [18], CNT-nanomechanical switches of aspect ratio around 10 [19], and CNTs of aspect ratio about 10–25 as atomic force microscope

(AFM) tips [20,21]. Owing to the hollow structure of CNTs, short CNTs are preferred in many cases to prevent undesirable kinking and buckling. Therefore, the vibrational behavior of short CNTs, say, of aspect ratio between 10 and 30, is of practical significance. For short CNTs, the existing results show that the noncoaxial intertube vibration of MWNTs will be excited at higher frequencies at which the characteristic wavelength of vibrational modes is just a few times the outermost diameter of MWNTs. For instance, for a 1.4 nm diameter double-walled carbon nanotube (DWNT) of aspect ratio between 10 and 30, the wavelength of the higher-order modes is just a few times the outermost diameter, and the associated vibrational modes are substantially noncoaxial. Since noncoaxial distortion could significantly affect some important physical (such as electronic and optical) properties of MWNTs, the study of noncoaxial vibration has attracted attention recently. A noncoaxial vibrational mode, firstly predicted by a simple linear multiple-beam model [14], is found to agree well with more recent atomistic simulations on the noncoaxial vibration of MWNTs [22,23].

Most CNTs to date have been synthesized with closed ends [4]. For an application of a MWNT, both its ends can be restricted only on the outer tube. For example, in a nanoelectromechanical system (NEMS), the small size and unique properties of CNTs suggest that they can be used in sensor devices with unprecedented sensitivity [24]. Other relevant issues to be clarified are the effects of differential boundary supports between the inner and outer tubes on the vibration of MWNTs and boundary effects on transverse vibration devices composed of rods in microelectromechanical systems (MEMSs) [25]. It is expected that the differences of boundary conditions, which are ignored in the existing beam model, would play an important role in the vibration of a DWNT when the vibrational modes at a resonant frequency between the two tubes are considered. Especially for short DWNTs, some changes of boundary conditions may affect the vibrational modes more sensitively. For this reason, the relevance of the existing

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model, in which both tubes have the same boundary conditions for the vibration of DWNTs, is questionable. To clarify this issue, free vibrations of DWNTs with differential boundary supports between inner and outer tubes are studied in this paper based on a double elastic-beam model developed in Refs. [10,14]. Unlike the previous work, however, the boundary conditions of DWNTs in the present paper are represented as different combinations of the inner and outer tubes, instead of the same boundary conditions used in all previous related works. The major goal of this paper is to study the free vibrations of DWNTs with different boundary conditions between inner and outer tubes. The first seven order resonant frequencies and related vibrational modes are calculated by the proposed method for a short DWNT with an inner diameter of 0.7 nm, an outer diameter of 1.4 nm, and length of 14 nm. Our results indicate that a special frequency exists, which can distinguish the coaxial and first noncoaxial resonant frequencies of a DWNT. This special frequency and the first noncoaxial resonant frequency are not affected by different combinations of boundary conditions. A new coaxial-noncoaxial vibrational mode is found in fixed-simple supports for inner and outer tubes of a DWNT. In addition, the changes of resonant frequency for various lengths of DWNTs and a reduced situation such as both inner and outer tubes fixed are discussed in detail.

2 Mode Functions for the Vibration of a Double-Walled Carbon Nanotube

Many studies have showed that the classical Euler elastic-beam offers a reliable model for the overall mechanical deformation of CNTs [5]. For example, static deflection of CNTs under point load is found to be well predicted by the beam model [6], and resonant frequencies and vibrational modes of CNTs given by the cantilever beam model are in good agreement with experimental data [12]. In particular, because elastic-beam models give simple general formulas in many important cases, such as critical stress for column buckling, resonant frequencies, and sound speeds, which clearly indicate major factors affecting the mechanical behavior of CNTs, they have the potential to identify key parameters and predict some physical phenomena.

It is known that the governing equations for a linear free vibration of a DWNT are

$$\begin{aligned} c_1(w_2 - w_1) &= EI_1 \frac{\partial^4 w_1}{\partial x^4} + \rho A_1 \frac{\partial^2 w_1}{\partial t^2} \\ -c_1(w_2 - w_1) &= EI_2 \frac{\partial^4 w_2}{\partial x^4} + \rho A_2 \frac{\partial^2 w_2}{\partial t^2} \end{aligned} \quad (1)$$

where x is the axial coordinate, t is time, $w_k = (x, t)$, I_k , and A_k ($k=1,2$) are the deflections, moments of inertia, and cross-sectional area of the inner tube and outer tube, respectively. We assume that two tubes have the same Young's modulus $E = 1$ TPa and the mass density $\rho = 2.3$ g/cm³ [9,10]. The van der Waals (vdW) interlayer interaction coefficient $c_1 = \partial^2 U / \partial \delta^2|_{\delta=\delta_0} \times 2R$ (where R is the inner tube radius of the DWNT) is derived from the interlayer potential per unit area U , which is expressed in terms of the interlayer spacing δ , namely,

$$U(\delta) = K \left[\left(\frac{\delta_0}{\delta} \right)^4 - 0.4 \left(\frac{\delta_0}{\delta} \right)^{10} \right] \quad (2)$$

where $K = -61.665$ meV/atom and $\delta_0 = 0.34$ is the equilibrium spacing of which the vdW force is zero. For example, if the inner tube diameter is 0.7 nm and the outer tube diameter is 1.4 nm, then $c_1 = 71.11$ GPa [10].

In general, the inner and outer tubes should have different vibrational modes $Y_k(x)$ ($k=1,2$). Then, the displacements of the inner and outer tubes can be represented by

$$w_1 = Y_1 e^{i\omega t} \quad w_2 = Y_2 e^{i\omega t} \quad (3)$$

where ω is a resonant frequency of the DWNT. Substituting expressions (3) into Eqs. (1), and dividing by $e^{i\omega t}$, yield

$$\begin{aligned} EI_1 Y_1^4 - \rho A_1 \omega^2 Y_1 &= c(Y_2 - Y_1) \\ EI_2 Y_2^4 - \rho A_2 \omega^2 Y_2 &= -c(Y_2 - Y_1) \end{aligned} \quad (4)$$

Furthermore, one can write the vibrational modes as

$$Y_1 = B_1 e^{\lambda x} \quad Y_2 = B_2 e^{\lambda x} \quad (5)$$

where B_k ($k=1,2$) and λ are constants that wait for a determination. Substituting expressions (5) into Eqs. (4), and dividing by $e^{\lambda x}$, yields a matrix equation,

$$\begin{pmatrix} c + EI_1 \lambda^4 - \rho A_1 \omega^2 & -c \\ -c & c + EI_2 \lambda^4 - \rho A_2 \omega^2 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = 0 \quad (6)$$

Accordingly, B_k ($k=1,2$) can be determined by the existence condition for a nonzero solution of Eq. (6), which leads to an eight order algebraic equation,

$$\left(\lambda^4 + \frac{c - \rho A_1 \omega^2}{EI_1} \right) \left(\lambda^4 + \frac{c - \rho A_2 \omega^2}{EI_2} \right) - \frac{c^2}{E^2 I_1 I_2} = 0 \quad (7)$$

In order to solve Eq. (7), let $\lambda^4 = \sigma$ give

$$\left(\sigma + \frac{c - \rho A_1 \omega^2}{EI_1} \right) \left(\sigma + \frac{c - \rho A_2 \omega^2}{EI_2} \right) - \frac{c^2}{E^2 I_1 I_2} = 0 \quad (8)$$

The quadratic equation (Eq. (8)) for the unknown quantity σ has two real roots,

$$\sigma_{1,2} = \frac{-((c - \rho A_1 \omega^2)/EI_1 + (c - \rho A_2 \omega^2)/EI_2) \pm \sqrt{\Delta}}{2} \quad (9)$$

where

$$\Delta = \left(\frac{c - \rho A_1 \omega^2}{EI_1} - \frac{c - \rho A_2 \omega^2}{EI_2} \right)^2 + 4 \frac{c^2}{E^2 I_1 I_2} > 0 \quad (10)$$

For any resonant frequency ω , σ_1 is always bigger than zero, but the sign of σ_2 is changed from minus to positive for increasing values of ω . A specific frequency ω_0 exists in Eq. (9) so that

$$\sigma_1 > 0 \quad \sigma_2 \begin{cases} < 0 & (\omega < \omega_0) \\ > 0 & (\omega > \omega_0) \end{cases} \quad (11)$$

For example, with a DWNT of length of 14 nm, outer tube diameter of 1.4 nm, and interspace between outer and inner tubes of 0.34 nm, Eq. (9) leads to

$$\omega_0 = 7.8456 \times 10^{12} \text{ Hz} \quad (12)$$

When the resonant frequency $\omega < \omega_0$, for $\lambda^4 = \sigma_1$ ($\sigma_1 > 0$), we have

$$\lambda_1 = \sigma_1^{1/4} \quad \lambda_2 = -\sigma_1^{1/4} \quad \lambda_3 = i\sigma_1^{1/4} \quad \lambda_4 = -i\sigma_1^{1/4} \quad (13)$$

For $\lambda^4 = \sigma_2$ ($\sigma_2 < 0$), the other four roots are complex numbers, namely,

$$\begin{aligned} \lambda_5 &= (1+i) \frac{\sqrt{2}}{2} (-\sigma_2)^{1/4} & \lambda_6 &= -(1+i) \frac{\sqrt{2}}{2} (-\sigma_2)^{1/4} \\ \lambda_7 &= (-1+i) \frac{\sqrt{2}}{2} (-\sigma_2)^{1/4} & \lambda_8 &= (1-i) \frac{\sqrt{2}}{2} (-\sigma_2)^{1/4} \end{aligned} \quad (14)$$

When the resonant frequency $\omega > \omega_0$, and then $\sigma_1 > 0$ and $\sigma_2 > 0$, too, all eight roots for $\lambda^4 = \sigma_{1,2}$ can be represented by real and imaginary numbers,

$$\lambda_1 = \sigma_1^{1/4} \quad \lambda_2 = -\sigma_1^{1/4} \quad \lambda_3 = i\sigma_1^{1/4} \quad \lambda_4 = -i\sigma_1^{1/4}$$

$$\lambda_5 = \sigma_2^{1/4} \quad \lambda_6 = -\sigma_2^{1/4} \quad \lambda_7 = i\sigma_2^{1/4} \quad \lambda_8 = -i\sigma_2^{1/4} \quad (15)$$

According to Eq. (5), the vibrational modes for inner and outer tubes are linear combinations, e.g.,

$$Y_1 = \sum_{i=1}^8 B_{1i} e^{\lambda_i x} \quad Y_2 = \sum_{i=1}^8 B_{2i} e^{\lambda_i x} \quad (16)$$

- (i) When $\omega < \omega_0$, from Eqs. (13) and (14) the vibrational mode functions Y_1 and Y_2 are

$$Y_k = A_{k1} \sin ax + A_{k2} \cos ax + A_{k3} \sinh ax + A_{k4} \cosh ax \\ + A_{k5} \sin bx \sin bx + A_{k6} \sinh bx \cos bx \\ + A_{k7} \cosh bx \sin bx + A_{k8} \cosh bx \cos bx \quad (k=1,2) \quad (17)$$

where the coefficients A_{1i}, A_{2i} ($i=1,2,\dots,8$) are linearly combined with B_{1i}, B_{2i} ($i=1,2,\dots,8$), and

$$a = (\sigma_1)^{1/4} \quad b = \frac{\sqrt{2}}{2}(-\sigma_2)^{1/4} \quad (18)$$

- (ii) When resonant frequency $\omega > \omega_0$, and then $\sigma_1 > 0$ and $\sigma_2 > 0$, from Eqs. (15) the vibrational mode functions Y_1 and Y_2 are

$$Y_k = A_{k1} \sin ax + A_{k2} \cos ax + A_{k3} \sinh ax + A_{k4} \cosh ax \\ + A_{k5} \sin bx + A_{k6} \cos bx + A_{k7} \sinh bx \\ + A_{k8} \cosh bx \quad (k=1,2) \quad (19)$$

where

$$a = (\sigma_1)^{1/4} \quad b = (\sigma_2)^{1/4} \quad (20)$$

On the other hand, substituting arbitrary λ_i ($i=1,2,\dots,8$) into any one of Eq. (6) (because the two equations in Eq. (6) are linear dependence) yields

$$B_{2i} = \frac{\lambda_i^4 EI_1 + c - \rho A_1 \omega^2}{c} B_{1i} \quad (i=1,2,\dots,8) \quad (21)$$

It is easily verified that A_{1i}, A_{2i} ($i=1,2,\dots,8$) obey

$$A_{2i} = \frac{s^4 EI_1 + c - \rho A_1 \omega^2}{c} A_{1i} \quad (i=1,2,\dots,8) \quad (22)$$

where

$$s = \begin{cases} \sigma_1 & (i=1,2,3,4) \\ \sigma_2 & (i=5,6,7,8) \end{cases} \quad (23)$$

Now, there are eight constants A_{1i} (or A_{2i}) ($i=1,2,\dots,8$) in Eqs. (17) and (19), which represent the vibrational modes of a DWNT to be determined by boundary conditions.

3 Vibrations of a Double-Walled Carbon Nanotube With a Free Inner Tube and a Fixed Outer Tube

Let us consider a DWNT with a free inner tube and a fixed outer tube; the length (l) is 14 nm and the outer tube diameter is 1.4 nm. For the free inner tube and fixed outer tube the boundary conditions satisfy the equations

$$\frac{\partial^2}{\partial x^2} w_1(0,t) = \frac{\partial^2}{\partial x^2} w_1(l,t) = 0$$

$$\frac{\partial^3}{\partial x^3} w_1(0,t) = \frac{\partial^3}{\partial x^3} w_1(l,t) = 0$$

$$w_2(0,t) = w_2(l,t) = 0$$

$$\frac{\partial}{\partial x} w_2(0,t) = \frac{\partial}{\partial x} w_2(l,t) = 0 \quad (24)$$

From Eqs. (24), It is easy to obtain the vibrational mode $Y_k(x)$ ($k=1,2$) and obey the boundary conditions as follows:

$$\frac{d^2}{dx^2} Y_1(0) = \frac{d^2}{dx^2} Y_1(l) = 0 \quad \frac{d^3}{dx^3} Y_1(0) = \frac{d^3}{dx^3} Y_1(l) = 0$$

$$Y_2(0) = Y_2(l) = 0 \quad \frac{d}{dx} Y_2(0) = \frac{d}{dx} Y_2(l) = 0 \quad (25)$$

- (i) When the resonant frequency $\omega < \omega_0 = 7.8456 \times 10^{12}$ Hz, substituting Y_k ($k=1,2$) represented in Eq. (17) into the boundary conditions (25) and noticing Eq. (22) gives eight linear algebraic equations for unknown constants A_{1i} ($i=1,2,\dots,8$) in the following forms:

$$\begin{aligned} & -a^2 A_{12} + a^2 A_{14} + 2b^2 A_{15} = 0 \\ & -a^3 A_{11} + a^3 A_{13} - 2b^3 A_{16} + 2b^3 A_{17} = 0 \\ & -a^2 A_{11} \sin al - a^2 A_{12} \cos al + a^2 A_{13} \sinh al \\ & + a^2 A_{14} \cosh al + 2b^2 A_{15} \cos bl \\ & - 2b^2 A_{16} \cosh bl \sin bl + 2b^2 A_{17} \cos bl \sinh bl \\ & - 2b^2 A_{18} \sin bl \sinh bl = 0 \\ & -a^3 A_{11} \cos al + a^3 A_{12} \sin al \\ & + a^3 A_{13} \cosh al + a^3 A_{14} \sinh al \\ & + 2b^3 (-\cosh bl \sin bl + \cos bl \sinh bl) A_{15} \\ & - 2b^3 (\cos bl \cosh bl + \sin bl \sinh bl) A_{16} \\ & + 2b^3 (\cos bl \cosh bl - \sin bl \sinh bl) A_{17} \\ & - 2b^3 (\cosh bl \sin bl + \cos bl \sinh bl) A_{18} = 0 \\ & ag A_{11} + ag A_{13} + bh A_{16} + bh A_{17} = 0 \\ & g A_{12} + g A_{14} + h A_{18} = 0 \\ & g A_{12} \cos al + g A_{14} \cosh al + h A_{18} \cos bl \cosh bl \\ & + g A_{11} \sin al + h A_{17} \cosh bl \sin bl + g A_{13} \sinh al \\ & + h A_{16} \cos bl \sinh bl + h A_{15} \sin bl \sinh bl = 0 \\ & ag A_{11} \cos al - ag A_{12} \sin al \\ & + ag A_{13} \cosh al + ag A_{14} \sinh al \\ & + bh A_{15} (\cosh bl \sin bl + \cos bl \sinh bl) \\ & + bh A_{16} (\cos bl \cosh bl - \sin bl \sinh bl) \\ & + bh A_{17} (\cos bl \cosh bl + \sin bl \sinh bl) \\ & - bh A_{18} (\cosh bl \sin bl - \cos bl \sinh bl) = 0 \end{aligned} \quad (26)$$

where g and h are the amplitude ratio A_{2i}/A_{1i} ($i=1,2,\dots,8$) of every harmonic vibration, which are defined as

$$g = \frac{\sigma_1^4 EI_1 + c - \rho A_1 \omega^2}{c} \quad h = \frac{\sigma_2^4 EI_1 + c - \rho A_1 \omega^2}{c} \quad (27)$$

Each order resonant frequency ω can be determined by the existence condition for a nonzero solution A_{1i} ($i=1,2,\dots,8$) of linear homogenous equations (Eqs. (26)) when the resonant frequency $\omega < \omega_0 = 7.8456 \times 10^{12}$ Hz.

- (ii) When the resonant frequency $\omega > \omega_0 = 7.8456 \times 10^{12}$ Hz, it is evident that the vibrational modes Y_k ($k=1,2$) are

Table 1 Resonant frequencies (10^{12} Hz) of a DWNT with a free inner tube and a fixed outer tube with an inner diameter of 0.7 nm, an outer diameter of 1.4 nm, and length of 14 nm

1	2	3	Mode 4	5	6	7
1.04	2.84	5.14	7.89	8.13	8.38	9.35

adopted by Eq. (19). From boundary conditions (25) and Eq. (22), the coefficients A_{1i} ($i=1,2,\dots,8$) satisfy linear homogenous equations,

$$\begin{aligned}
& -a^2A_{12} + a^2A_{14} + 2b^2A_{15} = 0 \\
& -a^3A_{11} + a^3A_{13} - 2b^3A_{16} + 2b^3A_{17} = 0 \\
& -a^2A_{11} \sin al - a^2A_{12} \cos al + a^2A_{13} \sinh al \\
& + a^2A_{14} \cosh al + 2b^2A_{15} \cos bl \\
& - 2b^2A_{16} \cosh bl \sin bl + 2b^2A_{17} \cos bl \sinh bl \\
& - 2b^2A_{18} \sin bl \sinh bl = 0 \\
& -a^3A_{11} \cos al + a^3A_{12} \sin al \\
& + a^3A_{13} \cosh al + a^3A_{14} \sinh al \\
& + 2b^3(-\cosh bl \sin bl + \cos bl \sinh bl)A_{15} \\
& - 2b^3(\cos bl \cosh bl + \sin bl \sinh bl)A_{16} \\
& + 2b^3(\cos bl \cosh bl - \sin bl \sinh bl)A_{17} \\
& - 2b^3(\cosh bl \sin bl + \cos bl \sinh bl)A_{18} = 0 \\
& agA_{11} + agA_{13} + bhA_{15} + bhA_{17} = 0 \\
& gA_{12} + gA_{14} + hA_{16} + hA_{18} = 0 \\
& gA_{12} \cos al + hA_{16} \cos bl + gA_{14} \cosh al + hA_{18} \cosh bl \\
& + gA_{11} \sin al + hA_{15} \sin bl + gA_{13} \sinh al \\
& + hA_{17} \sinh bl = 0 \\
& agA_{11} \cos al - agA_{12} \sin al + agA_{13} \cosh al \\
& + agA_{14} \sinh al + bhA_{15} \cos bl - bhA_{16} \sin bl \\
& + bhA_{17} \cosh bl + bhA_{18} \sinh bl = 0 \quad (28)
\end{aligned}$$

where g and h are the same as expression (27). The resonant frequency ω can be determined by the existence condition for a nonzero solution A_{1i} ($i=1,2,\dots,8$) of Eqs. (28) when the resonant frequency $\omega > \omega_0 = 7.8456 \times 10^{12}$ Hz.

From the lowest resonant frequency, the first seven order resonant frequencies are shown in Table 1. It can be found that from the fourth resonant frequency, all higher-order frequencies are bigger than $\omega_0 = 7.8456 \times 10^{12}$ Hz. Substituting the resonant frequencies ω_j ($j=1,2,3$) into Eqs. (18) and (22), then combining with Eq. (17), as well as substituting ω_j ($j=4,5,6,7$) into Eqs. (20) and (22), then combining with Eq. (19), one can obtain related vibrational modes for eight order resonant frequencies ω_j ($j=1,2,\dots,8$), which are shown in Fig. 1. It is seen from Fig. 1 that the first three vibrational modes are coaxial, and some noncoaxial vibrational modes occur when the resonant frequency $\omega > \omega_0 = 7.8456 \times 10^{12}$ Hz, such as the fourth, fifth, and seventh modes. The vibration at a higher resonant frequency causes complex noncoaxial distortion of the DWNT. Of course, the jump of the de-

flections between two tubes is bounded by the initial intertube spacing (about 0.34 nm). This is not a problem for small-deflection linear vibrations studied here.

Furthermore, let us consider the effect of the length of a DWNT on the resonant frequency. The first three order resonant frequencies of a DWNT with various lengths are listed in Table 2. It is observed that any resonant frequency decreases for increasing length of the DWNT.

4 Vibrations of a Double-Walled Carbon Nanotube With a Free Inner Tube and a Cantilever Outer Tube

In this case, it is evident that the governing equations for a linear free vibration of a DWNT are the same as Eqs. (1). The boundary conditions for a free inner tube and a cantilever outer tube can be written as

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} w_1(0, t) &= \frac{\partial^2}{\partial x^2} w_1(l, t) = 0 \\
\frac{\partial^3}{\partial x^3} w_1(0, t) &= \frac{\partial^3}{\partial x^3} w_1(l, t) = 0 \\
w_2(0, t) &= \frac{\partial}{\partial x} w_2(0, t) = 0 \\
\frac{\partial^2}{\partial x^2} w_2(l, t) &= \frac{\partial^3}{\partial x^3} w_2(l, t) = 0 \quad (29)
\end{aligned}$$

The vibrational modes $Y_k(x)$ ($k=1,2$) satisfy the boundary conditions, which are derived from Eqs. (29), namely,

$$\begin{aligned}
\frac{d^2}{dx^2} Y_1(0) &= \frac{d^2}{dx^2} Y_1(l) = 0 \quad \frac{d^3}{dx^3} Y_1(0) = \frac{d^3}{dx^3} Y_1(l) = 0 \\
Y_2(0) &= \frac{d}{dx} Y_2(0) = 0 \quad \frac{d^2}{dx^2} Y_2(l) = \frac{d^3}{dx^3} Y_2(l) = 0 \quad (30)
\end{aligned}$$

Using the same method and procedure shown in Sec. 3, the vibrational mode functions are given by expression (17) when resonant frequency $\omega < \omega_0 = 7.8456 \times 10^{12}$ Hz, and by expression (19) when $\omega > \omega_0 = 7.8456 \times 10^{12}$ Hz. The first seven order resonant frequencies are shown in Table 3.

It can be found that the sixth and seventh resonant frequencies are bigger than $\omega_0 = 7.8456 \times 10^{12}$ Hz in Table 3. In comparison with Table 1, any order resonant frequency of a DWNT with a free inner tube and a cantilever outer tube is smaller than the same order resonant frequency of the DWNT with a free inner tube and a fixed outer tube. The related vibrational modes are shown in Fig. 2. It is observed from Fig. 2 that the first five vibrational modes are coaxial, and when the resonant frequency $\omega > \omega_0 = 7.8456 \times 10^{12}$ Hz, the sixth and seventh modes are noncoaxial. Additionally, our results show that with the increase of resonant frequency, the vibrational modes of the inner and outer tubes gradually turn to noncoaxial vibrations from coaxial vibrations.

Table 4 lists the first three order resonant frequencies of a DWNT with various lengths for this end conditions. Like free vibrations of a single beam, any resonant frequency decreases for increasing length of the DWNT.

5 Vibrations of a Double-Walled Carbon Nanotube With a Simple Inner Tube and a Fixed Outer Tube

Besides governing Eqs. (1), the boundary conditions for a simple inner tube and a fixed outer tube can be written as

$$\begin{aligned}
w_1(0, t) &= w_1(l, t) = 0 \\
\frac{\partial^2}{\partial x^2} w_1(0, t) &= \frac{\partial^2}{\partial x^2} w_1(l, t) = 0
\end{aligned}$$

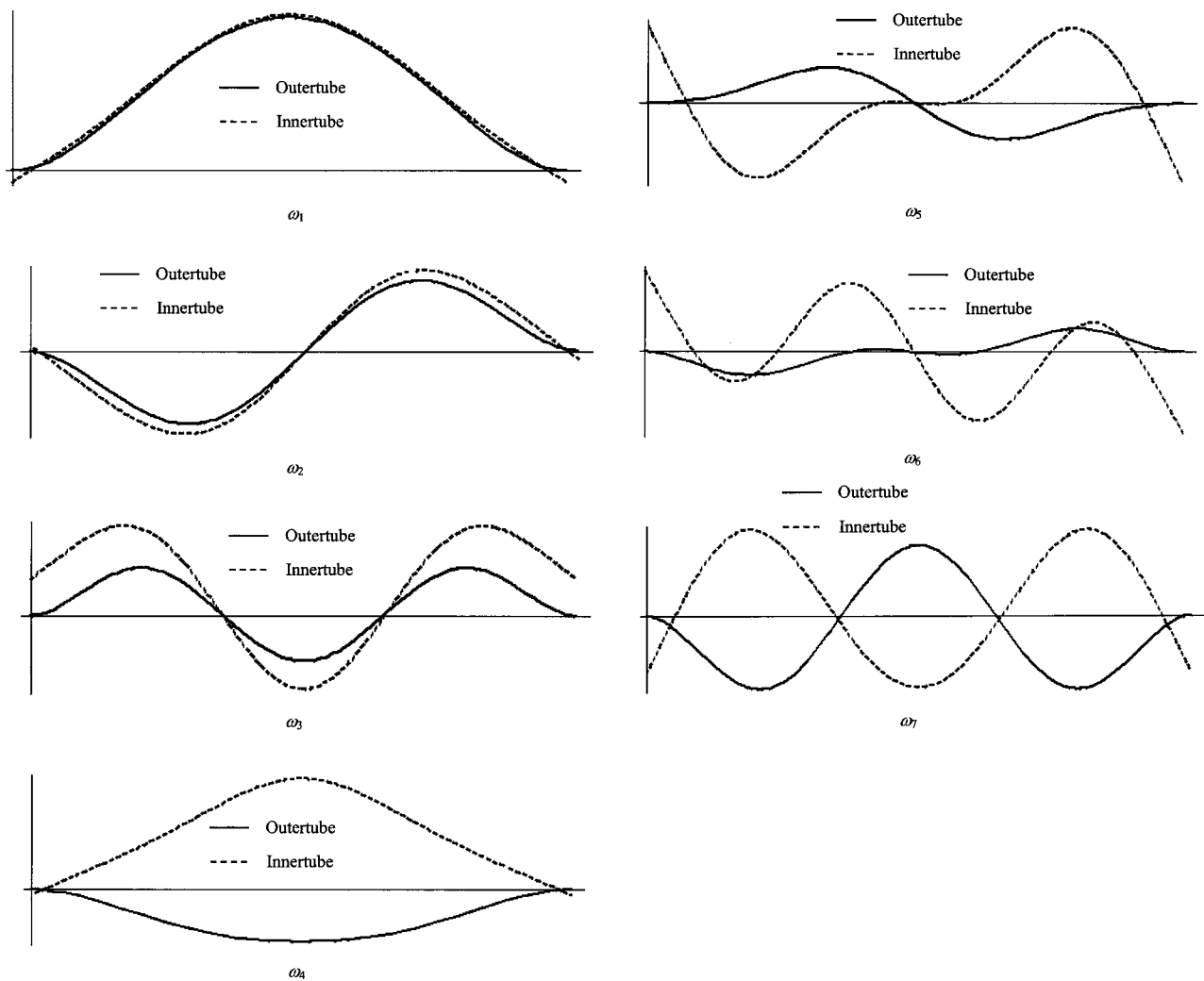


Fig. 1 The first seven vibrational modes of a DWNT with a free inner tube and a fixed outer tube

$$w_2(0, t) = w_2(l, t) = 0$$

$$\frac{\partial}{\partial x} w_2(0, t) = \frac{\partial}{\partial x} w_2(l, t) = 0 \quad (31)$$

From Eqs. (31), vibrational mode $Y_k(x)$ ($k=1,2$) satisfies the boundary conditions,

Table 2 Resonant frequencies (10^{12} Hz) of a DWNT with a free inner tube and a fixed outer tube for various lengths when the outer diameter is 1.4 nm

Length (nm)	14	16	18	20	24	26	28	30
ω_1	1.04	0.80	0.53	0.51	0.36	0.31	0.26	0.23
ω_2	2.84	2.20	1.75	1.42	0.99	0.85	0.73	0.64
ω_3	5.14	4.16	3.36	2.76	1.94	1.66	1.43	1.25

Table 3 Resonant frequencies (10^{12} Hz) of a DWNT with a free inner tube and a cantilever outer tube when the inner diameter is 0.7 nm, the outer diameter is 1.4 nm, and the length is 14 nm

	1	2	3	Mode 4	5	6	7
	0.17	1.04	2.89	5.29	6.55	7.89	8.17

$$Y_1(0) = Y_1(l) = 0 \quad \frac{d^2}{dx^2} Y_1(0) = \frac{d^2}{dx^2} Y_1(l) = 0$$

$$Y_2(0) = Y_2(l) = 0 \quad \frac{d}{dx} Y_2(0) = \frac{d}{dx} Y_2(l) = 0 \quad (32)$$

Still like the previous sections, here, the first seven order resonant frequencies are shown in Table 5. Because the fifth order resonant frequency $\omega_5 > \omega_0 = 7.8456 \times 10^{12}$ Hz, some vibrational modes must be noncoaxial when the resonant frequency $\omega \geq \omega_5$.

As shown in Fig. 3, the first four order vibrational modes are coaxial, and the fifth and sixth modes are noncoaxial. It is also observed from Fig. 3 that the seventh mode includes not only coaxial vibration in the middle of the DWNT, but also noncoaxial vibrations that occur in both end parts of the DWNT. It is a new vibrational phenomenon.

The first three order resonant frequencies of a DWNT with various lengths for this end conditions are listed in Table 6.

6 Vibrations of a Double-Walled Carbon Nanotube With Both Fixed Inner and Outer Tubes

In order to compare some results obtained by our model with other existing results, finally, let us consider vibration problems of a DWNT with both fixed inner and outer tubes. The governing equations are still in the form of Eqs. (1), and boundary conditions are reduced to

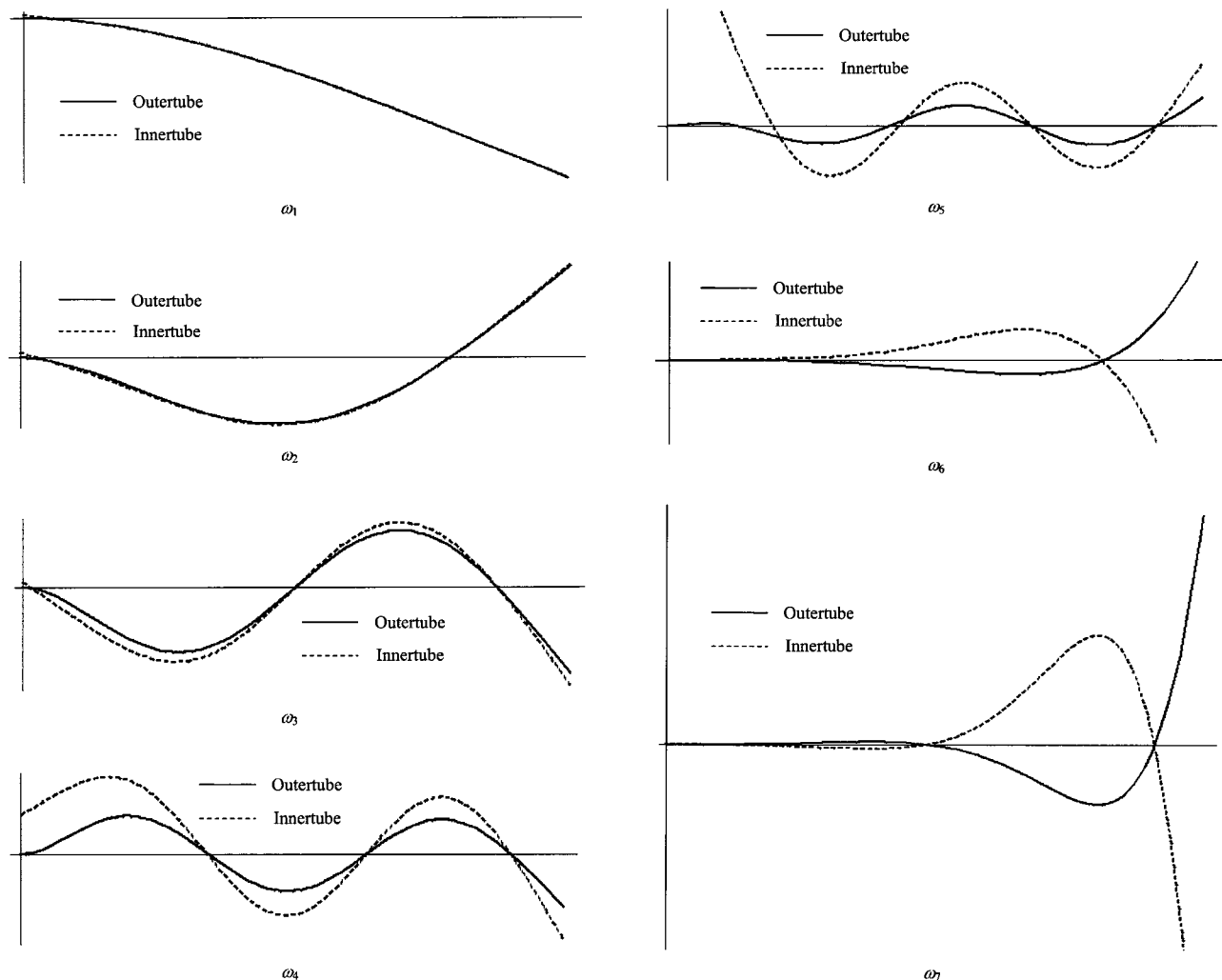


Fig. 2 The first seven vibrational modes of a DWNT with a free inner tube and a cantilever outer tube

$$w_k(0, t) = w_k(l, t) = 0 \quad \frac{\partial}{\partial x} w_k(0, t) = \frac{\partial}{\partial x} w_k(l, t) = 0 \quad (k = 1, 2) \quad (33)$$

From Eqs. (33), vibrational mode $Y_k(x)$ ($k=1, 2$) obeys

Table 4 Resonant frequencies (10^{12} Hz) of a DWNT with a free inner tube and a cantilever outer tube for various lengths when the outer diameter is 1.4 nm

Length (nm)	14	16	18	20	24	26	28	30
ω_1	0.17	0.13	0.10	0.082	0.057	0.046	0.042	0.036
ω_2	1.04	0.80	0.63	0.51	0.36	0.30	0.28	0.25
ω_3	2.89	2.23	1.77	1.44	1.00	0.85	0.74	0.62

Table 5 Resonant frequencies (10^{12} Hz) of a DWNT with a simple inner tube and a fixed outer tube when the inner diameter is 0.7 nm, the outer diameter is 1.4 nm, and the length is 14 nm

	1	2	3	Mode 4	5	6	7
	1.05	2.84	5.18	7.29	7.89	8.24	9.08

$$Y_k(0) = Y_k(l) = 0 \quad \frac{d}{dx} Y_k(0) = \frac{d}{dx} Y_k(l) = 0 \quad (k = 1, 2) \quad (34)$$

Through simulation computing the first six order resonant frequencies of the DWNT are shown in Table 7. Noncoaxial vibrational modes will occur when the resonant frequency $\omega \geq \omega_4$. In particular, frequencies ω_1 and ω_4 are almost consistent with the existing results [10], where it is shown that the first coaxial and noncoaxial resonant frequencies are 1.06×10^{12} Hz and 7.75×10^{12} Hz, respectively.

Six vibrational modes with respect to their resonant frequencies listed in Table 7 are shown in Fig. 4. It is seen from Fig. 4 that the first noncoaxial vibrational mode occurs when resonant frequency reaches the frequency ω_4 ($> \omega_0 = 7.8456 \times 10^{12}$ Hz). In addition, for both fixed inner and outer tubes of a DWNT, the inner-to-outer tube amplitude ratio (Y_1/Y_2) for any resonant frequency can be accurately calculated. For example, the amplitude ratios from the resonant frequencies ω_1 to ω_6 are shown in Table 8. It is found from Table 8 that the amplitude ratio of the lowest frequency ω_1 is 1.02, which is in good agreement with close to unity [10,14]. On the other hand, the amplitude ratio of the first noncoaxial vibrational mode (corresponding to the frequency ω_4) is -1.96 , which is almost equal to -2 . [10].

We can verify from computing simulation that for vibrations of a DWNT with both fixed inner and outer tubes, the first four terms in mode expression (17) and (19) are dominant when coaxial vi-

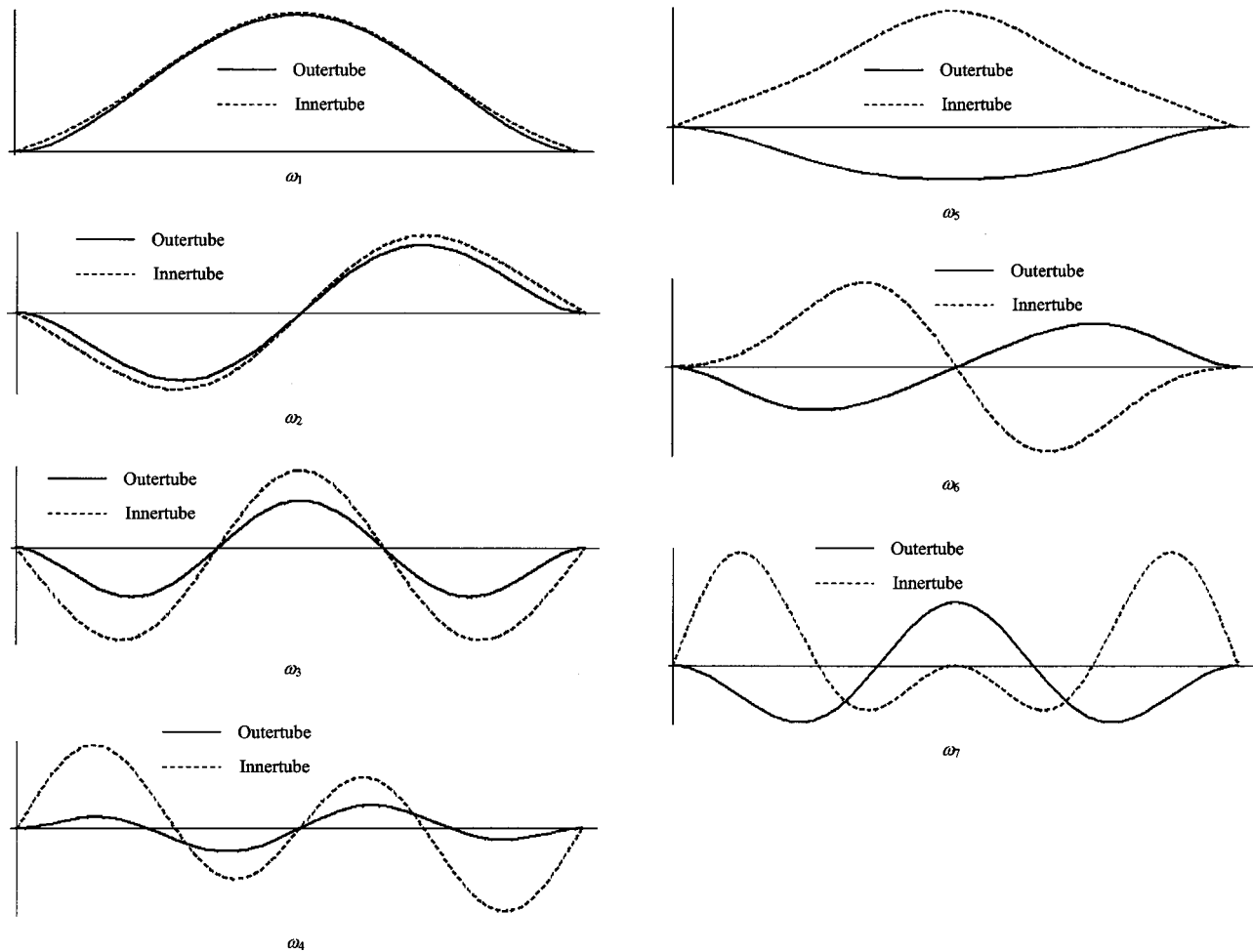


Fig. 3 The first seven vibrational modes of a DWNT with a simple inner tube and a fixed outer tube

brations occur; in other words, the coefficients of the last four terms in Eq. (17) and (19) become very small. Quite the contrary, for noncoaxial vibrations, the conclusion is just like what we expect, the last four terms in mode expression (19) are dominant in these situations. Thus, vibrational modes of a DWNT with both fixed inner and outer tubes can be written as simply approximate expressions from Eqs. (17) and (19) after neglecting the small coefficients. In particular, a coaxial vibrational mode is

Table 6 Resonant frequencies (10^{12} Hz) of a DWNT with a simple inner tube and a fixed outer tube for various lengths when the outer diameter is 1.4 nm

Length (nm)	14	16	18	20	24	26	28	30
ω_1	1.05	0.81	0.64	0.52	0.36	0.31	0.27	0.23
ω_2	2.84	2.20	1.75	1.43	1.00	0.85	0.73	0.64
ω_3	5.18	4.17	3.36	2.76	1.94	1.66	1.43	1.25

Table 7 Resonant frequencies (10^{12} Hz) of a DWNT with both fixed inner and outer tubes when the inner diameter is 0.7 nm, the outer diameter is 1.4 nm, and the length is 14 nm

Mode					
1	2	3	4	5	6
1.08	2.94	5.49	7.90	8.13	8.24

$$Y_k = A_{k1} \sin ax + A_{k2} \cos ax + A_{k3} \sinh ax + A_{k4} \cosh ax \quad (k = 1, 2) \quad (35)$$

and a noncoaxial vibrational mode

$$Y_k = A_{k5} \sin bx + A_{k6} \cos bx + A_{k7} \sinh bx + A_{k8} \cosh bx \quad (k = 1, 2) \quad (36)$$

where a and b are also defined by Eqs. (18) ($\omega < \omega_0$) or Eqs. (20) ($\omega > \omega_0$). Comparing Eqs. (35) and (36) with the involved modes in Refs. [10,14], they are completely the same.

7 Conclusions and Discussions

This paper studies free vibrations of a DWNT modeled as elastic beams due to different boundary conditions between inner and outer tubes. According to a combination of different boundary conditions between the two tubes, such as fixed-free, cantilever-free, fixed-simple, and fixed-fixed end conditions, an original and feasible method for constructing vibrational mode and computing resonant frequency is proposed. Our main results and some discussions are summarized as follows.

- (1) For free vibrations of a DWNT, a special frequency ω_0 exists in a series of resonant frequencies. All vibrational modes must be coaxial when the resonant frequency is smaller than ω_0 , and some noncoaxial vibrations will occur when their resonant frequencies are bigger than ω_0 . Additionally, the value of ω_0 is not changed by different combinations of boundary conditions between the inner and

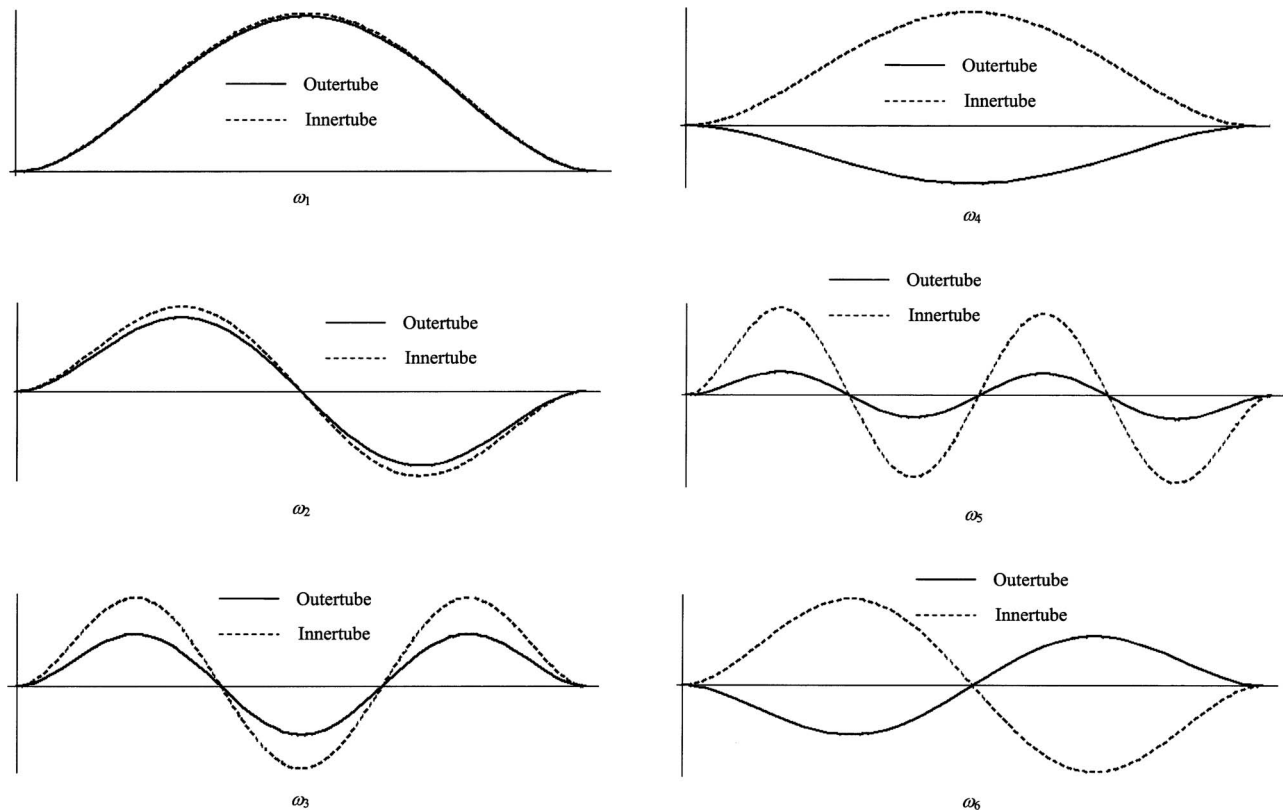


Fig. 4 The first six vibrational modes of a DWNT with both fixed inner and outer tubes

outer tubes of a DWNT, for example, a short DWNT of length of 14 nm, with the inner and outer diameters of 0.7 nm and 1.4 nm, respectively, then $\omega_0 = 7.8456 \times 10^{12}$ Hz.

- (2) The different boundary conditions between the inner and outer tubes of a DWNT have little effect on the first non-coaxial resonant frequency, which is around 7.9×10^{12} Hz for a short DWNT of length of 14 nm, with the inner diameter of 0.7 nm and outer diameter of 1.4 nm. On the other hand, the first coaxial resonant frequency (the lowest order frequency) changes with boundary conditions. The stronger the restrictions of boundary conditions are, the lower the first order frequencies are. Any order resonant frequency decreases with increasing length of the DWNT.
- (3) The first order noncoaxial mode of a DWNT occurs at an invariable resonant frequency exceeding ω_0 firstly (around 7.9×10^{12} Hz). It is a typically noncoaxial vibration whose mode looks like an open mouth if the outer tube is fixed at its both ends.
- (4) For a DWNT with an outer tube fixed at its both ends, the first three coaxial vibrational modes are almost the same except for the boundary areas. The vibration at a higher resonant frequency causes a complex noncoaxial distortion, especially in a DWNT with a free inner tube and a cantilever outer tube.

Table 8 The inner-to-outer tube amplitude ratios for the first six resonant frequencies of a DWNT with both fixed inner and outer tubes (the inner diameter is 0.7 nm, the outer diameter is 1.4 nm, and the length is 14 nm)

ω	1.08	2.94	5.49	7.90	8.13	8.24
Y_1/Y_2	1.02	1.15	1.69	-1.96	2.70	-1.75

- (5) Our present model for a vibrational DWNT with a simple inner tube and a fixed outer tube predicts a new vibrational phenomenon. The seventh mode includes not only coaxial vibration in the middle of the DWNT, but also noncoaxial vibrations that occur in both end parts of the DWNT.
- (6) Any resonant frequency decreases with increasing length of a DWNT. Especially, the higher resonant frequency declines more quickly along with increasing length of the DWNT. For example, Fig. 5 shows the attenuation of the first three order resonant frequencies for the increasing length of a DWNT with a free inner tube and a fixed outer

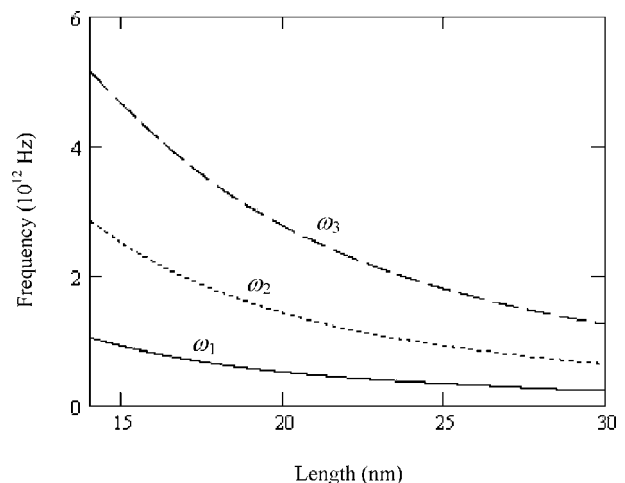


Fig. 5 Resonant frequency attenuation for the increasing length of a DWNT with a free inner tube and a fixed outer tube with an inner diameter of 0.7 nm and an outer diameter of 1.4 nm

tube. For other different boundary conditions, the resonant frequency attenuations are similar to those in Fig. 5.

Our proposed model and method are applied here to free vibrations of a DWNT with different boundary conditions between inner and outer tubes. Some similar results are expected for the role in other vibrational problems of MWNTs with different boundary conditions among tubes. In addition, we would emphasize that it is interesting to research and determine the main vibrational modes of a MWNT by a forced vibration analysis.

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Moving Load and Prestress Identification Using Wavelet-Based Method

A novel moving force and prestress identification method based on finite element and wavelet-based method for bridge-vehicle system is developed. A two-axle vehicle model and simple-supported beam with prestressing force are studied. Finite element method is flexible in modeling structures with complex boundaries while the wavelet-analysis method has the characteristic of multiresolution and the ability to detect abrupt changes. Both methods are used in this work to identify the moving loads and prestressing force from the "measured" bridge responses, which may be strain or acceleration. Numerical simulations demonstrate the efficiency of the method under the effects of measurement noise, road roughness, sampling rate, and the arrangement of sensors with good accuracy. Results indicate that the proposed method has the advantages of both high computational performance and fine identification resolution. [DOI: 10.1115/1.2793134]

Keywords: wavelet, bridge, vehicle, moving loads, inverse problem, prestress, regularization

1 Introduction

Vehicle axle load is one of the most important factors for bridge design. The dynamic responses of a bridge can be significant and Cebon [1] concluded that the dynamic wheel loads may increase the road surface damage by a factor of 2–4 over that due to static wheel loads. Traditional ways to acquire the vehicle axle loads using weighbridge cause delay and subject to bias, while early work on weigh-in-motion technique can only measure the equivalent static loads.

In recent years, the technique of dynamic wheel load identification has been developed rapidly. Existing methods can be broadly classified into two categories with one [2] based on a continuous bridge model and modal superposition technique to decouple the equation of motion, and with the subsequent solution using optimization scheme such as genetic algorithms [3]. The second category is based on discrete bridge model with finite element method to decouple the equation of motion, such as the state space approach [4] and the finite element method (FEM) [5]. The modal superposition technique has good accuracy for identification but it demands heavy computation when multiple vehicles cross a multispan bridge structure. The FEM approach is flexible when dealing with vehicle axle loads moving on top of a bridge-vehicle system with complex boundary conditions. The method has efficient computational performance and good identification accuracy especially with the orthogonal function smoothing technique to obtain the velocities and accelerations from the measured strains [6]. No existing method is found in the literature on prestress identification. Lu and Law [7,8] considered the prestressed force in each element as a system parameter and they successfully identified the prestressed force from the measured dynamic responses.

There has been increasing interest in the wavelet-based approach in recent years due to its success in several applications. Amaratunga et al. [9] developed the Wavelet–Galerkin method to solve one dimensional partial differential equation instead of the finite difference method. The connection coefficients in the

method are described by Latta et al. [10]. Ghanem and Romeo [11] used wavelet-based method for the identification of linear and nonlinear time-varying dynamic systems. A wavelet-based spectral finite element was developed by Mitra and Gopalakrishnan [12] for studying elastic wave propagation in 1D connected waveguides, and the treatment of boundaries for finite domain analysis is given for the wavelet-based approach.

Wavelet-based method is also widely used in system identification and damage detection for structures. Sone et al. [13] used the continuous wavelet transform to identify the structural parameters from the measured acceleration responses. Based on wavelet decompositions, Zabel [14] developed an algorithm for direct parameter estimation from the wavelet coefficients of the measured data as well as their integrals and derivatives. A wavelet-based method for modal parameter identification considering uncertainty is proposed by Yan et al. [15]. Zhu and Law [16] employed the continuous wavelet transform to identify the crack of a bridge beam subject to a moving load. All the applications above show the benefits of the wavelet-based method.

A new moving force identification technique using wavelet-based method is developed in this paper. The bridge is modeled as an Euler–Bernoulli beam with simple supports. The vehicle axle loads as well as the prestressed force are identified from the measured strains (or accelerations) of the bridge. The main benefit with the wavelet-based method is in the solution of the coupled equation of motion of the bridge-vehicle system with the Daubechies wavelet whereby both the exciting forces and measured responses are decomposed. The equations are then translated into the wavelet space in which the wavelet coefficients are computed. The results are then transformed back into the physical space to reconstruct the forces. This method has the advantages of both high computational performance and fine identification resolution, especially for systems with sudden irregularities in the responses due to crack or other system failures.

2 Model of the Vehicle-Bridge System

2.1 Vehicular Axle Loads. A vehicle with four degrees of freedom moving at a uniform speed v over a simply supported bridge deck is shown in Fig. 1. The equation of motion of the vehicle is derived using the Lagrange formulation as follows:

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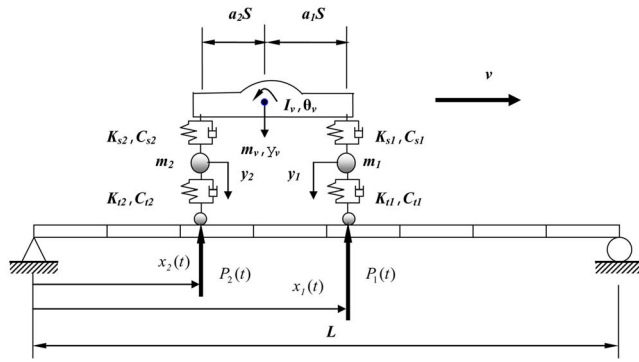


Fig. 1 The vehicle-bridge system

$$\begin{bmatrix} \mathbf{M}_{V1} & 0 \\ 0 & \mathbf{M}_{V2} \end{bmatrix} \ddot{\mathbf{Y}} + \begin{bmatrix} \mathbf{C}_{V11} & \mathbf{C}_{V12} \\ \mathbf{C}_{V21} & \mathbf{C}_{V22} \end{bmatrix} \dot{\mathbf{Y}} + \begin{bmatrix} \mathbf{K}_{V11} & \mathbf{K}_{V12} \\ \mathbf{K}_{V21} & \mathbf{K}_{V22} \end{bmatrix} \mathbf{Y} = - \begin{Bmatrix} 0 \\ \mathbf{P}(t) \end{Bmatrix} + \begin{Bmatrix} 0 \\ \mathbf{P}_0 \end{Bmatrix} \quad (1)$$

where $\mathbf{Y} = \{y_V, \theta_V, y_1, y_2\}^T$ is the vector of response of the vehicle. $\mathbf{P}(t) = \{\mathbf{P}_1(t), \mathbf{P}_2(t)\}^T$ is the vehicle-bridge interaction force vector. $\mathbf{P}_0$ is the static load vector of the vehicle. $\mathbf{M}_{V1}, \mathbf{M}_{V2}, \mathbf{C}_{V11}, \mathbf{C}_{V12}, \mathbf{C}_{V21}, \mathbf{C}_{V22}, \mathbf{K}_{V11}, \mathbf{K}_{V12}, \mathbf{K}_{V21}, \mathbf{K}_{V22}$ are the mass, damping, and stiffness matrices of the vehicle, respectively, and they are given in the Appendix.

2.2 Modeling of the Bridge. Consider a group of loads $P_i(t)$ moving on top of a bridge deck modeled as an Euler-Bernoulli beam with simple supports. The equation of motion can be written as

$$\rho A \frac{\partial^2 w(x,t)}{\partial t^2} + C \frac{\partial w(x,t)}{\partial t} + P_N \frac{\partial^2 w(x,t)}{\partial x^2} + EI \frac{\partial^4 w(x,t)}{\partial x^4} = \sum_{i=1}^{N_p} P_i(t) \delta(x - v_i t) \quad i = 1, 2, \dots, N_p \quad (2)$$

where A is the cross-sectional area and ρ is the mass per unit length. C and EI are the damping and flexural rigidity of the beam, respectively. P_N is the axial prestressed force. $w(x,t)$ is the displacement response, which varies with location x and time t . v_i is the speed of the i th moving force $P_i(t)$. $\delta(t)$ is the Dirac delta function. N_p is the number of moving loads.

Employing the Hermitian cubic interpolation shape functions and with the assumption of Rayleigh damping, the equation of motion of the prestressed beam can be rewritten as

$$\mathbf{M}_b \ddot{\mathbf{R}} + \tilde{\mathbf{C}}_b \dot{\mathbf{R}} + \tilde{\mathbf{K}}_b \mathbf{R} = \mathbf{H}_b \mathbf{P} \quad (3)$$

where $\mathbf{R}, \dot{\mathbf{R}},$ and $\ddot{\mathbf{R}}$ are the nodal displacement, velocity, and acceleration vectors of bridge, respectively. $\mathbf{H}_b \mathbf{P}$ is the equivalent nodal load vector from the bridge-vehicle interaction force with

$$\mathbf{H}_b = \begin{Bmatrix} 0 & \cdots & 0 & \cdots & \mathbf{H}_1 & \cdots & 0 \\ 0 & \cdots & \mathbf{H}_2 & \cdots & 0 & \cdots & 0 \end{Bmatrix}^T \quad (4)$$

$\mathbf{H}_b$ is an $n \times N_p$ matrix, where n is the number of degree of freedom of bridge after considering the boundary condition. The shape function H_i can be written in the global coordinate as

$$H_i = \begin{Bmatrix} 1 - 3 \left(\frac{x_j(t) - (i-1)l}{l} \right)^2 + 2 \left(\frac{x_j(t) - (i-1)l}{l} \right)^3 \\ (x_j(t) - (i-1)l) \left(\frac{x_j(t) - (i-1)l}{l} - 1 \right)^2 \\ 3 \left(\frac{x_j(t) - (i-1)l}{l} \right)^2 - 2 \left(\frac{x_j(t) - (i-1)l}{l} \right)^3 \\ (x_j(t) - (i-1)l) \left(\left(\frac{x_j(t) - (i-1)l}{l} \right)^2 - \left(\frac{x_j(t) - (i-1)l}{l} \right) \right) \end{Bmatrix}^T \quad (5)$$

where $x_j(t)$ is the location of j th force on the i th element at time t with $(i-1)l \leq x_j(t) < il$. l is the length of the beam element.

$\mathbf{M}_b, \tilde{\mathbf{C}}_b,$ and $\tilde{\mathbf{K}}_b$ are mass, damping, and stiffness matrices of the prestressed beam. α and β are the constants of Rayleigh damping with

$$\tilde{\mathbf{C}}_b = \alpha \mathbf{M}_b + \beta \tilde{\mathbf{K}}_b \quad \tilde{\mathbf{K}}_b = \mathbf{K}_b - \mathbf{K}_g$$

$$\mathbf{K}_g = \frac{P_N}{30l} \begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix} \quad (6)$$

where $\mathbf{K}_g$ is the geometric matrix due to the prestressing effect, which can be calculated as

$$[\mathbf{K}_g] = \int_0^l \left[\frac{\partial H_i}{\partial x} \right] P_N \left[\frac{\partial H_i}{\partial x} \right] dx \quad (7)$$

where H_i is the shape function of beam element, as given in Eq. (5).

3 Wavelet-Galerkin Approximation

3.1 Daubechies Compactly Supported Wavelets. The Daubechies wavelets and associated scaling functions $\varphi_{j,k}(t)$ are obtained by translation and dilation of functions $\psi(t)$ and $\varphi(t)$, respectively.

$$\psi_{J,k}(t) = 2^{J/2} \psi(2^J t - k) \quad J, k \in \mathbb{Z} \quad (8)$$

$$\varphi_{J,k}(t) = 2^{J/2} \varphi(2^J t - k) \quad J, k \in \mathbb{Z} \quad (9)$$

where J is the resolution. The scaling function $\varphi(t)$ and wavelet function $\psi(t)$ can be derived from the dilation equation as

$$\varphi(t) = \sum_k a_k \varphi(2t - k) \quad (10)$$

$$\psi(t) = \sum_k (-1)^k a_{1-k} \varphi(2t - k) \quad (11)$$

where a_k, a_{1-k} are the filter coefficients and they are fixed for specific wavelet or scaling function basis. It is noted that only a finite number of a_k, a_{1-k} are nonzero for compactly supported wavelets.

The scaling function $\varphi(t)$ and wavelet function $\psi(t)$ have the following properties:

$$\int_{-\infty}^{\infty} \varphi(t) dt = 1 \quad (12)$$

$$\int_{-\infty}^{\infty} \varphi(t-j) \varphi(t-k) dt = \delta_{j,k} \quad j, k \in \mathbb{Z} \quad (13)$$

$$\int_{-\infty}^{\infty} t^m \psi(t) dt = 0 \quad m = 0, 1, \dots, L/2 - 1 \quad (14)$$

where m denotes the number of vanish moments and L is the order of Daubechies wavelet with $L=2m$.

The translation of the scaling and wavelet functions on each fixed scale forms the orthogonal subspaces,

$$V_J = \{2^{J/2} \varphi(2^J t - k), J \in \mathbb{Z}\} \quad (15)$$

$$W_J = \{2^{J/2} \psi(2^J t - k), J \in \mathbb{Z}\} \quad (16)$$

such that V_J forms a sequence of embedded subspaces,

$$\{0\}, \dots, \subset V_{-1} \subset V_0 \subset V_1, \dots, \subset L^2(R) \quad \text{and} \quad V_{J+1} = V_J \oplus W_J \quad (17)$$

where $\oplus$ is the operator for the addition of two subspaces. At a certain resolution J , the approximation of a function $f(t)$ in $L^2(R)$ space using $\varphi_{J,k}(t)$ as basis can be denoted as

$$P_J(f) = \sum_k \tilde{\alpha}_{J,k} \varphi_{J,k}(t) \quad J, k \in \mathbb{Z} \quad (18)$$

where $P_J(f)$ is the approximation of $f(t)$ and $\tilde{\alpha}_{J,k}$ is the approximation coefficient. Let $Q_J(f)$ be the detail of the function using $\psi_{J,k}(t)$ as basis at the same level J , and

$$Q_J(f) = \sum_k \tilde{\beta}_{J,k} \psi_{J,k}(t) \quad J, k \in \mathbb{Z} \quad (19)$$

where $\tilde{\beta}_{J,k}$ is the detail coefficient. The approximation $P_{J+1}(f)$ of the next level ($J+1$) of resolution is given by

$$P_{J+1}(f) = P_J(f) + Q_J(f) \quad (20)$$

This forms the basis of multiresolution analysis associated with wavelet approximation.

3.2 Signal Decomposition Using Wavelet-Based Method.

The Wavelet–Galerkin approximation to the signal $f(t)$ at a certain resolution J can be expressed as

$$h(t) = \sum_k \tilde{\alpha}_{J,k} 2^{J/2} \varphi(2^J t - k) \quad J, k \in \mathbb{Z} \quad (21)$$

from Eqs. (8) and (18). Substituting $y=2^J t$ into Eq. (21), we obtain

$$h(y) = \sum_k \alpha_{J,k} \varphi(y - k) \quad \alpha_{J,k} = 2^{J/2} \tilde{\alpha}_{J,k} \quad J, k \in \mathbb{Z} \quad (22)$$

If y takes up only integer values, the approximation is discretized at all dyadic points with $t=2^{-J}y$ as

$$h(i) = h(i\Delta y) = h_i \quad i = 0, 1, 2, \dots, N_T \quad (23)$$

where N_T is the number of time instances. Equation (22) can be rewritten as

$$h_i = \sum_k \alpha_k \varphi_{i-k} = \sum_k \alpha_{i-k} \varphi_k \quad (24)$$

with $\varphi_k = \varphi(k)$. In matrix form, this becomes

$$\begin{Bmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ \vdots \\ \vdots \\ h_{N_T-1} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & \varphi_{L-2} & \cdots & \varphi_2 & \varphi_1 \\ \varphi_1 & 0 & 0 & \cdots & 0 & \cdots & \varphi_3 & \varphi_2 \\ \varphi_2 & \varphi_1 & 0 & \cdots & 0 & \cdots & \varphi_4 & \varphi_3 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ \varphi_{L-2} & \varphi_{L-3} & \varphi_{L-4} & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \varphi_{L-2} & \varphi_{L-3} & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \varphi_{L-3} & \cdots & \varphi_1 & 0 \end{bmatrix} \times \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \vdots \\ \vdots \\ \alpha_{N_T-1} \end{Bmatrix} \quad (25)$$

The periodic boundary condition has been included in Eq. (25) for the finite domain analysis denoted as

$$\begin{Bmatrix} \alpha_{-1} = \alpha_{N_T-1} \\ \alpha_{-2} = \alpha_{N_T-3} \\ \vdots \\ \alpha_{-L+2} = \alpha_{N_T-L+2} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \alpha_{N_T} = \alpha_0 \\ \alpha_{N_T+1} = \alpha_1 \\ \vdots \\ \alpha_{N_T+L-2} = \alpha_{L-2} \end{Bmatrix} \quad (26)$$

4 Moving Forces and Prestress Identification

4.1 Identification Theory. Employ the Wavelet–Galerkin approximation to the i th nodal displacement $R_i(t)$ and j th moving force $P_j(t)$ in Eq. (3). The expression is similar to that in Eq. (21) as

$$R_i(t) = \sum_k d_{i,k} \varphi(2^J t - k) \quad i = 1, 2, \dots, n \quad (27)$$

$$P_j(t) = \sum_k g_{j,k} \varphi(2^J t - k) \quad j = 1, 2, \dots, N_P \quad (28)$$

where $d_{i,k}$ and $g_{j,k}$ are the approximation coefficients for the i th nodal displacement and j th moving force, respectively. The i th nodal velocity and acceleration can be obtained from Eq. (27) as

$$\dot{R}_i(t) = \sum_k d_{i,k} \dot{\varphi}(2^J t - k) \quad i = 1, 2, \dots, n \quad (29)$$

$$\ddot{R}_i(t) = \sum_k d_{i,k} \ddot{\varphi}(2^J t - k) \quad i = 1, 2, \dots, n \quad (30)$$

In the identification with the prestressed beam model, Eq. (7) is substituted into Eq. (3) to get

$$M_b \ddot{R} + (\alpha M_b + \beta K_b) \dot{R} + K_b R = [\beta K'_g \dot{R} + K'_g R \ H_b] \begin{Bmatrix} P_N \\ P \end{Bmatrix} \quad (31)$$

where $K_g = P_N K'_g$. Further, substitute Eqs. (27)–(30) into Eq. (31) and let $y=2^J t$. Taking the inner product of both sides of Eq. (31) with $\varphi(y-j)$ and employing the orthogonal property in Eq. (13), we have

$$\mathbf{M}_b 2^J \mathbf{\Pi}^{(2)} + (\alpha \mathbf{M}_b + \beta \mathbf{K}_b) 2^J \mathbf{\Pi}^{(1)} + \mathbf{K}_b \begin{Bmatrix} d_{1,k} \\ \vdots \\ d_{N,k} \end{Bmatrix} = \begin{bmatrix} \beta K'_g \mathbf{\Pi}^{(1)} + K'_g \begin{Bmatrix} d_{1,k} \\ \vdots \\ d_{N,k} \end{Bmatrix} \mathbf{H}_b \\ P_N \\ g_{1,k} \\ \vdots \\ g_{N_p,k} \end{bmatrix} \quad (32)$$

where

$$\mathbf{\Pi}^{(1)} = \begin{Bmatrix} \sum_{k=j-L+2}^{j+L-2} d_{1,k} \Omega_{k-j}^1 \\ \vdots \\ \sum_{k=j-L+2}^{j+L-2} d_{N,k} \Omega_{k-j}^1 \end{Bmatrix}; \quad \mathbf{\Pi}^{(2)} = \begin{Bmatrix} \sum_{k=j-L+2}^{j+L-2} d_{1,k} \Omega_{k-j}^2 \\ \vdots \\ \sum_{k=j-L+2}^{j+L-2} d_{N,k} \Omega_{k-j}^2 \end{Bmatrix}$$

and Ω_{k-j}^1 and Ω_{k-j}^2 are the first and second connection coefficients, respectively, which can be expressed as

$$\Omega_{k-j}^1 = \int \phi(y-k) \phi(y-j) dy \quad (33)$$

$$\Omega_{k-j}^2 = \int \ddot{\phi}(y-k) \phi(y-j) dy \quad (34)$$

It should be noted that the solution of Eq. (32) needs to compute the inverse of the system matrices at each time step, which is more computational economical when compared with existing methods, which need to have the inversion of matrices including all time steps in the whole time duration of moving load identification.

4.2 Identification Procedure. Any of the measured acceleration, velocity, or strain can be used for the identification. The deflection of the bridge at position x and time t can be expressed as

$$w(x, t) = \mathbf{H}(x) \mathbf{R}(t) \quad (35)$$

where $\mathbf{H}(x) = \{0 \cdots \mathbf{H}_i(x)^T 0 \cdots 0\}$ with $(i-1)l \leq x(t) < il$. $\mathbf{H}(x)$ is a $1 \times n$ vector with zero entries except at the degrees of freedom corresponding to the nodal displacements of the i th beam element in which x is located. The components of the vector $\mathbf{H}_i(x)$ are calculated similar to Eq. (5) with $x(t)$ replacing $x_j(t)$. Thus, the acceleration of bridge at position x and time t can be expressed as

$$\ddot{w}(x, t) = \mathbf{H}(x) \ddot{\mathbf{R}}(t) \quad (36)$$

Also, the strain at a point x and time t can be written as follows:

$$\varepsilon(x, t) = -z \frac{\partial^2 w(x, t)}{\partial x^2} = -z \frac{\partial^2 \mathbf{H}(x) \mathbf{R}(t)}{\partial x^2} \quad (37)$$

where z represents the distance from the neutral axis of the beam to the strain gauge.

The wavelet coefficients $d_{i,k}$ of the i th nodal displacement can be obtained from any of Eqs. (35)–(37). Equation (32) is utilized to obtain the wavelet coefficients $g_{j,k}$ of the loads and P_N , and the moving forces can be reconstructed from Eq. (25). The complete sequence of the identification process is as follows:

Step 1. Calculate the system matrices in Eq. (1) for the vehicle and the bridge as well as the geometric matrix for the prestressing force.

Step 2. Select a Daubechies compactly supported wavelet such as $D6$, $D8$, etc. Choose a resolution J and compute the first and second order connection coefficients Ω_{k-j}^1 and Ω_{k-j}^2 from Eqs. (33) and (34).

Table 1 Parameters of the bridge-vehicle systems

Prestressed bridge	Vehicle ^a	
$L=30$ m	$\mathbf{I}_v=1.47 \times 10^5$ kg m ²	$m_v=17,735$ kg
$E=2.5 \times 10^{10}$ N/m ²	$a_1=0.519$	$a_2=0.481$
$\rho A=5.0 \times 10^3$ kg/m	$m_1=1500$ kg	$m_2=1000$ kg
$h_0=1.0$ m; $b=0.6$ m	$k_{s1}=2.47 \times 10^6$ N/m	$k_{s2}=4.23 \times 10^6$ N/m
$f_1=1.03$ Hz; $f_2=4.75$ Hz	$k_{t1}=3.74 \times 10^6$ N/m	$k_{t2}=4.60 \times 10^6$ N/m
$f_3=10.11$ Hz	$c_{s1}=3.00 \times 10^4$ N/m s	$c_{s2}=4.00 \times 10^4$ N/m s
$\xi=0.02$ for all modes	$c_{t1}=3.90 \times 10^3$ N/m s	$c_{t2}=4.30 \times 10^3$ N/m s
$P_N=8.2247 \times 10^6$ N	$S=4.27$ m	

^aReference [5].

Step 3. Obtain the wavelet coefficients $d_{i,k}$ from the measured acceleration or strain response.

Step 4. The wavelet coefficients of the forces $g_{j,k}$ are calculated from Eq. (32), and the moving force signals can be identified by reconstructing according to Eq. (25). The prestressing force can be directly identified from Eq. (32).

5 Regularization

The wavelet coefficients of the forces $g_{j,k}$, obtained from Eq. (32) using a straightforward least-squares method, would be unbound. A regularization technique [17] can be used to solve the ill-posed problem in the form of minimizing the function

$$\chi(\mathbf{P}, \lambda) = \|\mathbf{B}\mathbf{P} - \mathbf{U}\|^2 + \lambda \|\mathbf{P}\|^2 \quad (38)$$

where matrices $\mathbf{U}$, $\mathbf{B}$, and $\mathbf{P}$ represent the left-hand side of Eq. (32), and the first and second matrices on the right-hand side of the equation, respectively. λ is a non-negative regularization parameter corresponding to the smallest relative percentage error calculated from Eq. (40). The S-curve method can be employed to determine the optimal parameter.

The solution of Eq. (38) is obtained by the damped least-squares method as

$$\mathbf{P} = (\mathbf{B}^T \mathbf{B} + \lambda \mathbf{I})^{-1} \mathbf{B}^T \mathbf{U} \quad (39)$$

where $\mathbf{I}$ is the identity matrix and singular-value decomposition is used in the pseudoinverse calculation.

6 Numerical Simulation

The effects of noise level, road surface roughness, sampling rate related to the wavelet resolution, and the arrangement of sensors on the accuracy of the identified results are investigated. The improved reduced system method [5] is employed for model reduction and the number of master degree of freedoms is always equal to the number of measuring points in the study.

White noise is added to the calculated responses to simulate the polluted measurements as

$$\xi_i = \xi_{ic} (1 + E_p \mathbf{N}_{\text{oise}})$$

where ξ_i and ξ_{ic} are the vectors of measured and calculated responses at the i th measuring point, E_p is the noise level, and $\mathbf{N}_{\text{oise}}$ is a standard normal distribution vector with zero mean and unit standard deviation.

Table 2 Sensor arrangements

No. of sensors	Location
$N_s=4$	1/8L, 3/8L, 5/8L, 7/8L
$N_s=6$	1/8L, 1/4L, 3/8L, 1/2L, 5/8L, 3/4L
$N_s=7$	1/8L, 1/4L, 3/8L, 1/2L, 5/8L, 3/4L, 7/8L

Table 3 Percentage error of identified forces from different sampling rate and resolution ($N_s=7$). Note that * denotes exceptional large error.

Wavelet resolution	Without noises			With Class C road roughness			With noise (5%)		
	Axle 1	Axle 2	P_N	Axle 1	Axle 2	P_N	Axle 1	Axle 2	P_N
$J=11$	23.1	12.3	*91.5	21.8	11.5	*95.1	23.1	25.3	*91.5
$J=10$	23.1	14.2	*81.8	21.8	13.9	*81.3	23.1	25.4	*81.8
$J=8$	21.3	16.7	*64.7	21.3	16.2	*64.3	21.5	17.1	*64.8
$J=7$	28.6	24.5	*52.7	29.8	24.6	*52.8	28.7	26.5	*53.7
$J=6$	35.4	32.1	*49.8	35.1	32	*49.9	37.3	34.5	*56.3
$J=5$	*49.5	*46.3	*46.8	*49.1	*46.2	*46.8	*53.4	*47.4	*71

The relative percentage error (RPE) in the identified results is calculated from Eq. (40), where $\|\cdot\|$ is the norm of matrix, and $P_{\text{identified}}$ and P_{true} are the identified and the true force time histories, respectively.

$$\text{RPE} = \frac{\|P_{\text{identified}} - P_{\text{true}}\|}{\|P_{\text{true}}\|} \times 100\% \quad (40)$$

Velocity of the vehicle is 30 m/s, and parameters of the vehicle and the prestressed beam are given in Table 1.

The measuring points are located at the bottom of the beam and their locations are shown in Table 2 for different arrangements of sensors. Only displacement response is used in the study. The polluted response is "denoised" using standard command in MATLAB before the identification. Eight finite beam elements for the bridge model are used, as previous research indicates that discretized into eight elements would be sufficient for accurate identification in a straight bridge deck.

6.1 Effect of Wavelet Resolution. The effect of resolution of the wavelet analysis on the identification of the forces is studied. The resolution J is related to the sampling time interval Δt as $\Delta t = T/(N_T - 1)$ and $N_T = T \times 2^J$, where T is the total time interval for the vehicle to travel across the bridge deck and N_T is the number of total time instances round to the nearest integer. Resolution J varies from 5 to 11 with the corresponding sampling rate varies from 2^5 to 2^{11} . Seven sensors as listed in Table 2 are used in the identification. The error of identification as calculated from Eq. (40) for the cases with and without noise and with Class C road roughness [18] included in the analysis is listed in Table 3.

The case with $J=5$ is less accurate with larger error, and the error decreases, in general, with increase in the wavelet resolution up to $J=8$ where the error of identification is relatively stable for $J=8$ and larger. The effect of 5% noise and road roughness is similar with insignificant decrease in the error of identification with increase in the resolution, while the effect of noise is similar to the road surface roughness. The error with the prestressing force is very large for the whole range of resolution studied. However, inspection of Fig. 2 shows that the identified moving force time histories vary around the true time histories except in the short duration after the entry and before the exit of the vehicle. The identified prestressing force in Fig. 3 is also very accurate fluctuating slightly around the true value in the latter half of the time duration and with large fluctuations in the first 0.3 s of the time duration. This contributes to most of the error of identification. Therefore, values in Table 3 should be taken cautiously and reference to the identified time histories is always necessary.

6.2 Effect of Sensor Arrangements. The wavelet resolution is selected as $J=10$, and the three sensor arrangements as listed in Table 2 are used for the identification, and the identified results for the cases with and without noise and with Class C road surface roughness are listed in Table 4. The first and third sensor arrangements are symmetrical about the midspan while the second arrangement has more sensors on one half of the beam. The error of

identification is shown in Table 4 and the identified force time histories and the prestressed force time history are shown in Figs. 4 and 5.

The error of identification clearly shows a larger value for the nonsymmetric sensor arrangement while the two symmetric arrangements give similar error of identification. Four sensors are checked to give acceptable force time histories, as shown in Fig. 4 and Table 4. Results not shown also indicate that a less fine res-

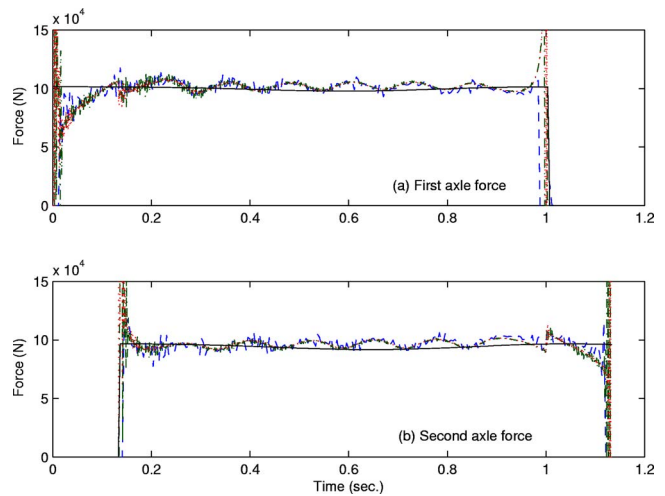


Fig. 2 Identified moving forces with 5% noise and different resolutions (—true, --- $J=8$, -.- $J=10$, ... $J=11$)

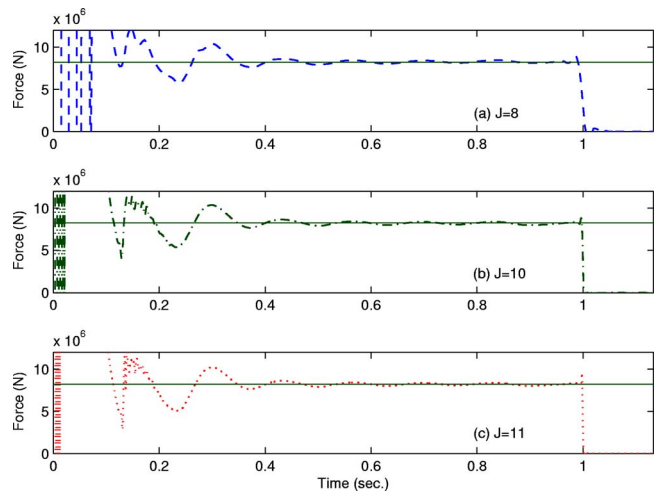


Fig. 3 Identified prestress force with 5% noise and different resolutions (—true)

Table 4 Percentage error of identified forces for different sensor arrangement ($J=10$)

No. of sensors	Without noises			With Class C road roughness			With noise (5%)		
	Axle 1	Axle 2	P_N	Axle 1	Axle 2	P_N	Axle 1	Axle 2	P_N
$N_s=7$	23.1	14.2	81.8	21.8	13.9	81.3	23.1	25.4	81.8
$N_s=6$	29.5	24.1	69.9	28.9	23.9	69.3	28.4	29.8	85.8
$N_s=4$	26.7	30.4	67.9	26.6	29.9	68.2	29.8	33.3	87.2

olution of $J=8$ gives similar accuracy in the force time histories indicating that a higher resolution does not necessarily give more accurate results.

The identified force time histories are very close to the true value, and the case with seven sensors could have very good identified prestressing force identified in the latter half of the time duration, as shown in Fig. 5.

7 Discussions

This method does not require an assumption on the initial condition of the system as only the responses are transformed into wavelet space and the periodic boundary condition has been in-

cluded in such transformation, as seen in Eq. (26). Since only the wavelet coefficient of the displacement response $d_{i,k}$ is required in the identification equation, any one of Eqs. (35)–(37) can give these coefficients and thus this method has no restriction on the type of measured response for the identification. Results not shown indicate that the accuracy of identification is larger when only the moving forces are identified instead of having both types of forces identified simultaneously. This may be due to the different sensitivities of the two types of forces in the same identification giving rise to a larger error.

8 Conclusions

A new method of moving load and prestress identification is developed using the wavelet-based method in which the approximation of the measured response is used to form the identification equation. This method is for general system identification making use of any types of measured dynamic responses and no assumption is needed on the initial condition of the system. Simulation study with and without measurement noise and with road surface roughness gives acceptable results with different wavelet resolutions. The error of identification is largest in the identified time histories in the short duration after the entry and before the exit of the vehicle, while accurate results can be obtained in most of the time duration with a proper selection of wavelet resolution and sensor arrangement.

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Appendix: System Matrices of Vehicle and the Interaction Forces

$$\mathbf{M}_{V1} = \begin{bmatrix} m_v & 0 \\ 0 & I_v \end{bmatrix} \quad \mathbf{M}_{V2} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$\mathbf{C}_{V11} = \begin{bmatrix} C_{s1} + C_{s2} & (-C_{s1}a_1 + C_{s2}a_2')S \\ (-C_{s1}a_1 + C_{s2}a_2)S & (C_{s1}a_1^2 + C_{s2}a_2^2)S^2 \end{bmatrix}$$

$$\mathbf{C}_{V12} = \begin{bmatrix} -C_{s1} & -C_{s2} \\ C_{s1}a_1S & -C_{s2}a_2S \end{bmatrix} \quad \mathbf{C}_{V21} = \begin{bmatrix} -C_{s1} & C_{s1}a_1S \\ -C_{s2} & -C_{s2}a_2S \end{bmatrix}$$

$$\mathbf{C}_{V22} = \begin{bmatrix} C_{s1} & 0 \\ 0 & C_{s2} \end{bmatrix}$$

$$\mathbf{K}_{V11} = \begin{bmatrix} K_{s1} + K_{s2} & (-K_{s1}a_1 + K_{s2}a_2)S \\ (-K_{s1}a_1 + K_{s2}a_2)S & (K_{s1}a_1^2 + K_{s2}a_2^2)S^2 \end{bmatrix}$$

$$\mathbf{K}_{V12} = \begin{bmatrix} -K_{s1} & -K_{s2} \\ K_{s1}a_1S & -K_{s2}a_2S \end{bmatrix}$$

$$\mathbf{K}_{V21} = \begin{bmatrix} -K_{s1} & K_{s1}a_1S \\ -K_{s2} & -K_{s2}a_2S \end{bmatrix} \quad \mathbf{K}_{V22} = \begin{bmatrix} K_{s1} & 0 \\ 0 & K_{s2} \end{bmatrix}$$

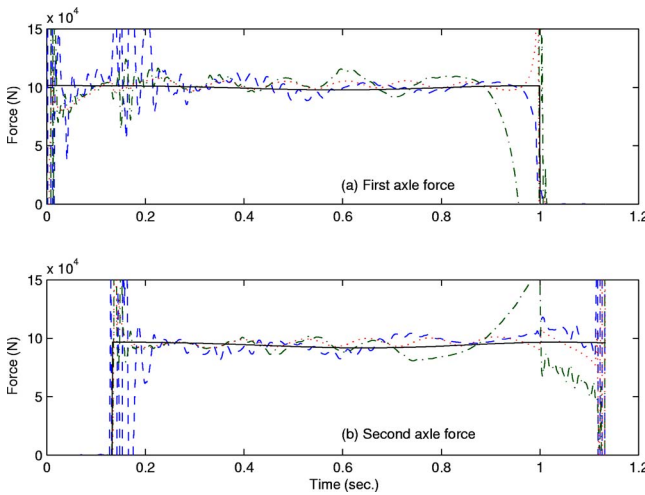


Fig. 4 Identified moving forces on the prestressed bridge from different sensor arrangements with 5% noise and $J=10$ (—true, --- $N_s=4$, -.-.- $N_s=6$, ... $N_s=7$)

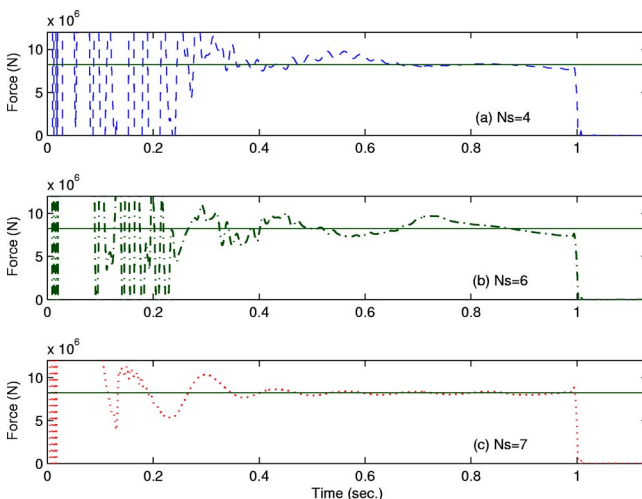


Fig. 5 Identified prestress force from different sensor arrangements and 5% noise with $J=10$ (—true)

$$\mathbf{P} = \begin{Bmatrix} P_1(t) \\ P_2(t) \end{Bmatrix} \quad \mathbf{P}_0 = \begin{Bmatrix} (m_1 + a_2 m_v)g \\ (m_2 + a_1 m_v)g \end{Bmatrix}$$

where S is the axle spacing, $\{K_{si}, C_{si}, (i=1,2)\}$ are the stiffness and the damping of the two suspensions, m_v and I_v are the mass and the mass moment of inertia of the vehicle, and m_1 and m_2 are the masses of the bogie. a_1 and a_2 denote the dimensions of the vehicle, as given in Fig. 1.

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Postyield Cyclic Buckling Criteria for Aluminum Shear Panels

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Aluminum shear panels can dissipate significant amount of energy through hysteresis provided strength deterioration due to buckling is avoided. A detailed experimental study of the inelastic behavior of the full-scale models of shear panels of 6063-O and 1100-O alloys of aluminum is conducted under slow cyclic loading of increasing displacement levels. The geometric parameters that determine buckling of the shear panels, such as web depth-to-thickness ratio, aspect ratio of panels, and number of panels, were varied among the specimens. Test results were used to predict the onset of buckling with proportionality factor f in Gerard's formulation of inelastic buckling. Moreover, a logarithmic relationship between buckling stress and slenderness ratio of the panel was observed to predict experimental data closely. These relations can be further used to determine the geometry of shear panels, which will limit the inelastic web buckling at design shear strains. [DOI: 10.1115/1.2793135]

Keywords: buckling, shear, aluminum, inelastic, cyclic, postyield

Introduction

Shear panels of soft alloys of aluminum can be effectively used as a device to dissipate energy through hysteresis for a number of engineering applications. One such application is in the area of earthquake resistant design of structures where these devices are used as a means to dissipate seismic energy and control the seismic response of the structure. With thick webs of shear panels of aluminum alloys of low yield values, not only the problem of elastic buckling is avoided but the onset of inelastic buckling can be delayed even past the yielding. Postyield buckling of panels seriously limits their energy dissipation potential with severe pinching of hysteretic loops. Therefore, shear panels are to be designed to avoid buckling at operating shear strains for various applications [1,2]. The purpose of this study is to experimentally investigate the buckling behavior of aluminum shear panels of low slenderness ratio which buckle after yielding and to develop a criterion for postyield shear buckling of such shear panels.

Inelastic Shear Buckling Stress

The plastic buckling analysis has been attempted using the classical theories of plasticity, which involved the incremental (or flow) and/or the deformation theory of plasticity [3–6]. The solutions for simple cases of plate problems for uniaxial and biaxial monotonic loading have been derived. Azhari and Bradford [7] employed both deformation and flow theory in the complex finite strip eigenvalue method for plastic buckling of plates. However, these analytical studies are too complex and computationally intensive making them difficult to use for design purposes. The objective of this study is to provide simple expressions for cyclic plastic buckling of aluminum shear panels based on experimental investigation.

For stresses beyond the proportional limit, the critical buckling stresses by elastic theory (e.g., Euler theory) give exaggerated values. In order to get satisfactory results, the behavior of the material beyond the proportional limit must be considered. At these higher stresses, the modulus of elasticity, or slope of the stress-strain curve, varies depending on the strain level and can be

represented by the tangent modulus of elasticity E_t . Substituting E_t for Young's modulus, E in Euler's buckling formula, inelastic buckling stress τ_b can be given as follows [8]:

$$\tau_b = \frac{\pi^2 E_t}{\lambda^2} \quad (1)$$

where λ is the characteristic slenderness ratio. Assuming that the edges are partially restrained against rotation for a panel of shorter dimension, a , and longer dimension, b , characteristic slenderness ratio can be given as per the following expression [9]:

$$\lambda = \frac{a}{t_w} \sqrt{\frac{1.6}{1 + 0.7(a/b)^2}} \quad (2)$$

Clark and Rolf [10] showed that rather than using tangent modulus which varies with stress, Eq. (1) can be conveniently reduced to a linear function of the equivalent slenderness ratio λ , as shown below:

$$\tau_b = B_s - D_s \lambda \quad (3)$$

where B_s and D_s are the material parameters that depend on the yield shear stress of the material. Sharp and Clark [9] summarized the observed behavior of thin aluminum shear webs of plate girders under monotonic loading, which formed the basis of design provisions of the Aluminum Association [11]. However, this relation does not provide good predictions of inelastic buckling stress in the strain-hardening region.

Gerard [12] extended the concept of use of secant modulus (in place of tangent modulus) in determining critical shear stresses above the proportional limit and formulated the plastic web buckling problem as follows:

$$\tau_b = \eta(\tau) \tau_E \quad (4)$$

where $\eta(\tau)$ is a plastic-reduction factor, which is related to postelastic behavior of the plate, and τ_E is the elastic buckling stress. Gerard proposed an empirical equation for η as a function of the ratio of shear secant modulus G_s and shear modulus G of the shear panel, i.e.,

$$\eta = f(G_s/G) \quad (5)$$

where f is a proportionality constant to be determined from experimental data. These relations were developed for monotonic loading; however, they can be extended for reversed cyclic loading. Secant shear modulus G_s is now defined as $G_s = \tau_b / \bar{\gamma}_b$, where

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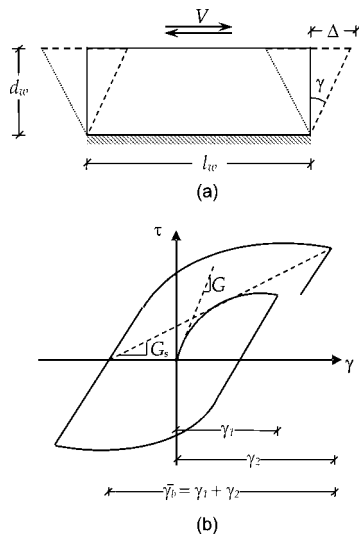


Fig. 1 Details of parameters used in Gerard's buckling criterion: (a) shear deformation of shear panel and (b) definition of secant shear modulus G_s and inelastic shear strain $\bar{\gamma}_b$

τ_b is the shear stress and $\bar{\gamma}_b$ is the shear strain, as shown in Fig. 1. In this manner, Gerard's approach for the inelastic buckling criterion is explicitly expressed in terms of applied cyclic shear strain, which can be directly used with deformation-based design provisions for shear panels.

Kasai and Popov [13] tested various steel shear links employed in eccentrically braced frames (EBFs) under reversed cyclic loads and observed that the inelastic shear buckling stress can be adequately represented by Gerard's formulation with proportionality factor f being 3.7. Similarly, in another study on cyclic load tests on shear panel of low yield alloy of aluminum (3003-O), Gerard's formulation for inelastic buckling was found in excellent agreement with experimental results and the factor ($f=3.76$) was found nearly constant for all the test specimens [14]. However, in this preliminary study, all the specimens had identical geometry of shear web panels and, therefore, the observed value of f needed to be verified with a larger dataset of specimens of differing geometries. The present paper revisits the earlier results with expanded dataset of full-scale models of shear panels of different geometric parameters.

Experimental Program

Test Specimens. The energy dissipation capacity of aluminum shear panels depends on the mechanical properties of the material to a great extent. A highly ductile material is needed to meet the large inelastic strain demand required in these applications. Soft alloys of aluminum are less susceptible to web buckling problems because of their low yield strength, which enables the usage of thicker webs for the same strength. Widely available Alloys 6063 and 1100 of aluminum for structural applications were used for fabrication of I-shaped specimens with transverse stiffeners. This alloy was chosen for its availability in flat sections of required thickness. This material was not commercially available in the fully soft annealed condition. Instead, a more common T6 temper of 6063 alloy, which is solution heat treated and then artificially aged, was obtained and annealed in the furnace. This annealing process is believed to eliminate the history of prior straining above a reference temperature (such as welding) and stress relieves in the test specimens [15]. Annealing resulted in changing the temper T6 to softer temper O, thus reducing the values of yield stress and ultimate stress of the material. The specimens were raised to a temperature of 413°C and kept at that temperature for 2 h. Then, they were allowed to cool gradually at a rate of

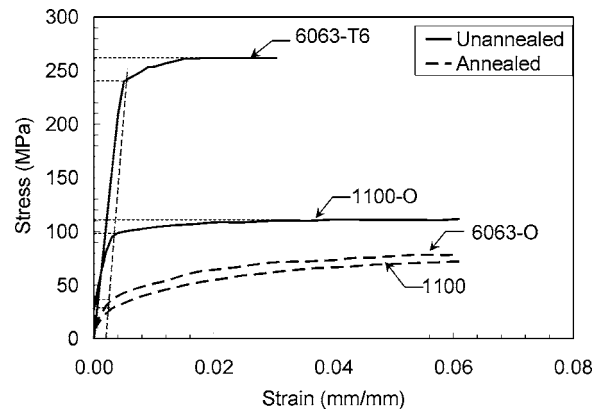


Fig. 2 Stress-strain curves of unannealed and annealed aluminum alloys used in this study

28°C per hour in the heat treating oven. However, no attempt was made to assess the residual stress and its distribution in the specimens before and after the annealing process in the present study.

Figure 2 shows the stress-strain behavior of unannealed and annealed aluminum alloys used in the present study. The proof stress for unannealed temper T6 corresponding to 0.2% of strain was 240 MPa, which was reduced to 35 MPa after annealing. The stress-strain curve unannealed tensile coupon tests result in a curve with a sharp knee in contrast to more rounded with much lower yield stress in the case of annealed coupons. Also, elongation of the coupons was increased from around 15% to 30% after annealing (Table 1). In addition to reduction in the yield stress, effect of strain hardening of the material was more pronounced due to annealing.

I sections of specimens were fabricated mainly using three aluminum strips—two separate strips for each of the flanges and one strip for the web. The flanges were welded to the web from the inside face of the flange using tungsten inert gas (TIG) welding process [16]. Transverse stiffeners were employed in specimens to delay the initiation of the web buckling and were rigid enough so that inclined waves of the buckled plate do not run across the stiffener. To maintain postbuckling capacities of shear panels, each transverse stiffener is proportioned to avoid web buckling with the stiffener and must remain effective even after the web buckles to support the tension field as well as to prevent the tendency of flange to move toward each other. The stiffeners were groove welded to both flanges as well as to the web (Fig. 3).

Nineteen specimens of panels (Specimen 1 as trial specimen) with web thickness of 4.5 mm, 6.5 mm, and 7.6 mm were fabricated with aspect ratios of 0.75, 1.00, and 1.25. For each combination of aspect ratio and web depth-to-thickness ratio, two-paneled as well as three-paneled specimens were fabricated using transverse stiffeners. The clear depth of web and width of flange of all specimens were 152.4 mm and 100 mm, respectively (Fig. 3). Similarly, the thickness of flange was 6.5 mm for specimens with a web thickness of 4.5 mm and was increased to 10 mm for specimens with thicker webs. The geometric properties of all test specimens are summarized in Table 2. Since the flat sections of required thickness for Specimens 14–19 were not available in

Table 1 Uniaxial tensile coupon test results

No.	Alloy	Condition	Percentage elongation	Yield stress (MPa)	Ultimate stress (MPa)
1	6063-T6	Unannealed	15.26	240	261
2	6063-O	Annealed	31.08	35	85
3	1100-O	Unannealed	16.42	99	112
4	1100	Annealed	33.32	25	82

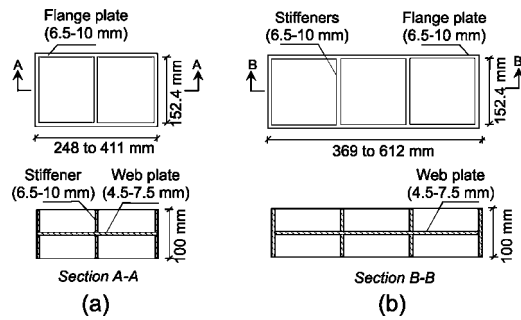


Fig. 3 Details of test specimens: (a) two-paneled specimen (b) three-paneled specimen

Alloy 6063, plates of Alloy 1100-O were used. The material used for flanges of test specimens was the same as that of the web. All the test specimens were annealed before being used in the experiment.

Test Setup. A testing system used in the study was designed, as shown in Fig. 4. The load application system consisted of servo-hydraulic closed loop actuator (MTS manufactured) with a force capacity of ± 500 kN and a displacement stroke of ± 125 mm. Loading was applied through a controller unit and a function generator that enabled the servocontrolled actuator to produce preprogrammed displacement histories. The lateral shear load was transferred from the actuator to the specimen through an I-shaped steel beam, which moved back and forth with the actuator. The specimen was bolted securely to the bottom flange of the top beam. The bottom flange of the specimen was bolted to the top flange of an I-shaped steel beam at the bottom which was welded to a steel plate, firmly held to the horizontal strong floor of the laboratory. The lateral out-of-plane movement of the top movable beam was restrained by providing side supports with ball bearings on both sides of its web. This arrangement prevented out-of-plane movement, bending or twisting. In order to prevent the movement of the top beam in the vertical plane, roller bearings were provided on the top flange of the top steel beam, as shown in Fig. 4. The setup was so fabricated that the lateral shear load was applied at the mid-depth level of the shear panel.

The instrumentation consisted of transducers, which included a load cell, linear variable differential transformers (LVDTs), and strain gauges. A set of 45 deg-strain rosettes was used to measure the shear strains at the center of the panel, which was also used to

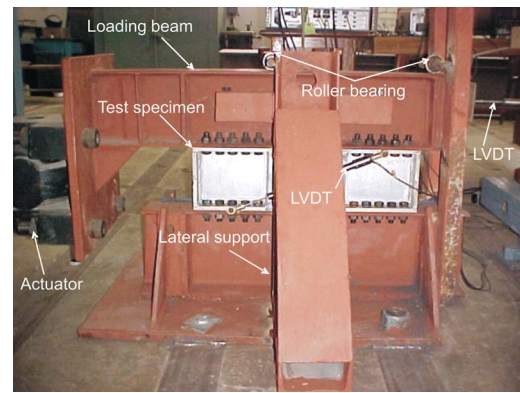


Fig. 4 Test setup with locations of LVDTs and strain gauges

determine the initial modulus of shear rigidity G of the material. The measurement of force in the specimen was accomplished directly via a load cell located in the actuator arm. A pair of LVDTs was diagonally mounted on either face of the specimen to measure the shearing deformation of the web of specimen. An additional LVDT was mounted on the loading beam to measure the horizontal movement of the actuator.

Displacement History. As stated earlier, the objective of this study is to investigate the force-deformation behavior of the shear links under slow cyclic loading. Slow cyclic implies that load or deformation cycles are imposed on a test specimen in a slow, controlled, and predetermined manner, and dynamic effects as well as rate of deformation effects are not considered [17]. Displacement histories consisted of symmetric reversed cycles of increasing displacements in predetermined steps at a frequency of 0.01 Hz in the ramp wave form in displacement controlled regime. Cycles were performed at shear strain levels of ± 0.005 , ± 0.01 , ± 0.02 , ± 0.05 , ± 0.10 , ± 0.15 , and ± 0.20 . Shear strain is calculated as the ratio of horizontal shear displacement of the panel to the clear depth of web plate. The push displacement applied by the actuator was taken as positive and the pull displacement was considered as negative. Each displacement cycle was repeated for three times, as shown in Fig. 5. Such loading program is representative of low cycle fatigue caused by short duration events, such as earthquakes, blasts, etc.

Table 2 Geometric properties of all the specimens

Sp. No.	Alloy ^a	n	t_w (mm)	C (mm)	α	β	l_w (mm)	A_w (mm ²)	t_s (mm)
1	6063-T6	2	6.5	190.5	1.25	23.5	397.0	2580.5	6.5
2	6063-T6	2	4.5	114.3	0.75	38.1	248.1	1116.45	6.5
3	6063-T6	2	4.5	152.4	1.00	38.1	324.3	1459.35	6.5
4	6063-T6	2	4.5	190.5	1.25	38.1	400.5	1802.25	6.5
5	6063-T6	3	4.5	114.3	0.75	38.1	368.9	1660.05	6.5
6	6063-T6	3	4.5	152.4	1.00	38.1	483.9	2177.55	6.5
7	6063-T6	3	4.5	190.5	1.25	38.1	597.5	2688.75	6.5
8	6063-T6	2	6.5	114.3	0.75	23.5	248.1	1612.65	6.5
9	6063-T6	2	6.5	152.4	1.00	23.5	324.3	2107.95	6.5
10	6063-T6	2	6.5	190.5	1.25	23.5	400.5	2603.25	6.5
11	6063-T6	3	6.5	114.3	0.75	23.5	368.9	2397.85	6.5
12	6063-T6	3	6.5	152.4	1.00	23.5	483.9	3145.35	6.5
13	6063-T6	3	6.5	190.5	1.25	23.5	597.5	3883.75	6.5
14	1100-O	2	7.6	114.3	0.75	20.0	252.6	1919.76	7.6
15	1100-O	2	7.6	152.4	1.00	20.0	328.8	2498.88	7.6
16	1100-O	2	7.6	190.5	1.25	20.0	411.0	3123.6	10.0
17	1100-O	3	7.6	114.3	0.75	20.0	374.9	2849.24	7.6
18	1100-O	3	7.6	152.4	1.00	20.0	489.2	3717.92	7.6
19	1100-O	3	7.6	190.5	1.25	20.0	611.5	4647.4	10.0

^aAlloy specification for plates used to fabricate the test specimens, which were annealed before testing.

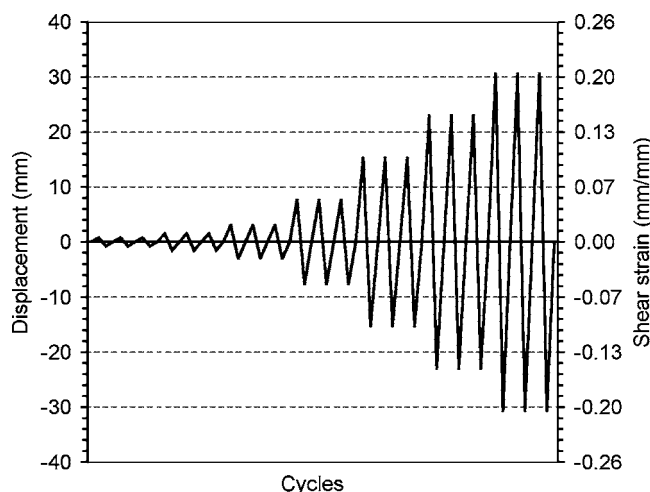


Fig. 5 Displacement (or strain) history

Discussion on Experimental Results

Figure 6 shows shear-stress–shear-strain hysteretic response of two-paneled and three-paneled test specimens with varied alloy type and web depth-to-thickness ratio up to a cyclic shear strain of 20%. Specimens made of 6063 alloy and web depth-to-thickness ratio of 38.1 exhibited pinching of hysteretic loops due to inelastic buckling, thereby reducing their energy dissipation capacity. Both two-paneled and three-paneled test specimens made of 6063 alloy and web depth-to-thickness ratio of 23.5 showed higher shear strength due to noticeable strain-hardening behavior. However, specimens using 1100 alloy did not exhibit significant strain-hardening behavior. Most of the test specimens buckled at a shear strain of 10% due to large web depth-to-thickness ratio and specimens having smaller web depth-to-thickness ratio continued to exhibit stable and full hysteretic loops without pinching at larger strain levels without buckling. Table 3 summarizes the results of experimental study of aluminum shear panels of all test specimens.

Load-Deformation Behavior. Specimens 1–19 showed no apparent distress in the panels at low levels of strain (0.005, 0.01, 0.02, and 0.05 strains). Specimens 2, 3, 4, 6, and 7 showed onset of buckling at a 0.10 strain level. The buckled configuration caused pinched flanges at both ends of the specimens. The onset of web buckling initiated the deterioration of the hysteric performance. At the end of 0.10 strain cycles, substantial out-of-plane web buckling was observed. During next cycle at 0.15 strain level, rapid deterioration of the resistance to shear loading was observed, which kept on accentuating with each additional cycle. The end stiffeners were visibly distressed at this stage due to excessive strains. The initiation of the web tearing was observed in the web along the buckles and at the points where buckles formed in either direction intersected. Extreme buckled shapes of the specimens were observed at this stage at a strain level of 0.20 (Fig. 7). Other specimens showed buckling at the higher strain levels: 0.15 for Specimens 5, 13, and 19; 0.20 for Specimens 10 and 15; and 0.25 for Specimen 18. Further, Specimens 8, 9, 11, 12, 14, 16, and 17 had such a configuration that they did not experience buckling up to 0.25 strain levels.

Effect of End Stiffeners. The tension field in a shear panel is traditionally believed to be resisted by the flanges and transverse stiffeners. However, a recent study shows that their role is rather limited [18]. Since the panels adjacent to an interior panel of a specimen having three panels are able to resist the tension field, they can be counted on to furnish the necessary support. As the end panel does not have such support, end stiffeners undergo large bending while resisting the bending effects of tributary tension

field. The end stiffeners help in controlling the amplitude of web buckling and thereby reduce the severity of resistance deterioration of the panel upon cycling. The end stiffeners appeared to be much more bent due to tension field, while intermediate transverse stiffeners do not show much bending.

Effect of Aspect Ratio α . The function of transverse stiffener is to subdivide the panel web into smaller panels, thereby increasing the shear buckling stress. The effect of providing stiffener is to delay the onset of web buckling. Delaying the web buckling allowed the webs to continue to strain harden and permitted the specimens to reach higher stress level. The web of the aluminum section was reinforced with transverse stiffeners to increase its resistance to buckling. The onset of buckling in Specimen 6 ($\alpha = 1$) was observed at a strain level of 0.1 while it was observed at a strain level of 0.15 in Specimen 5 ($\alpha = 0.75$). Similar observation was made in Specimen 18 ($\alpha = 1.0$ and $\gamma_b = 0.25$) and Specimen 19 ($\alpha = 1.25$ and $\gamma_b = 0.15$). Hence, reduction in the spacing of transverse stiffeners results into the lower value of aspect ratio α of the panel, thus resulting in increase in web buckling deformation angle γ_b .

Effect of Web Depth-to-Thickness Ratio β . For web depth-to-thickness ratios of 23.5 and 20, some specimens such as Specimens 8, 9, 11, 12, 14, and 17 showed no buckling at all even at strains up to 0.20 or sometimes even completely avoiding web buckling until the tearing of web plate. Specimen 4 having β value as 38.1 buckled at 0.1 strain while Specimen 10 having β value as 23.5 buckled at 0.2 strain, whereas Specimen 16 with β value as 20 did not buckle until tearing of plate. Thus, as web depth-to-thickness ratio is decreased, the tendency of buckling of the panel is delayed to larger strain levels.

Effect of Number of Panels. In Specimen 5 (three paneled), larger buckling deformation angle (0.15 strain) was noticed as compared to Specimen 2 (0.1 strain) having two panels while all other parameters were the same. Similar behavior was noticed in Specimen 18 ($\gamma_b = 0.25$) and Specimen 15 ($\gamma_b = 0.2$). Thus, it can be stated that three-paneled specimen buckled at large strain level as compared to two-paneled specimen with other parameters remaining the same. Specimen 4 resisted 78.3 kN while corresponding three-paneled Specimen 7 resisted 129.2 kN and similar observation was made in other specimens also. It is observed that the ultimate load level achieved in three-paneled specimens is about 1.5 times the corresponding two-paneled specimens with other parameters remaining constant. This may be due to the tension field developed in the central panel resisted by the adjacent outer panel web, which is not present in the case of two-paneled specimens.

Criteria for Postyield Shear Buckling

Test results presented in Table 2 can be used to predict the proportionality factor in Gerard's formulation for the onset of inelastic buckling, as discussed earlier. On plotting the experimental data as shown in Fig. 8, it is clear that the data points lie in a "triangular" banded region with proportionality factor f ranging from 3.0 to 7.0. This is primarily due to large variations in geometric configuration of shear panels, especially due to two panels versus three panels. It can be observed that the value of G_s/G of Specimen 18 has been decreased as compared to Specimen 15 due to an increase in the number of panels with other geometric parameters remaining the same. Similar reduction in G_s/G was observed in Specimen 5 as compared to Specimen 2 due to an increase in the strain level at the onset of buckling resulting in lower value of G_s . Thus, the suggested range of values of f takes into account the effect of number of panels into consideration as well. However, for convenience, a best-fit line has been plotted for the dataset, which suggests the value of f to be 4.92. Using this value, strain at the onset of postyield buckling γ_b can be obtained as

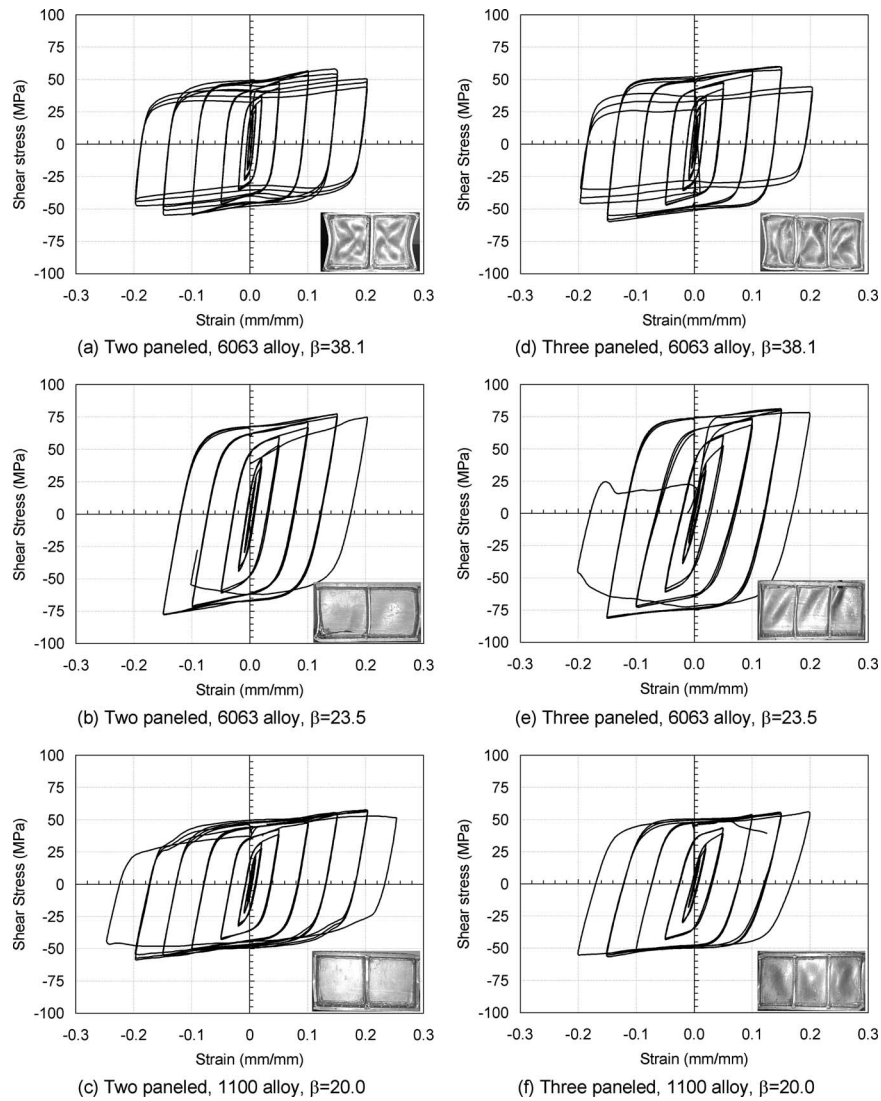


Fig. 6 Hysteretic response and buckled configurations of test specimens with varied number of panels, type of alloy, and web width-to-thickness ratio

Table 3 Aluminum shear panel test results

Sp. No.	λ	τ_E (MPa)	τ_b (MPa)	γ_b	G_s (MPa)	G (MPa)	G_s/G	τ_b/τ_E	f	$\bar{\gamma}_b$
1	21.7	1466.1	49.3	0.05	537.0	23,220 ^a	0.023	0.034	1.454	0.09
2	26.9	953.2	53.0	0.10	353.3	21,860	0.016	0.056	3.440	0.15
3	32.7	644.5	49.4	0.10	329.3	21,720	0.015	0.077	5.055	0.15
4	35.2	555.4	39.3	0.10	262.1	23,220 ^a	0.011	0.071	6.271	0.15
5	26.9	953.2	59.6	0.15	238.4	23,060	0.010	0.063	6.048	0.25
6	32.7	644.5	51.4	0.10	342.4	24,220	0.014	0.080	5.637	0.15
7	35.2	555.4	48.1	0.10	320.7	25,220	0.013	0.087	6.811	0.15
8	16.5	2516.5	b	b	b	24,260	b	b	b	b
9	20.1	1701.3	b	b	b	27,200	b	b	b	b
10	21.7	1466.1	74.3	0.20	212.2	23,200	0.009	0.051	5.539	0.30
11	16.5	2516.5	b	b	b	21,187	b	b	b	b
12	20.1	1701.3	b	b	b	23,220 ^a	b	b	b	b
13	21.1	1466.1	79.4	0.15	317.6	20,274	0.016	0.054	3.457	0.25
14	14.1	3459.3	b	b	b	31,480	b	b	b	b
15	17.2	2338.9	56.8	0.20	162.3	21,220	0.008	0.024	3.175	0.2
16	18.5	2015.5	b	b	b	28,640 ^a	b	b	b	b
17	14.1	3459.3	b	b	b	29,544	b	b	b	b
18	17.2	2338.9	58.9	0.25	131.0	32,340	0.004	0.025	6.222	0.25
19	18.5	2015.5	55.0	0.15	157.2	28,640 ^a	0.005	0.027	4.974	0.25

^aAverage value of shear modulus considered because of erratic strain gauge data.

^bNo incident of buckling.



Fig. 7 State of test specimens before and after the testing: (a) two paneled specimen and (b) three-paneled specimen

$$\gamma_b = 4.92 \frac{\tau_E}{G} \quad (6)$$

Elastic critical stress is given by [19]

$$\tau_E = k_s \frac{\pi^2 E}{12(1-\nu^2)} \left(\frac{1}{\beta} \right)^2 \quad (7)$$

where E is Young's modulus, ν is Poisson's ratio, and k_s is the buckling coefficient, which depends on aspect ratio of the web subpanel formed by the transverse stiffeners and its boundary restraint conditions. It is reasonable to assume clamped edge conditions for the web panel, as the stiffeners welded to the web and the flanges provide significant restraint to the web. For finite rectangular plate with clamped edges [19],

$$k_s = \begin{cases} 5.6 + \frac{8.98}{\alpha^2} & \text{for } \alpha \leq 1 \\ 8.98 + \frac{5.6}{\alpha^2} & \text{for } \alpha \geq 1 \end{cases} \quad (8)$$

Comparing Eq. (7) to Eq. (1) and using the values of k_s as given in Eq. (8), the slenderness ratio λ can be expressed as follows:

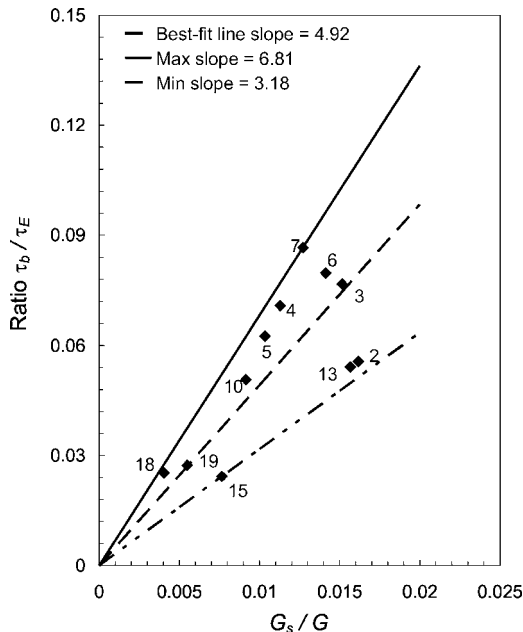


Fig. 8 Experimental relationship between G_s/G and η

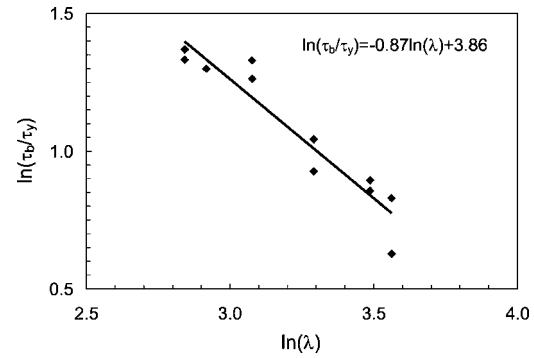


Fig. 9 Log-log plot between slenderness ratio and inelastic shear buckling stress

$$\lambda = \begin{cases} \alpha \beta \sqrt{\frac{1.2}{1 + 0.63 \alpha^2}} & \text{for } \alpha \leq 1 \\ \beta \sqrt{\frac{1.2}{1 + (0.63/\alpha^2)}} & \text{for } \alpha \geq 1 \end{cases} \quad (9)$$

Using Eq. (1) for τ_E by taking $E_t = E$ and value of Poisson's ratio ν as 0.33 in Eq. (6), the web buckling deformation angle γ_b can be expressed as a function of slenderness ratio λ of Eq. (9) as follows:

$$\gamma_b = \frac{129.17}{\lambda^2} \quad (10)$$

Equation (10) can be used to determine the spacing of transverse stiffeners to avoid web buckling by taking equal γ_b to an expected peak-to-peak web deformation angle for fully reversed cycles of loading shown in Fig. 5.

A linear relationship was observed in the log-log plot between slenderness ratio λ of Eq. (9) and ratio of inelastic buckling shear stress τ_b to shear yield stress τ_y , as shown in Fig. 9. Shear yield stress τ_y can be defined as 0.6 times of yield stress of material, $\sigma_{0.2}$ (i.e., stress corresponding to a proof strain of 0.2%). Hence, the shear buckling stress τ_b of aluminum panel in the region beyond the yield limit can be expressed in terms of its slenderness ratio λ as follows:

$$\frac{\tau_b}{\tau_y} = \frac{47.5}{\lambda^{0.87}} \quad (11)$$

Shear buckling curve of aluminum panels obtained using Eq. (11) is compared with Euler's elastic curve, curves proposed by Gerard [12], and the Aluminum Association [11], as shown in Fig. 10. Two buckling curves as per Gerard's formulation correspond to the minimum and maximum observed values of G_s/G and 0.15, respectively (Table 3). Gerard's buckling curve clearly gives the lower bound value of inelastic shear buckling stress. The inelastic buckling curve proposed by the Aluminum Association [11] lies well below the experimental prediction; however, it matched with Euler's elastic buckling curve at higher slenderness ratio. The proposed postyield buckling curve as given by Eq. (11) lies within Gerard's buckling band and hence, the prediction of inelastic shear buckling stress for panels of low slenderness ratio is quite reasonable. However, further investigation is needed to justify the validity of the proposed expression in the intermediate region for shear panels of medium slenderness.

Figure 11 shows an array describing α and β values for panels which buckled at a strain level of 0.15. The region without hatching is the zone in which no buckling took place. Thus, by taking the values of α and β for shear panels in this zone, the postyield buckling can be completely avoided for the specified strain of 0.15.

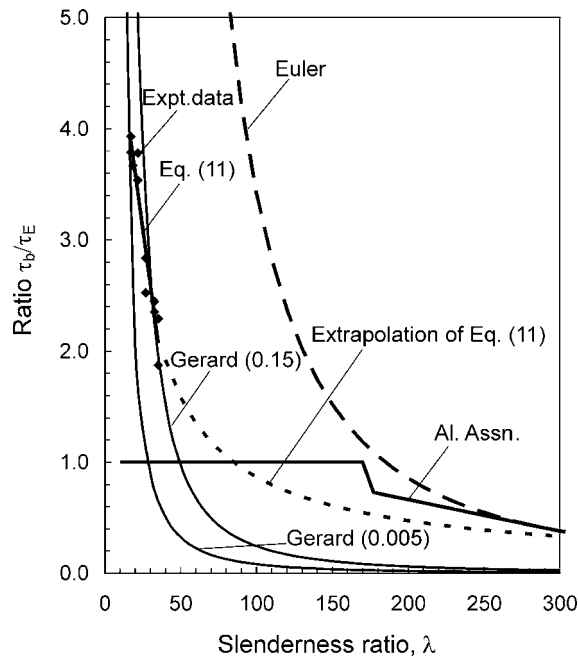


Fig. 10 Comparison of shear buckling stress curves

Conclusions

This paper presents the basic information on strength and stiffness characteristics, deformation capacities, cyclic strain-hardening effects, and deterioration behavior at large deformations of aluminum shear panels subjected low cycle fatigue, typically associated with extreme events of short duration, such as earthquakes, blast, etc., which are less repetitive at a constant magnitude. The specimens showed very ductile behavior and excellent energy dissipation potential with stable and full hysteric loops without pinching with shear strains up to 0.20. The deleterious effects of web buckling beyond yield limit can be controlled by reducing the spacing between the transverse stiffeners and thus delaying the onset of web buckling to larger strain levels. As web depth-to-thickness ratio is decreased, the tendency of buckling of the panel is significantly delayed to larger strain levels even until the tearing of web plate.

Experimental study revealed that the proportionality factor f in Gerard's formulation varied from 3.0 to 7.0 for shear panels of low slenderness ratio and differing geometries. An expression connecting the web buckling deformation angle $\bar{\gamma}_b$ and the web panel aspect ratio α and the web panel depth-to-thickness ratio β was determined experimentally. It can be used to determine the spacing of transverse stiffeners to avoid web buckling of shear panels. A linear relationship between the ratios of inelastic buckling stress with slenderness ratio of the panel was also established in the log-log plot. A zone of aspect ratio α and web depth-to-

Alloy	1100	6063	6063	
α β	20	23.5	38.1	
1.25				3 panels buckled
1.00				
0.75				2 panels buckled

Fig. 11 Buckling matrix of aluminum shear panels up to 15% strain

thickness ratio β has been identified in which postyield buckling of aluminum shear panels can be completely avoided.

Acknowledgment

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Nomenclature

- A_w = area of web
- B_s, D_s = material parameters defined by Aluminum Association
- a = shorter dimension of panel
- b = longer dimension of panel
- C = clear spacing of stiffeners
- d_w = clear depth of web
- E = Young's modulus
- E_t = tangent modulus
- f = proportionality constant as defined in Gerard's formulation
- G = shear modulus
- G_s = shear secant modulus
- k_s = buckling coefficient
- l_w = length of web
- n = number of panels
- t_w = thickness of web
- t_s = thickness of stiffeners
- V = lateral load
- α = ratio of stiffener spacing to clear depth of web
- β = web depth-to-thickness ratio
- Δ = lateral displacement
- η = plastic-reduction factor
- λ = characteristic slenderness ratio
- τ = shear stress
- γ = elastic shear strain
- γ_b = inelastic cyclic shear strain at buckling
- $\bar{\gamma}_b$ = inelastic cyclic shear strain at buckling in Gerard's buckling criterion
- $\sigma_{0.2}$ = proof stress corresponding to 0.2% strain
- τ_b = inelastic buckling stress
- τ_E = elastic buckling stress
- τ_y = yield shear stress

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Bifurcations of Equilibria in Potential Systems at Bimodal Critical Points

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Bifurcations of equilibria at bimodal branching points in potential systems are investigated. General formulas describing postbuckling paths and conditions for their stability are derived in terms of the original potential energy. Formulas describing unfolding of bimodal branching points due to a change of system parameters are given. A full list of possible cases for postbuckling paths, their stability, and unfolding depending on three system coefficients is presented. In order to calculate these coefficients, one needs the derivatives of the potential energy and eigenvectors of the linearized problem taken at the bifurcation point. The presented theory is illustrated by a mechanical example on stability and postbuckling behavior of an articulated elastic column having four degrees of freedom and depending on three problem parameters (stiffness coefficients at the hinges). For some of the bimodal critical points, numerical results are obtained illustrating influence of parameters on postbuckling paths, their stability, and unfolding. A surprising phenomenon that a symmetric bimodal column loaded by an axial force can buckle with a stable asymmetric mode is recognized. An example with a constrained sum of the stiffnesses of the articulated column shows that the maximum critical load (optimal design) is attained at the bimodal point. [DOI: 10.1115/1.2793136]

1 Introduction

This paper is devoted to analysis of bimodal branching points of stable trivial equilibrium in multiple degrees-of-freedom potential systems with the symmetry. These points were studied in a number of books and papers [1–5]. In the books [1,2], a rather general method how to analyze postbuckling paths and their stability is presented. This method involves diagonalization procedure of the potential energy and elimination of passive coordinates, i.e., some transformations of the original potential energy are needed. References [3,4] deal with the unfolding of bimodal branching points of general two degrees-of-freedom systems with symmetry. The bimodal critical points and their unfolding for two degrees-of-freedom systems with double symmetry were studied in Chap. X of the well-known book [5] on bifurcation theory. However, in these works, the full list of possible bifurcations was not given.

Some early examples on bimodal critical points were presented in Refs. [6–8]. It turns out that bimodal branching points are closely related to structural optimization problems [1]. Bimodal optimal columns (in continuous formulation) were recognized in Ref. [9]. Since that time, bi- and multimodality (multiplicity of eigenmodes at the same critical load) became a popular topic in structural optimization under stability constraints [10–14].

In this paper, we intend to give a complete theory of bimodal bifurcations in potential systems with symmetries. We present the full classification of possible cases for postbuckling paths and their stability depending on three coefficients. It is important that all the formulas derived in this paper are given in terms of the original potential energy of the system with multiple degrees of freedom. Then, we study unfolding of bimodal branching points due to change of problem parameters. Our approach is straightforward, explicit, and practical allowing to analyze bifurcations and stability of postbuckling paths, as well as their unfolding, based on calculation of the derivatives of the potential energy and eigen-

vectors of the linearized problem, taken at the bifurcation point. The presented theory is illustrated by a mechanical example on stability and postbuckling behavior of an articulated bimodal elastic column having four degrees of freedom and depending on three parameters.

2 Potential Systems

Consider a potential system with a state vector $\mathbf{q} = (q_1, q_2, \dots, q_n)$. Equilibria of such a system are determined by critical points of the potential energy function $V(\mathbf{q})$ at which first variation of the potential energy with respect to the state vector is zero:

$$\delta V(\mathbf{q}) = 0 \quad (1)$$

An equilibrium is stable if it is a minimum of the potential. The sufficient stability condition is that the second variation of the potential is positive for all small variations $\delta \mathbf{q}$:

$$\delta^2 V(\mathbf{q}) > 0 \quad (2)$$

with the unstrict inequality $\delta^2 V(\mathbf{q}) \geq 0$ giving the necessary condition. The equilibrium condition (1) can be written in the form

$$\nabla V = 0 \quad \nabla = \left(\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \dots, \frac{\partial}{\partial q_n} \right) \quad (3)$$

The stability condition (2) requires positive definiteness of the Hessian matrix

$$\mathbf{C}(\mathbf{q}) = \begin{bmatrix} \partial^2 V / \partial q_1^2 & \partial^2 V / \partial q_1 \partial q_2 & \cdots & \partial^2 V / \partial q_1 \partial q_n \\ \partial^2 V / \partial q_1 \partial q_2 & \partial^2 V / \partial q_2^2 & \cdots & \partial^2 V / \partial q_2 \partial q_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial^2 V / \partial q_1 \partial q_n & \partial^2 V / \partial q_2 \partial q_n & \cdots & \partial^2 V / \partial q_n^2 \end{bmatrix} > 0 \quad (4)$$

with the second derivatives taken at the equilibrium point $\mathbf{q}$. For the second variation of the potential, one has $\delta^2 V = \frac{1}{2} \mathbf{C} \delta \mathbf{q} \cdot \delta \mathbf{q}$, where a dot denotes the inner product in $\mathbb{R}^n$. The symmetric matrix $\mathbf{C}$ is called the stiffness matrix for elastic systems.

We consider systems with the potential $V(\mathbf{q})$ having the property

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$$V(\mathbf{q}) = V(-\mathbf{q}) \quad (5)$$

This means that the system is symmetric under the inversion of the state vector $\mathbf{q} \rightarrow -\mathbf{q}$ (pendulum systems, straight beams, plates, etc.). Clearly, $\mathbf{q}=0$ is an equilibrium for such systems. Moreover, the Taylor expansion of the potential $V(\mathbf{q})$ in the neighborhood of $\mathbf{q}=0$ contains only even order terms.

3 Unimodal (Pitchfork) Bifurcation

Consider a system with the potential smoothly dependent on a parameter ϵ such that the trivial equilibrium $\mathbf{q}=0$ is stable for $\epsilon < 0$ and unstable for $\epsilon > 0$. For example, ϵ is a deviation of the loading parameter from a critical value. At $\epsilon=0$, the stability condition (4) is violated, and the stiffness matrix $\mathbf{C}_0 = \mathbf{C}(0)$ becomes singular and positive semidefinite ($\mathbf{C}_0 \geq 0$). In the case of unimodal (pitchfork) bifurcation, there is only one eigenvector $\mathbf{u} = (u_1, u_2, \dots, u_n)$ satisfying the equation

$$\mathbf{C}_0 \mathbf{u} = 0 \quad (6)$$

This eigenvector $\mathbf{u}$ corresponds to the zero eigenvalue of the matrix $\mathbf{C}_0$ and is defined up to an arbitrary nonzero scalar factor. Properties of the unimodal bifurcation are well known. However, in this section, we provide the derivation that facilitates the further analysis of the bimodal case.

For small $\mathbf{q}$ and ϵ , the potential is given by the Taylor expansion

$$V = \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 V}{\partial q_i \partial q_j} q_i q_j + \frac{1}{4!} \sum_{i,j,k,l=1}^n \frac{\partial^4 V}{\partial q_i \partial q_j \partial q_k \partial q_l} q_i q_j q_k q_l + \dots + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^3 V}{\partial q_i \partial q_j \partial \epsilon} q_i q_j \epsilon + \dots \quad (7)$$

where all the derivatives are taken at $\mathbf{q}=0$ and $\epsilon=0$. Here, we used condition (5) implying that all terms of odd order in $\mathbf{q}$ vanish; an arbitrary constant term of the potential is taken to be zero. The main (second order) term in expansion (7) can be represented as $V = \frac{1}{2} \mathbf{C}_0 \mathbf{q} \cdot \mathbf{q} + \dots$. By using this expression in the equation for equilibria (3), we find

$$\nabla V = \mathbf{C}_0 \mathbf{q} + \dots = 0 \quad (8)$$

Hence, according to Eq. (6), nontrivial equilibria for small ϵ are given asymptotically by

$$\mathbf{q}(\epsilon) \approx \alpha \mathbf{u} \quad (9)$$

where α is an unknown function of ϵ .

In order to find α , consider the equation $\mathbf{u} \cdot \nabla V = 0$ following directly from Eq. (3). By using expansion (7), we obtain

$$\mathbf{u} \cdot \nabla V = \frac{1}{3!} \sum_{i,j,k,l=1}^n \frac{\partial^4 V}{\partial q_i \partial q_j \partial q_k \partial q_l} q_i q_j q_k u_l + \sum_{i,j=1}^n \frac{\partial^3 V}{\partial q_i \partial q_j \partial \epsilon} q_i u_j \epsilon + \dots = 0 \quad (10)$$

In Eq. (10), the two lowest order terms are presented, and the term $\mathbf{u} \cdot \mathbf{C}_0 \mathbf{q} = \mathbf{C}_0 \mathbf{u} \cdot \mathbf{q}$ vanishes due to condition (6). Substituting Eq. (9) into Eq. (10) and neglecting higher order terms, we obtain the equation for α as

$$\frac{v_{1111}}{6} \alpha^3 + v_{11\epsilon} \alpha \epsilon = 0 \quad (11)$$

where the coefficients v_{1111} and $v_{11\epsilon}$ are

$$v_{1111} \equiv (\mathbf{u} \cdot \nabla)^4 V = \sum_{i,j,k,l=1}^n \frac{\partial^4 V}{\partial q_i \partial q_j \partial q_k \partial q_l} u_i u_j u_k u_l$$

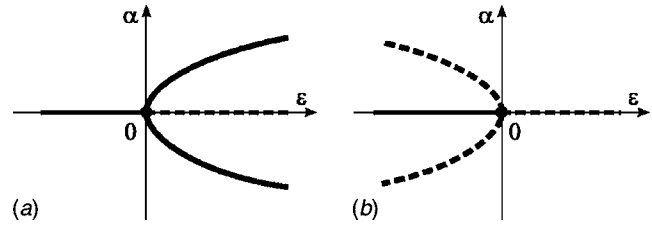


Fig. 1 Pitchfork bifurcation: (a) supercritical ($v_{1111} > 0$) and (b) subcritical ($v_{1111} < 0$). Stable equilibria are shown by solid lines.

$$v_{11\epsilon} \equiv (\mathbf{u} \cdot \nabla)^2 \frac{\partial V}{\partial \epsilon} = \sum_{i,j=1}^n \frac{\partial^3 V}{\partial q_i \partial q_j \partial \epsilon} u_i u_j \quad (12)$$

(here, $\mathbf{u} \cdot \nabla = \sum_{i=1}^n u_i (\partial / \partial q_i)$ is the derivative along the direction $\mathbf{u}$ in state space).

Nonzero solutions of Eq. (11) are

$$\alpha = \pm \sqrt{-\frac{v_{11\epsilon}}{v_{1111}} 6 \epsilon} \quad (13)$$

Nontrivial equilibria exist only if the expression under the square root is positive. Thus, if $v_{11\epsilon}/v_{1111} < 0$, then two nontrivial solutions exist for $\epsilon > 0$. If $v_{11\epsilon}/v_{1111} > 0$, then two nontrivial solutions exist for $\epsilon < 0$. These two cases are called supercritical and subcritical bifurcations, respectively [5].

Let us study stability of the equilibria $\mathbf{q} = \alpha \mathbf{u}$ for small ϵ . The equilibrium is stable if the stiffness matrix $\mathbf{C}$ is positive definite or, equivalently, the second variation $\delta^2 V = \frac{1}{2} \mathbf{C} \delta \mathbf{q} \cdot \delta \mathbf{q}$ is positive for all $\delta \mathbf{q}$. Here, the stiffness matrix $\mathbf{C}(\mathbf{q})$ is evaluated at $\mathbf{q} = \alpha \mathbf{u}$. Up to zero order terms, $\mathbf{C}(\mathbf{q}) = \mathbf{C}_0 + \dots$ with the positive semidefinite matrix $\mathbf{C}_0$ such that $\mathbf{C}_0 \delta \mathbf{q} \cdot \delta \mathbf{q} = 0$ only for $\delta \mathbf{q} \sim \mathbf{u}$. Hence, the stability condition must be checked only along the degenerate direction $\delta \mathbf{q} = \mathbf{u}$:

$$\begin{aligned} \delta^2 V &= \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 V}{\partial q_i \partial q_j} \bigg|_{\mathbf{q}=\alpha \mathbf{u}, \epsilon} \delta q_i \delta q_j \\ &\approx \frac{1}{4} \sum_{i,j,k,l=1}^n \frac{\partial^4 V}{\partial q_i \partial q_j \partial q_k \partial q_l} \bigg|_{\mathbf{q}=0, \epsilon=0} u_i u_j (\alpha u_k) (\alpha u_l) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^3 V}{\partial q_i \partial q_j \partial \epsilon} \bigg|_{\mathbf{q}=0, \epsilon=0} u_i u_j \epsilon \end{aligned} \quad (14)$$

where we used the expansion similar to Eq. (7) and neglected higher order terms. By using notation (12), we write the stability condition $\delta^2 V > 0$ in the form

$$\frac{v_{1111}}{2} \alpha^2 + v_{11\epsilon} \epsilon > 0 \quad (15)$$

For the trivial equilibrium ($\alpha=0$), the stability condition yields $v_{11\epsilon} \epsilon > 0$. Since we assumed that the trivial equilibrium is stable for $\epsilon < 0$, one obtains

$$v_{11\epsilon} < 0 \quad (16)$$

For the nontrivial solutions (13), we substitute $\epsilon = -v_{1111} \alpha^2 / (6 v_{11\epsilon})$ into Eq. (15) and get

$$\frac{v_{1111}}{3} \alpha^2 > 0 \quad (17)$$

This gives the well-known property of the pitchfork bifurcation [1,5]: The nontrivial equilibrium is stable in supercritical bifurcations ($v_{1111} > 0$) and unstable in subcritical bifurcations ($v_{1111} < 0$), see Fig. 1.

4 Bifurcation at a Bimodal Critical Point

The goal of this paper is to study bifurcation at a so-called *bimodal* critical point, when there are two linearly independent eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ (unstable modes) satisfying Eq. (6). For small $\mathbf{q}$ and ϵ , Eq. (8) gives asymptotic form of the bifurcating equilibria. Hence, $\mathbf{q}$ is a null-space vector of $\mathbf{C}_0$ given by an arbitrary linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$:

$$\mathbf{q}(\epsilon) \approx \alpha \mathbf{u}_1 + \beta \mathbf{u}_2 \quad (18)$$

where α and β are unknown functions of ϵ . Consider the equations $\mathbf{u}_1 \cdot \nabla V = 0$ and $\mathbf{u}_2 \cdot \nabla V = 0$ following directly from Eq. (3). Similar to the unimodal case, by using expressions (10) and (18) and neglecting higher order terms, we obtain the equations for α and β as

$$(v_{11\epsilon}\alpha + v_{12\epsilon}\beta)\epsilon + \frac{v_{1111}}{6}\alpha^3 + \frac{v_{1112}}{2}\alpha^2\beta + \frac{v_{1122}}{2}\alpha\beta^2 + \frac{v_{1222}}{6}\beta^3 = 0 \quad (19)$$

$$(v_{12\epsilon}\alpha + v_{22\epsilon}\beta)\epsilon + \frac{v_{1112}}{6}\alpha^3 + \frac{v_{1122}}{2}\alpha^2\beta + \frac{v_{1222}}{2}\alpha\beta^2 + \frac{v_{2222}}{6}\beta^3 = 0$$

Here, we introduced the notation

$$v_{abcd} = (\mathbf{u}_a \cdot \nabla)(\mathbf{u}_b \cdot \nabla)(\mathbf{u}_c \cdot \nabla)(\mathbf{u}_d \cdot \nabla)V \quad v_{ab\epsilon} = (\mathbf{u}_a \cdot \nabla)(\mathbf{u}_b \cdot \nabla) \frac{\partial V}{\partial \epsilon} \quad (20)$$

with the derivatives evaluated at $\mathbf{q}=0$ and $\epsilon=0$ (for comparison, see Eq. (12)).

Equations (19) can be solved as follows. Expressing ϵ from either of Eq. (19), we find

$$\epsilon = c\beta^2 \quad (21)$$

where

$$c = -\frac{v_{1111}\gamma^3 + 3v_{1112}\gamma^2 + 3v_{1122}\gamma + v_{1222}}{6(v_{11\epsilon}\gamma + v_{12\epsilon})} = -\frac{v_{1111}\gamma^3 + 3v_{1112}\gamma^2 + 3v_{1222}\gamma + v_{2222}}{6(v_{12\epsilon}\gamma + v_{22\epsilon})} \quad (22)$$

and $\gamma = \alpha/\beta$. It is also possible to express ϵ through α from Eqs. (19) with a coefficient depending on the inverse ratio $1/\gamma = \beta/\alpha$.

The second equality in Eq. (22) yields the quartic equation for γ as

$$c_4\gamma^4 + c_3\gamma^3 + c_2\gamma^2 + c_1\gamma + c_0 = 0 \quad (23)$$

with the coefficients

$$\begin{aligned} c_0 &= v_{1222}v_{22\epsilon} - v_{2222}v_{12\epsilon} & c_1 &= 3v_{1122}v_{22\epsilon} - 2v_{1222}v_{12\epsilon} - v_{2222}v_{11\epsilon} \\ c_2 &= 3v_{1112}v_{22\epsilon} - 3v_{1222}v_{11\epsilon} \\ c_3 &= v_{1111}v_{22\epsilon} + 2v_{1112}v_{12\epsilon} - 3v_{1122}v_{11\epsilon} \\ c_4 &= v_{1111}v_{12\epsilon} - v_{1112}v_{11\epsilon} \end{aligned} \quad (24)$$

Equation (23) has two or four real roots, see Sec. 5 for the proof that the situation when all four roots are complex is impossible, i.e., isola point does not exist. The vanishing leading coefficient ($c_4=0$) corresponds to $1/\gamma = \beta/\alpha = 0$, which yields $\beta=0$.

The obtained results can be summarized as follows.

THEOREM 1. *Nontrivial equilibria near a bimodal critical point $\epsilon=0$ have the asymptotic form $\mathbf{q}(\epsilon) \approx \alpha \mathbf{u}_1 + \beta \mathbf{u}_2$, with $\alpha = \gamma\beta$ and $\beta = \pm \sqrt{\epsilon/c}$. There exist two or four branches of nontrivial equilibria given by two or four real solutions γ of quartic equation (23), and c given by expression (22). Each branch determines two symmetric equilibria, which differ by the sign; the branch is subcritical if $c < 0$ (equilibria appear for $\epsilon < 0$) and supercritical if $c > 0$ (equilibria appear for $\epsilon > 0$).*

We note that the maximum number of postbuckling paths was counted [5] but formulas for the coefficients (21)–(24) are new.

Let us study stability of the equilibria $\mathbf{q} = \alpha \mathbf{u}_1 + \beta \mathbf{u}_2$ for small ϵ . The equilibrium is stable if the stiffness matrix $\mathbf{C}$ is positive definite or, equivalently, the second variation $\delta^2 V = \frac{1}{2} \mathbf{C} \delta \mathbf{q} \cdot \delta \mathbf{q}$ is positive for all $\delta \mathbf{q}$ with the stiffness matrix $\mathbf{C}(\mathbf{q})$ evaluated at $\mathbf{q} = \alpha \mathbf{u}_1 + \beta \mathbf{u}_2$. As in the unimodal case (see Sec. 3), the stability condition must be checked only along the degenerate directions. In the bimodal case, degenerate directions are $\delta \mathbf{q} = a \mathbf{u}_1 + b \mathbf{u}_2$ with arbitrary constants a and b . Up to lowest order terms, we have

$$\begin{aligned} \delta^2 V &= \frac{1}{2} \sum_{i,j=1}^n \left. \frac{\partial^2 V}{\partial q_i \partial q_j} \right|_{\alpha \mathbf{u}_1 + \beta \mathbf{u}_2, \epsilon} \delta q_i \delta q_j \approx \frac{1}{4} \sum_{i,j,k,l=1}^n \left. \frac{\partial^4 V}{\partial q_i \partial q_j \partial q_k \partial q_l} \right|_{\mathbf{q}=0, \epsilon=0} (a u_{1i} + b u_{2i})(a u_{1j} + b u_{2j})(a u_{1k} + b u_{2k})(a u_{1l} + b u_{2l}) \\ &+ \frac{1}{2} \sum_{i,j=1}^n \left. \frac{\partial^3 V}{\partial q_i \partial q_j \partial \epsilon} \right|_{\mathbf{q}=0, \epsilon=0} (a u_{1i} + b u_{2i})(a u_{1j} + b u_{2j}) \epsilon \\ &= \frac{1}{2} \left(v_{11\epsilon} \epsilon + \frac{v_{1111}}{2} \alpha^2 + v_{1112} \alpha \beta + \frac{v_{1122}}{2} \beta^2 \right) a^2 \\ &+ \left(v_{12\epsilon} \epsilon + \frac{v_{1112}}{2} \alpha^2 + v_{1122} \alpha \beta + \frac{v_{1222}}{2} \beta^2 \right) ab + \frac{1}{2} \left(v_{22\epsilon} \epsilon + \frac{v_{1122}}{2} \alpha^2 + v_{1222} \alpha \beta + \frac{v_{2222}}{2} \beta^2 \right) b^2 \end{aligned} \quad (25)$$

where we used expansion (7) and notation (20); u_{1i} and u_{2i} are the components of the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$. Then, the stability condition $\delta^2 V > 0$ (for arbitrary nonzero a and b) takes the form of positive definiteness of the 2×2 matrix

$$\begin{pmatrix} v_{11\epsilon} \epsilon + (v_{1111}/2) \alpha^2 + v_{1112} \alpha \beta + (v_{1122}/2) \beta^2 & v_{12\epsilon} \epsilon + (v_{1112}/2) \alpha^2 + v_{1122} \alpha \beta + (v_{1222}/2) \beta^2 \\ v_{12\epsilon} \epsilon + (v_{1112}/2) \alpha^2 + v_{1122} \alpha \beta + (v_{1222}/2) \beta^2 & v_{22\epsilon} \epsilon + (v_{1122}/2) \alpha^2 + v_{1222} \alpha \beta + (v_{2222}/2) \beta^2 \end{pmatrix} > 0 \quad (26)$$

It was assumed that the trivial equilibrium ($\alpha=\beta=0$) is stable for $\epsilon<0$. In this case, the stability condition (26) yields the inequalities

$$v_{11\epsilon} < 0 \quad v_{22\epsilon} < 0 \quad v_{11\epsilon}v_{22\epsilon} - v_{12\epsilon}^2 > 0 \quad (27)$$

For nontrivial equilibria $\alpha=\gamma\beta$, $\epsilon=c\beta^2$, condition (26) is equivalent to

$$\begin{pmatrix} v_{11\epsilon}c + (v_{1111}/2)\gamma^2 + v_{1112}\gamma + (v_{1122}/2) & v_{12\epsilon}c + (v_{1112}/2)\gamma^2 + v_{1122}\gamma + (v_{1222}/2) \\ v_{12\epsilon}c + (v_{1112}/2)\gamma^2 + v_{1122}\gamma + (v_{1222}/2) & v_{22\epsilon}c + (v_{1122}/2)\gamma^2 + v_{1222}\gamma + (v_{2222}/2) \end{pmatrix} > 0 \quad (28)$$

Thus, an equilibrium with the branch corresponding to a given γ is stable if the matrix (28) is positive definite. If this matrix has a negative eigenvalue, the equilibrium is unstable.

5 No Isola Point Exists

We show that it is impossible to have four complex roots γ of the quartic equation (23).

First, we choose the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ such that in expressions (19) and (20) $v_{12\epsilon}=0$. With this choice, the 2×2 matrix with the elements $v_{ij\epsilon}$ is reduced to the diagonal form. Then, according to Eq. (24), we have

$$\frac{c_2}{c_4} = 3 \frac{v_{1222}}{v_{1112}} - 3 \frac{v_{22\epsilon}}{v_{11\epsilon}} \quad \frac{c_0}{c_4} = - \frac{v_{1222}v_{22\epsilon}}{v_{1112}v_{11\epsilon}} \quad (29)$$

Now, let us assume that all four roots of the polynomial (23) are complex and equal to $x_1 \pm iy_1$, $x_2 \pm iy_2$. Then, $c_0/c_4 = (x_1^2 + y_1^2)(x_2^2 + y_2^2) > 0$. Hence, from conditions (27) and expressions (29), we obtain $v_{1222}/v_{1112} < 0$ and $c_2/c_4 < 0$. Under these inequalities, it is easy to show that $(c_2/c_4)^2 \geq 36c_0/c_4$. On the other hand, $c_2/c_4 = x_1^2 + y_1^2 + x_2^2 + y_2^2 + 4x_1x_2$. Since $c_2/c_4 < 0$, we have $(c_2/c_4)^2 \leq (4x_1x_2)^2 = 16x_1^2x_2^2 < 16c_0/c_4$. But, this contradicts to the inequality $(c_2/c_4)^2 \geq 36c_0/c_4$ derived above. Therefore, Eq. (23) always has real roots.

This means that there is no isola point, i.e., there exist nontrivial paths bifurcating from the trivial state at the bimodal critical point.

6 Symmetric Systems

In many practical problems, a system possesses an additional symmetry represented by the following invariance condition for the potential:

$$V(\mathbf{q}) = V(S(\mathbf{q})) \quad (30)$$

with a linear map $S(\mathbf{q})$ satisfying the relation $S(S(\mathbf{q})) = \mathbf{q}$ (of course, S is assumed to be different from $\mathbf{q} \rightarrow -\mathbf{q}$). This condition may reflect axial or spatial symmetry of the system. For example, consider a beam of variable cross section with the material distribution symmetric with respect to the middle and identical boundary conditions taken at $x = \pm a$, where x is the axial coordinate with the origin at the beam center. Then, $S(w(x)) = w(-x)$, where $\mathbf{q} \equiv w(x)$ is a deflection function of the beam. Keeping in mind this example, we say that $\mathbf{q}$ is a symmetric or antisymmetric form if $S(\mathbf{q}) = \mathbf{q}$ or $S(\mathbf{q}) = -\mathbf{q}$, respectively. If $S(\mathbf{q}) \neq \pm \mathbf{q}$, we say that the form is of mixed type.

In a unimodal (pitchfork) bifurcation, the unstable mode $\mathbf{u}$ must be either symmetric or antisymmetric: $\mathbf{u} = \pm S(\mathbf{u})$. It cannot be of mixed type since that would automatically provide two linearly independent unstable modes $\mathbf{u}$ and $S(\mathbf{u})$.

Let us consider a bimodal bifurcation. We can always choose the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ to be symmetric or antisymmetric: $S(\mathbf{u}_{1,2}) = \pm \mathbf{u}_{1,2}$. We assume that $\mathbf{u}_1$ is symmetric, while $\mathbf{u}_2$ is antisymmetric.

According to symmetry condition (30), the coefficients (20) do not change if we substitute $\mathbf{u}_1$ and $\mathbf{u}_2$ by $S(\mathbf{u}_1) = \mathbf{u}_1$ and $S(\mathbf{u}_2) = -\mathbf{u}_2$, respectively. The following coefficients vanish:

$$v_{12\epsilon} = v_{1112} = v_{1222} = 0 \quad (31)$$

since they change their sign under the substitution $\mathbf{u}_2 \rightarrow -\mathbf{u}_2$. Note that Eq. (19) with conditions (31) coincides with the corresponding equation for a two degrees-of-freedom system with double symmetry studied earlier [3–5].

For the sake of convenience, we introduce the normalization conditions for the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ such that

$$v_{11\epsilon} = -1 \quad v_{22\epsilon} = -1 \quad (32)$$

which is possible since according to Eq. (27) $v_{11\epsilon} < 0$ and $v_{22\epsilon} < 0$.

Solving system (19) with Eqs. (31) and (32) gives the unknown α and β corresponding to three types of nontrivial equilibria:

$$\alpha^2 = \frac{6\epsilon}{v_{1111}} \quad \beta = 0 \quad (33)$$

$$\alpha = 0 \quad \beta^2 = \frac{6\epsilon}{v_{2222}} \quad (34)$$

$$\alpha^2 = \frac{v_{2222} - 3v_{1122}}{v_{1111}v_{2222} - 9v_{1122}^2} 6\epsilon \quad \beta^2 = \frac{v_{1111} - 3v_{1122}}{v_{1111}v_{2222} - 9v_{1122}^2} 6\epsilon \quad (35)$$

Solutions (33), (34), and (35) with different signs of α and β define two symmetric, two antisymmetric, and four mixed-type equilibria (18), respectively. Symmetric equilibria are subcritical or supercritical for negative and positive values of v_{1111} , respectively. The type of antisymmetric equilibria is determined similarly by the sign of v_{2222} . Mixed-type equilibria (35) exist if the quantities $v_{2222} - 3v_{1122}$ and $v_{1111} - 3v_{1122}$ have the same sign. Under this condition, mixed-type equilibria are subcritical or supercritical for negative and positive signs of the fractional factor in Eq. (35), respectively.

The stability condition (26) takes the form

$$\begin{pmatrix} -\epsilon + v_{1111}\alpha^2/2 + v_{1122}\beta^2/2 & v_{1122}\alpha\beta \\ v_{1122}\alpha\beta & -\epsilon + v_{1122}\alpha^2/2 + v_{2222}\beta^2/2 \end{pmatrix} > 0 \quad (36)$$

For symmetric equilibria (33), eigenvalues of the matrix (36) are

$$\lambda_1 = \frac{v_{1111}}{3}\alpha^2 \quad \lambda_2 = \frac{3v_{1122} - v_{1111}}{6}\alpha^2 \quad (37)$$

By using inequalities (27), we obtain the stability conditions as

$$v_{1111} > 0 \quad 3v_{1122} - v_{1111} > 0 \quad (38)$$

For antisymmetric equilibria (34), the eigenvalues are

$$\lambda_1 = \frac{v_{2222}}{3}\beta^2 \quad \lambda_2 = \frac{3v_{1122} - v_{2222}}{6}\beta^2 \quad (39)$$

and the stability conditions become

Table 1 Classification of bifurcations at a bimodal critical point

No.	v_{1111}	v_{2222}	$v_{1111}v_{2222}-9v_{1122}^2$	$v_{1111}-3v_{1122}$	$v_{2222}-3v_{1122}$
1	+	+	-	-	-
2	+	-	-	-	-
3	-	+	-	-	-
4	-	-	-	-	-
5	-	-	+	-	-
6	+	+	+	+	+
7	+	+	-	+	+
8	+	-	-	+	+
9	-	+	-	+	+
10	-	-	-	+	+
11	+	+	-	-	+
12	+	+	-	+	-
13	+	-	-	+	-
14	-	+	-	-	+
15	-	-	-	-	+
16	-	-	-	+	-

$$v_{2222} > 0 \quad 3v_{1122} - v_{2222} > 0 \quad (40)$$

Finally, for mixed-type equilibria (35), we find the eigenvalues

$$\lambda_{1,2} = \frac{v_{1111}\alpha^2 + v_{2222}\beta^2 \pm \sqrt{(v_{1111}\alpha^2 - v_{2222}\beta^2)^2 + 36v_{1122}^2\alpha^2\beta^2}}{6} \quad (41)$$

and the stability conditions

$$v_{1111} > 0 \text{ and } v_{2222} > 0 \quad v_{1111}v_{2222} - 9v_{1122}^2 > 0 \quad (42)$$

The obtained results allow classifying all types of bimodal bifurcations by the signs of specific quantities (depending on derivatives of the potential) evaluated at $\epsilon=0$ and $\mathbf{q}=0$, see Table 1. We note that only three numbers, namely, v_{1111} , v_{2222} , and v_{1122} , govern the postbuckling behavior.

Bifurcation diagrams corresponding to 16 cases of Table 1 are shown in Figs. 2 and 3, see the diagrams corresponding to $\chi=0$. Due to the symmetry with respect to the planes $\alpha=0$ and $\beta=0$, we show only the quarter domain $\alpha, \beta \geq 0$ of the $(\epsilon, \alpha, \beta)$ space. In the figures, stable equilibria are shown by thick lines. Thin solid and dashed lines correspond to unstable equilibria with one and two negative eigenvalues of the matrix (36), respectively. S, A, and M are abbreviations for symmetric, antisymmetric, and mixed-type equilibria, respectively. Pictures in Figs. 2 and 3 ($\chi=0$) are based on relations (33)–(36), (38), (40), and (42).

One can see from Figs. 2 and 3 ($\chi=0$) that stable nontrivial equilibria exist in six cases (the Cases 1–3, 6, 11, and 12). An equilibrium of any type can be stable: symmetric, antisymmetric, or mixed type. In the remaining ten cases, all nontrivial equilibria are unstable. These cases describe limit points leading to dynamic snaps since beyond these critical points, there is no stable solution. Nontrivial supercritical equilibria can be unstable for bimodal bifurcations, while for unimodal bifurcations, they are always stable. However, stable nontrivial equilibria are always supercritical. If symmetric or antisymmetric equilibrium is stable, mixed-type equilibrium is unstable, and if mixed-type equilibrium is stable, the symmetric and antisymmetric equilibria are unstable.

Note that Cases 1, 6, 7, 11, and 12 of Table 1 were recognized and qualitatively described [5].

7 Unfolding of Bifurcations at Bimodal Critical Points

Now, let us consider a symmetric system, as in the previous section, with the potential smoothly depending on m parameters $\epsilon_1, \epsilon_2, \dots, \epsilon_m$. We assume that, for fixed $\epsilon_2 = \dots = \epsilon_m = 0$, the bimodal bifurcation takes place in one-parameter system $\epsilon = \epsilon_1$, as described above. For small nonzero (but fixed) values of the parameters $\epsilon_2, \dots, \epsilon_m$, the system behavior depending on $\epsilon = \epsilon_1$ can change qualitatively, i.e., we can observe unfolding of the bimodal

bifurcation. The parameters $\epsilon_2, \dots, \epsilon_m$ can be treated as imperfections that keep the system symmetry. Nontrivial equilibria in this case are described by the asymptotic formula (18). The unknown coefficients α and β are determined by the equations

$$\begin{aligned} (-\epsilon + \tilde{v}_{11\epsilon})\alpha + v_{1111}\alpha^3/6 + v_{1122}\alpha\beta^2/2 &= 0 \\ (-\epsilon + \tilde{v}_{22\epsilon})\beta + v_{1122}\alpha^2\beta/2 + v_{2222}\beta^3/6 &= 0 \end{aligned} \quad (43)$$

where

$$\tilde{v}_{11\epsilon} = \sum_{k=2}^n (\mathbf{u}_1 \cdot \nabla)^2 \frac{\partial V}{\partial \epsilon_k} \epsilon_k \quad \tilde{v}_{22\epsilon} = \sum_{k=2}^n (\mathbf{u}_2 \cdot \nabla)^2 \frac{\partial V}{\partial \epsilon_k} \epsilon_k \quad (44)$$

with the derivatives taken at $\epsilon_1 = \epsilon_2 = \dots = \epsilon_m = 0$ and $\mathbf{q}=0$. Equations (43) differ from Eqs. (19) and (31) only by small constants $\tilde{v}_{11\epsilon}$ and $\tilde{v}_{22\epsilon}$ dependent on the unfolding parameters $\epsilon_2, \dots, \epsilon_m$. By solving system (43), we find the coefficients α and β corresponding to nontrivial equilibria. Similar to Eqs. (33)–(35), there can be symmetric, antisymmetric, and mixed-type solutions:

$$\alpha^2 = \frac{6(\epsilon - \tilde{v}_{11\epsilon})}{v_{1111}} \quad \beta = 0 \quad (45)$$

$$\alpha = 0 \quad \beta^2 = \frac{6(\epsilon - \tilde{v}_{22\epsilon})}{v_{2222}} \quad (46)$$

$$\alpha^2 = 6 \frac{(\epsilon - \tilde{v}_{11\epsilon})v_{2222} - 3(\epsilon - \tilde{v}_{22\epsilon})v_{1122}}{v_{1111}v_{2222} - 9v_{1122}^2} \quad (47)$$

$$\beta^2 = 6 \frac{(\epsilon - \tilde{v}_{22\epsilon})v_{1111} - 3(\epsilon - \tilde{v}_{11\epsilon})v_{1122}}{v_{1111}v_{2222} - 9v_{1122}^2}$$

If $\tilde{v}_{11\epsilon} \neq \tilde{v}_{22\epsilon}$, the branches of symmetric and antisymmetric equilibria (45) and (46) do not intersect in the space $(\epsilon, \alpha, \beta)$. This means that the bimodality is destroyed. As for the equilibria of mixed type (47), they coincide with the symmetric ones (45) at the points

$$(s): \alpha_s^2 = -\frac{6(\tilde{v}_{11\epsilon} - \tilde{v}_{22\epsilon})}{v_{1111} - 3v_{1122}} \quad \beta_s = 0 \quad \epsilon_s = \frac{\tilde{v}_{22\epsilon}v_{1111} - 3\tilde{v}_{11\epsilon}v_{1122}}{v_{1111} - 3v_{1122}} \quad (48)$$

Similarly, mixed-type equilibria (47) coincide with the antisymmetric ones (46) at the points

$$(a): \alpha_a = 0 \quad \beta_a^2 = \frac{6(\tilde{v}_{11\epsilon} - \tilde{v}_{22\epsilon})}{v_{2222} - 3v_{1122}} \quad \epsilon_a = \frac{\tilde{v}_{11\epsilon}v_{2222} - 3\tilde{v}_{22\epsilon}v_{1122}}{v_{2222} - 3v_{1122}} \quad (49)$$

At these points, the secondary (postcritical) bifurcations occur. Critical points (48) and (49) exist if the quantities α^2 and β^2 determined by the corresponding expressions are positive.

With a change of parameters $\epsilon_2, \dots, \epsilon_m$, the bimodal bifurcation splits into a series of unimodal bifurcations. For understanding the structure of the bifurcating equilibria, let us plot solutions (45)–(47) in the $(\epsilon, \alpha^2, \beta^2)$ space. Each of these solutions is represented by a straight line, Fig. 4. The line corresponding to the symmetric equilibria lies in the (ϵ, α^2) plane, the line corresponding to the antisymmetric equilibria lies in the (ϵ, β^2) plane, and the line corresponding to the mixed-type equilibria intersects the two previous lines. Of course, only the $\alpha^2 \geq 0, \beta^2 \geq 0$ part of the space has physical meaning. Therefore, we can distinguish four qualitatively different situations. If $\alpha_s^2 < 0$ and $\beta_a^2 < 0$, then the mixed-type equilibrium line does not intersect the physical domain (equilibria of mixed type do not exist). If $\alpha_s^2 > 0$ and $\beta_a^2 < 0$, then the mixed-type equilibrium half-line belongs to the physical domain (equilibria of mixed type exist and appear in the bifurcation of symmetric equilibria); this is the case shown in Fig. 4. If $\alpha_s^2 < 0$

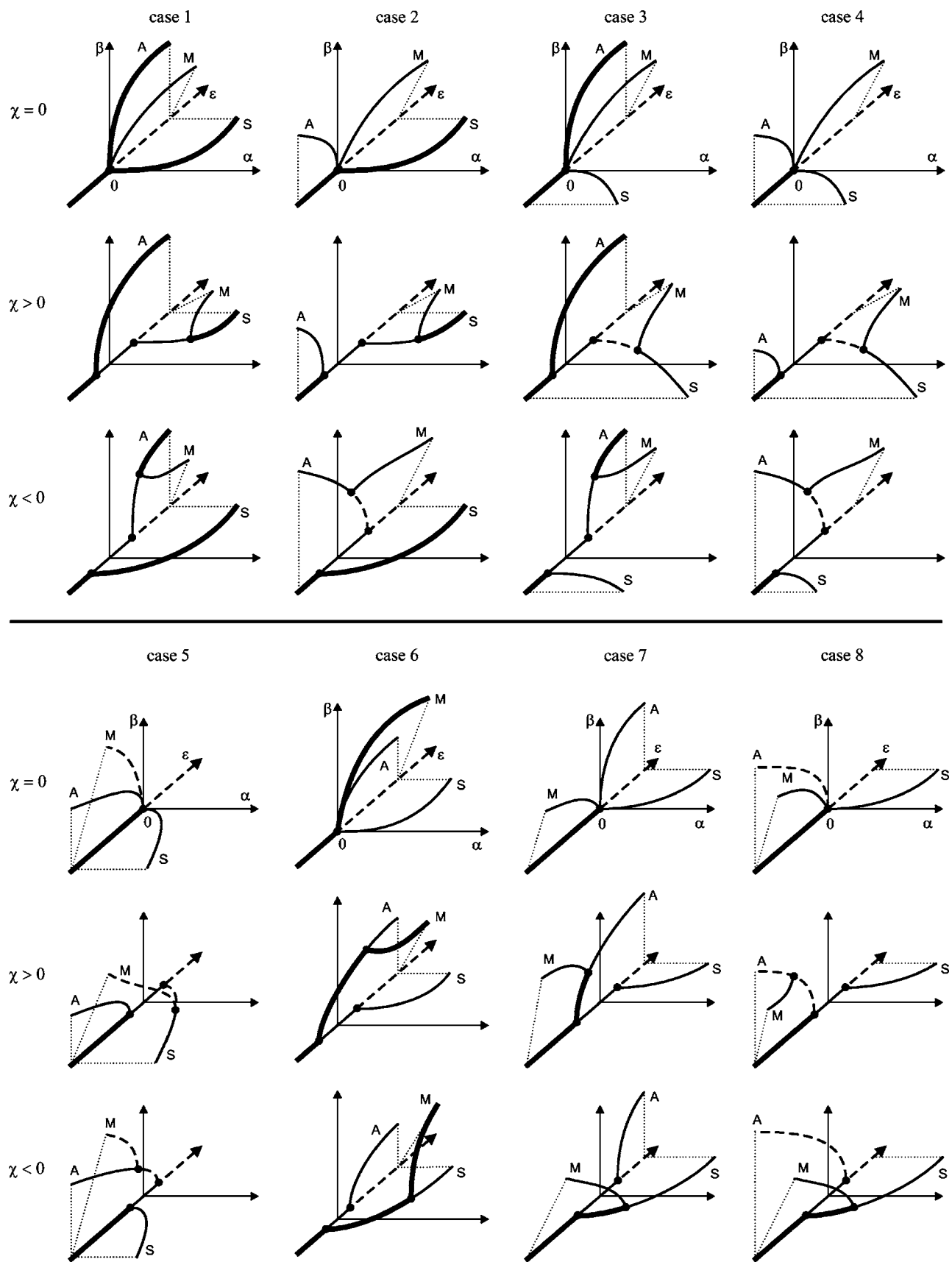


Fig. 2 Unfolding of the bimodal bifurcation: Cases 1–8

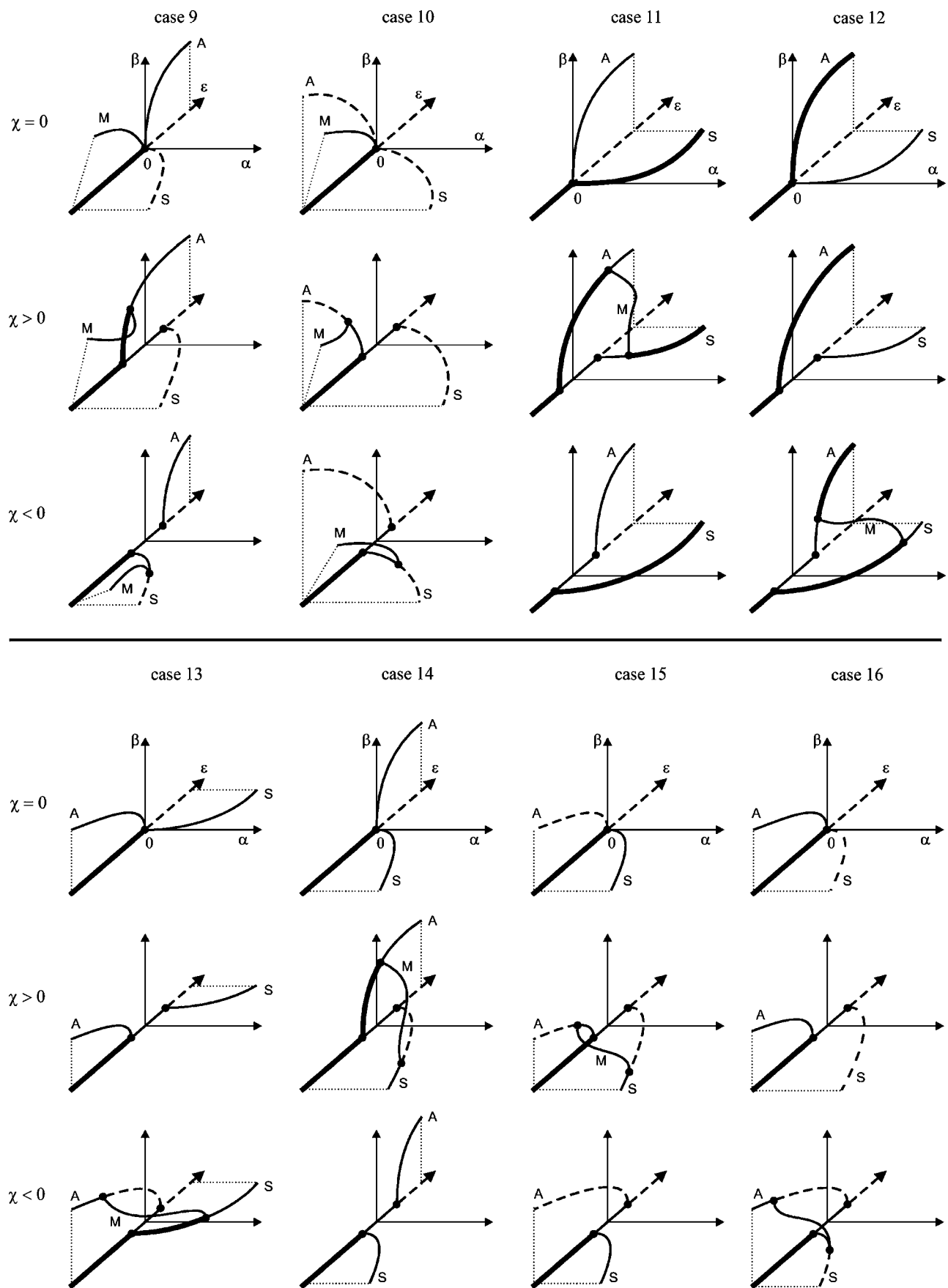


Fig. 3 Unfolding of the bimodal bifurcation: Cases 9–16

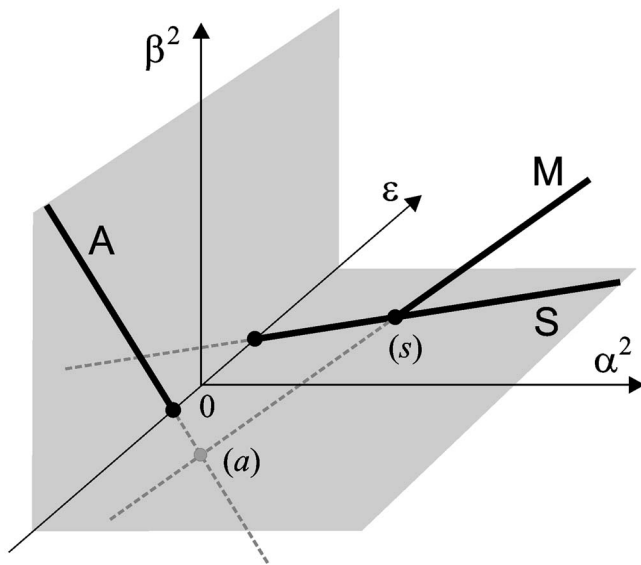


Fig. 4 Structure of nontrivial equilibria for nearly bimodal critical point

and $\beta_a^2 > 0$, then the mixed-type equilibrium half-line belongs to the physical domain (equilibria of mixed type exist and appear in the bifurcation of antisymmetric equilibria). Finally, if $\alpha_s^2 > 0$ and $\beta_s^2 > 0$, then the mixed-type equilibrium segment between the points $(\epsilon_s, \alpha_s^2, \beta_s^2)$ and $(\epsilon_a, \alpha_a^2, \beta_a^2)$ belongs to the physical domain (equilibria of mixed type exist and are bounded by the bifurcations of symmetric and antisymmetric equilibria).

In the $(\epsilon, \alpha, \beta)$ space, the equilibrium lines become curves, which are orthogonal to the planes $\alpha=0$ and $\beta=0$. Eliminating ϵ from Eqs. (47), we obtain

$$(v_{2222} - 3v_{1122})\beta^2 - (v_{1111} - 3v_{1122})\alpha^2 = 6\chi \quad \chi = \bar{v}_{11\epsilon} - \bar{v}_{22\epsilon} \quad (50)$$

On the (α, β) plane, this is a hyperbola, ellipse, or empty set depending on the signs of the coefficients $v_{1111} - 3v_{1122}$, $v_{2222} - 3v_{1122}$, and χ .

Unfolding of bifurcations for 16 cases of Table 1 is depicted in Figs. 2 and 3. These figures are based on relations (45)–(50). Note that the bifurcation points of the trivial equilibrium correspond to $\epsilon = \bar{v}_{11\epsilon}$ for the symmetric path and $\epsilon = \bar{v}_{22\epsilon}$ for the antisymmetric path. For the sake of simplicity, in the figures, we took $\bar{v}_{11\epsilon} > 0$ and $\bar{v}_{22\epsilon} < 0$, which does not change the pictures qualitatively.

As an example, let us consider unfolding of the first case in Table 1. The unperturbed situation ($\bar{v}_{11\epsilon} = \bar{v}_{22\epsilon} = 0$), is shown in Fig. 2 (Case 1, $\chi = 0$). From Table 1, it follows that the denominators in formulas (48) and (49) are positive. Hence, if $\chi = \bar{v}_{11\epsilon} - \bar{v}_{22\epsilon} > 0$, then there are only intersections (bifurcations) between symmetric and mixed-type equilibrium branches at the point (48), Fig. 2 (Case 1, $\chi > 0$). If $\chi < 0$, then only antisymmetric and mixed-type equilibrium branches intersect at the point (49), Fig. 2 (Case 1, $\chi < 0$). Therefore, in the unfolding picture, the mixed-type equilibria appear due to the secondary bifurcation of symmetric ($\chi > 0$) or antisymmetric ($\chi < 0$) equilibria.

It should be noted that the unfolding of a bimodal critical point is qualitatively different for systems without symmetry property (30). In the latter case, typically, there are no secondary pitchfork bifurcations.

In multiparameter case, stability criterion for the equilibria is the condition of positive definiteness of the matrix (36), where one must substitute $-\epsilon$ by $-\epsilon + \bar{v}_{11\epsilon}$ and $-\epsilon + \bar{v}_{22\epsilon}$ in the first and second diagonal elements, respectively.

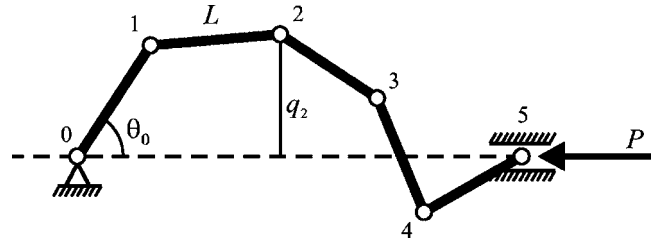


Fig. 5 Elastic articulated column loaded by an axial force

First, consider the trivial equilibrium $\alpha = \beta = 0$. In this case, the eigenvalues of the matrix (36) become

$$\lambda_1 = -\epsilon + \bar{v}_{11\epsilon} \quad \lambda_2 = -\epsilon + \bar{v}_{22\epsilon} \quad (51)$$

The bimodal critical point is defined by the conditions $\lambda_1 = \lambda_2 = 0$. With the use of Eq. (44), these conditions define a plane of codimension 2 in parameter space $(\epsilon_1, \dots, \epsilon_m)$. Hence, the codimension of a bimodal critical point equals 2 (this critical point can be typically found by adjusting values of two parameters). Here, the symmetry (30) is very important: Due to this symmetry, the off-diagonal elements $\bar{v}_{12\epsilon}$ vanish. These elements are nonzero in systems without symmetry (30) or if both unstable modes are symmetric (or antisymmetric). In that case, the codimension of a bimodal critical point equals 3, which agrees with general results of the singularity theory [15].

Stability of nontrivial equilibria can be studied similarly by computing eigenvalues of the 2×2 second variation matrix. However, in the perturbed case, we can avoid these computations by using known properties of unimodal bifurcations (Sec. 3), and the properties of postcritical paths for large $\epsilon_1 \gg \epsilon_2, \dots, \epsilon_m$ (at these values of ϵ_1 , the stability type of a postcritical path is the same as for $\epsilon_2 = \dots = \epsilon_m = 0$). The results of stability analysis are shown in Figs. 2 and 3. Recall that stable equilibria are shown by thick lines, while thin solid and dashed lines correspond to unstable equilibria with one and two negative eigenvalues of the matrix (36), respectively. For example, in Fig. 2 (Case 1, $\chi < 0$), the first bifurcation is supercritical (symmetric equilibria are stable), and antisymmetric equilibria appear when the unstable trivial equilibrium bifurcates (antisymmetric equilibria are unstable). After the secondary bifurcation, antisymmetric equilibria become stable as in the bimodal picture ($\chi = 0$), and unstable mixed-type equilibria appear. We can see that for higher values of ϵ , the stability properties of all the equilibria are the same as for the bimodal bifurcation for $\chi = 0$.

We remark that the unfoldings in Cases 5 and 10, as well as 15 and 16, are similar from the physical point of view since the unstable paths differ only by degrees of instability.

Note that for symmetric two degrees-of-freedom systems, classification of four cases with respect to the parameters $v_{1111} - 3v_{1122}$ and $v_{2222} - 3v_{1122}$ was given in [3], and Cases 1, 7, 11, and 12 ($\chi < 0$) of Figs. 2 and 3 were drawn in [4], while we have recognized 16 different cases, each of them corresponding to different pictures in 3D space.

8 Mechanical Example

As a mechanical example, we consider an elastic articulated column with elastically clamped ends loaded by an axial force P , Fig. 5. The column consists of five segments of length L connected by six elastic hinges with the bending stiffnesses $b_0, b_1, \dots, b_5$. Linear stability problem for the straight equilibrium of the column has been treated in [10]. We consider a symmetric structure with symmetric boundary conditions, so that $b_0 = b_5$, $b_1 = b_4$, and $b_2 = b_3$. Deflection of the column is determined by the vector of coordinates $\mathbf{q} = (q_1, q_2, q_3, q_4)$, which are related to the angles between the segments and the horizontal axis as

$$q_{i+1} - q_i = L \sin \theta_i \quad i = 0, \dots, 4 \quad q_0 = q_5 = 0 \quad (52)$$

The potential function of the column is

$$V = \sum_{i=0}^5 \left(\frac{b_i}{2} (\theta_i - \theta_{i-1})^2 - PL(1 - \cos \theta_i) \right) \quad \theta_{-1} = 0 \quad \theta_5 = 0 \quad (53)$$

For the sake of simplicity, we introduce nondimensional quantities

$$\tilde{q}_i = \frac{q_i}{L} \quad \tilde{P} = \frac{PL}{b_*} \quad \tilde{b}_i = \frac{b_i}{b_*} \quad \tilde{V} = \frac{V}{b_*} \quad (54)$$

where b_* is a reference stiffness.

Substituting Eq. (52) into Eq. (53) with the use of Eq. (54) and omitting tildes, we obtain the nondimensional potential function as

$$\begin{aligned} V = & \frac{b_0}{2} (\arcsin q_1)^2 + \frac{b_1}{2} (\arcsin(q_2 - q_1) - \arcsin q_1)^2 \\ & + \frac{b_2}{2} (\arcsin(q_3 - q_2) - \arcsin(q_2 - q_1))^2 + \frac{b_2}{2} (\arcsin(q_4 - q_3) \\ & - \arcsin(q_3 - q_2))^2 + \frac{b_1}{2} (\arcsin q_4 + \arcsin(q_4 - q_3))^2 \\ & + \frac{b_0}{2} (\arcsin q_4)^2 - P(5 - \sqrt{1 - q_1^2} - \sqrt{1 - (q_2 - q_1)^2} \\ & - \sqrt{1 - (q_3 - q_2)^2} - \sqrt{1 - (q_4 - q_3)^2} - \sqrt{1 - (q_4)^2}) \end{aligned} \quad (55)$$

For small values of the coordinates q_i , the potential can be expanded in Taylor series

$$\begin{aligned} V = & \frac{b_0}{2} q_1^2 + \frac{b_1}{2} (q_2 - 2q_1)^2 + \frac{b_2}{2} (q_3 - 2q_2 + q_1)^2 + \frac{b_2}{2} (q_4 - 2q_3 + q_2)^2 \\ & + \frac{b_1}{2} (-2q_4 + q_3)^2 + \frac{b_0}{2} q_4^2 - \frac{P}{2} (q_1^2 + (q_2 - q_1)^2 + (q_3 - q_2)^2 \\ & + (q_4 - q_3)^2 + q_4^2) + \dots \end{aligned} \quad (56)$$

The second order terms given in Eq. (56) define the stiffness matrix $\mathbf{C}$. Equation (6) for the linear buckling problem takes the form

$$(b_0 + 4b_1 + b_2 - 2P)u_1 + (-2b_1 - 2b_2 + P)u_2 + b_2u_3 = 0$$

$$(-2b_1 - 2b_2 + P)u_1 + (b_1 + 5b_2 - 2P)u_2 + (-4b_2 + P)u_3 + b_2u_4 = 0 \quad (57)$$

$$b_2u_1 + (-4b_2 + P)u_2 + (b_1 + 5b_2 - 2P)u_3 + (-2b_1 - 2b_2 + P)u_4 = 0$$

$$b_2u_2 + (-2b_1 - 2b_2 + P)u_3 + (b_0 + 4b_1 + b_2 - 2P)u_4 = 0$$

Due to symmetry of the column, Eq. (57) possesses symmetric and antisymmetric solutions. For the symmetric solution, we take $u_4 = u_1$, $u_3 = u_2$. Then, from the first two (or the last two) equations (57), we get the quadratic equation for buckling loads

$$P_s^2 - P_s(b_0 + 2b_1 + b_2) + b_0b_1 + b_0b_2 + b_1b_2 = 0 \quad (58)$$

Both roots of this equation are positive, and the smaller root gives the critical buckling load if buckling is symmetric.

For the antisymmetric solution, we take $u_4 = -u_1$, $u_3 = -u_2$ and similarly obtain the quadratic equation

$$P_a^2 - P_a \left(\frac{3}{5}b_0 + 2b_1 + 3b_2 \right) + \frac{1}{5}b_0b_1 + \frac{9}{5}b_0b_2 + 5b_1b_2 = 0 \quad (59)$$

The smaller root of this equation yields the critical buckling load if buckling is antisymmetric.

The condition of bimodality is that the smaller P_s is equal to the smaller P_a . So, we have

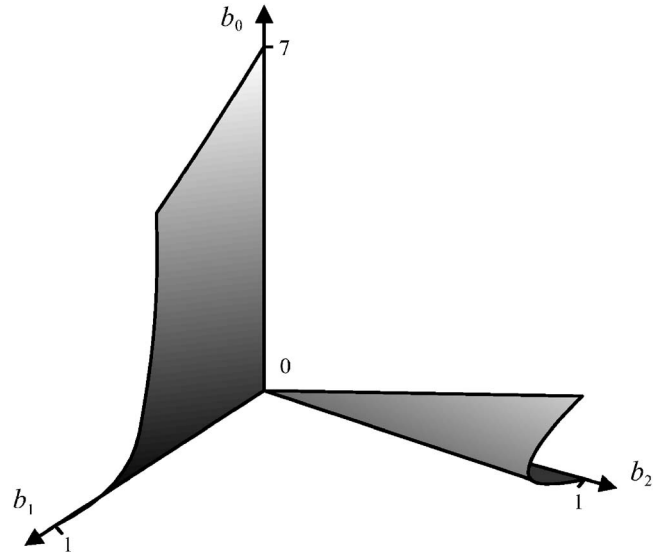


Fig. 6 Stiffness parameters for columns undergoing bimodal buckling

$$\begin{aligned} 2b_2 - \frac{2}{5}b_0 + \sqrt{b_0^2 - 2b_0b_2 + 4b_1^2 + b_2^2} \\ = \frac{1}{5} \sqrt{9b_0^2 + 40b_0b_1 - 90b_0b_2 + 100b_1^2 - 200b_1b_2 + 225b_2^2} \end{aligned} \quad (60)$$

This equation defines a surface in three-dimensional space of the column stiffnesses (b_0, b_1, b_2) shown in Fig. 6. Each point on this surface corresponds to a column with the bimodal critical buckling load.

Note that for rigid clamping of the column (as b_0 tends to infinity), Eqs. (58) and (59) furnish the buckling loads

$$P_s = b_1 + b_2 \quad P_a = \frac{b_1}{3} + 3b_2 \quad (61)$$

Thus, for rigid clamping, the bimodality condition is $b_1 = 3b_2$. This means that the bimodal surface tends to the plane $b_1 = 3b_2$ for the stiffness b_0 tending to infinity, see Fig. 6.

Let us study postbuckling behavior of the symmetric column for the parameters $b_0 = 1$, $b_1 = 0.25$, $b_2 = 1$, satisfying the bimodality condition (60). According to Eqs. (57)–(59), we compute the bimodal critical buckling load $P = 1$ and the corresponding eigenmodes (eigenvectors) $\mathbf{u}_1 = (1, 2, 2, 1)$ and $\mathbf{u}_2 = (1, 0.4, -0.4, -1)$. Expanding the potential function (55) up to fourth order terms and using Eq. (20), we compute the coefficients $v_{11\epsilon} = -4.0$ and $v_{22\epsilon} = -3.36$. Then, we normalize the eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ dividing them by $\sqrt{-v_{11\epsilon}}$ and $\sqrt{-v_{22\epsilon}}$, respectively, so that the condition (32) is satisfied. Using normalized eigenvectors in Eq. (20), we calculate the coefficients

$$v_{1111} = 0.25 \quad v_{2222} = 0.38605 \quad v_{1122} = 0.25 \quad (62)$$

The bifurcation belongs to Type 1 in Table 1. It means that both symmetric and antisymmetric solutions are supercritical and stable while the mixed-type solution is supercritical and unstable, see Fig. 2 (Case 1, $\chi = 0$). The nontrivial equilibria according to Eqs. (33)–(35) are given asymptotically as

$$\begin{aligned} \mathbf{q}_s = & \pm \sqrt{\epsilon} (0, 2.4494, 4.8989, 4.8989, 2.4494, 0) \\ \mathbf{q}_a = & \pm \sqrt{\epsilon} (0, 2.1507, 0.8602, -0.8602, -2.1507, 0) \\ \mathbf{q}_{m1} = & \pm \sqrt{\epsilon} (0, 2.4665, 2.7184, 1.6110, -0.3018, 0) \end{aligned} \quad (63)$$

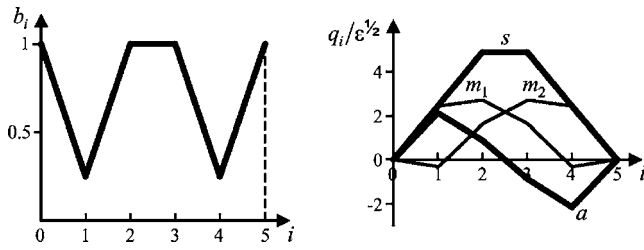


Fig. 7 Stiffnesses and buckling modes of the elastic column ($b_0=1, b_1=0.25, b_2=1$)

$$\mathbf{q}_{m2} = \pm \sqrt{\epsilon}(0, -0.3018, 1.6110, 2.7184, 2.4665, 0)$$

The stiffnesses of the bimodal column and corresponding new equilibrium states, divided by $\pm\sqrt{\epsilon}$, are presented in Fig. 7.

Let us study unfolding of this bifurcation due to change of the stiffness δb_0 . According to Eqs. (44) and (56), we find

$$\chi = \bar{v}_{11\epsilon} - \bar{v}_{22\epsilon} = 2\delta b_0(u_{11}^2 - u_{21}^2) = -0.0952\delta b_0 \quad (64)$$

Hence, if we decrease the stiffness $\delta b_0 < 0$, then the antisymmetric form of instability becomes critical, and the corresponding unfolding is shown in Fig. 2, Case 1 ($\chi > 0$). If the stiffness is increased $\delta b_0 > 0$, then the symmetric form of instability becomes critical with the unfolding shown in Fig. 2, Case 1 ($\chi < 0$).

For the stiffnesses $b_0=1, b_1=0.15, b_2=0.76465$, we compute the bimodal critical buckling load $P=0.84167$, the corresponding eigenmodes $\mathbf{u}_1=(1, 3.0554, 3.0554, 1)$ and $\mathbf{u}_2=(1, 0.3888, -0.3888, -1)$, and the coefficients $v_{11\epsilon}=-10.449$ and $v_{22\epsilon}=-3.3517$. Then, we normalize the eigenvectors and calculate the coefficients

$$v_{1111} = 0.29058 \quad v_{2222} = 0.35097 \quad v_{1122} = 0.05163 \quad (65)$$

The bifurcation belongs to Type 6 in Table 1. This means that symmetric and antisymmetric solutions are supercritical and unstable while the mixed-type solution is supercritical and stable, see Fig. 2 (Case 6, $\chi=0$). Thus, we have recognized a surprising effect that a symmetric bimodal column loaded by an axial force can buckle with a stable asymmetric mode!

According to Eqs. (33)–(35), the bifurcating equilibria are given asymptotically as

$$\begin{aligned} \mathbf{q}_s &= \pm \sqrt{\epsilon}(0, 1.4056, 4.2949, 4.2949, 1.4056, 0) \\ \mathbf{q}_a &= \pm \sqrt{\epsilon}(0, 2.2583, 0.8780, -0.8780, -2.2583, 0) \\ \mathbf{q}_{m1} &= \pm \sqrt{\epsilon}(0, 2.9661, 4.3570, 2.9848, -0.5632, 0) \\ \mathbf{q}_{m2} &= \pm \sqrt{\epsilon}(0, -0.5632, 2.9848, 4.3570, 2.9661, 0) \end{aligned} \quad (66)$$

The stiffnesses of the bimodal column and corresponding non-trivial equilibrium states, divided by $\pm\sqrt{\epsilon}$, are presented in Fig. 8.

If we study unfolding of this bifurcation due to change of the stiffness δb_1 , then according to Eqs. (44) and (56), we get

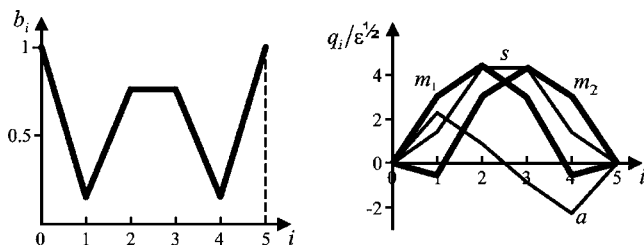


Fig. 8 Stiffnesses and buckling modes of the elastic column ($b_0=1, b_1=0.15, b_2=0.76465$)

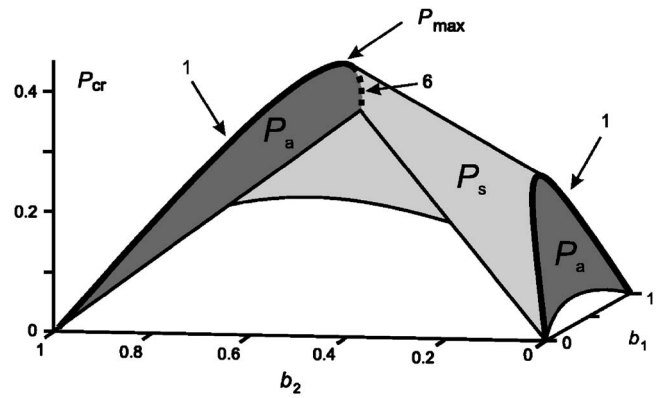


Fig. 9 Critical load depending on stiffness parameters

$$\chi = \bar{v}_{11\epsilon} - \bar{v}_{22\epsilon} = 2\delta b_1((u_{12} - 2u_{11})^2 - (u_{22} - 2u_{21})^2) = 1.3358\delta b_1 \quad (67)$$

Thus, if we decrease the stiffness $\delta b_1 < 0$, then the antisymmetric form of instability becomes critical, and the corresponding unfolding is shown in Fig. 2, Case 6 ($\chi > 0$). If the stiffness is increased $\delta b_1 > 0$, then the symmetric form of instability becomes critical with the unfolding is shown in Fig. 2, Case 6 ($\chi < 0$).

8.1 Bimodal Optimal Column. Let us consider columns under the condition

$$b_0 + b_1 + b_2 = \text{const} \quad (68)$$

This equality resembles the fixed total volume constraint for a continuous column. Figure 9 shows dependence of the critical load on b_1 and b_2 with b_0 given by Eq. (68) with $\text{const}=1$; due to homogeneity of Eqs. (58) and (59), the plot for any const can be obtained from Fig. 9 by scaling. Columns with bimodal critical loads correspond to edges, where the surfaces P_a and P_s intersect. The analysis (similar to the one given above) shows that the left bimodal arch contains two big parts corresponding to Bifurcations 1 and 6 according to the classification in Fig. 2; between these two parts, there is a tiny part corresponding to Bifurcation 11 (not shown in the figure). The right arch corresponds to the bifurcation of Type 1. The maximal critical load $P_{\max}=0.4465$ is attained at the bimodal point $b_0=0.4717, b_1=0.1021, b_2=0.4263$ with the bifurcation of Type 1. We note that the postbuckling behavior of the articulated optimal column is similar to that of the continuous optimal column [12]. Clearly, a bimodal optimal solution is the generic phenomenon. In different optimization problems, the bimodal solutions were found [6,10–14].

9 Conclusion

For general potential systems with symmetry having multiple degrees of freedom, we studied bifurcations at bimodal branching points. Formulas describing postbuckling paths and conditions for their stability are derived. We presented the full list of possible cases for postbuckling paths and their stability depending on three system coefficients v_{1111} , v_{2222} , and v_{1122} . In order to calculate these coefficients, we need to know the derivatives of the potential energy and eigenvectors of the linearized problem taken at the bifurcation point. Then, we studied unfolding of bimodal branching points due to change of system parameters. Classification and analysis of all possible cases given in Table 1 with Figs. 2 and 3 constitute the central result of the paper. It is remarkable that all the formulas derived in this paper are given in terms of the original potential energy.

The presented theory is illustrated by a mechanical example on stability and postbuckling behavior of a bimodal articulated elastic column having four degrees of freedom and depending on three stiffnesses at the hinges (problem parameters). It is shown that

bimodal critical points are described by smooth surfaces in parameter space. Numerical results are presented illustrating influence of problem parameters on postbuckling paths, their stability and unfolding. Two different kinds of postbuckling behavior are demonstrated. One is associated with stable symmetric and antisymmetric modes, and unstable mixed-type modes, while the second one is associated with stable mixed-type modes and unstable symmetric and antisymmetric modes. Thus, a surprising phenomenon that a symmetric bimodal column loaded by an axial force can buckle with a stable asymmetric mode is recognized.

A considered example with the constrained sum of the stiffnesses of the articulated column shows that the maximum critical load (optimal design) is attained at the bimodal point with the postbuckling behavior similar to that of the continuous optimal column.

We remark that we have studied bimodal bifurcations of the stable stability path of the potential system with increasing load parameter. Certainly, the case when the trivial equilibrium is unstable on both sides of the bifurcation point could also be useful. It would be interesting to recognize more physical systems and phenomena related to bimodal bifurcations.

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Development of Component-Level Damage Evolution Models for Mechanical Prognosis

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This paper presents component-level empirical damage evolution regression models based on loads and damage information that do for mechanical damage prediction what the Paris law does for predicting crack growth under fatigue loading. Namely, these regression models combine information about the current damage state and internal system loads to predict the progress of damage to failure. One of the drawbacks of Paris-like crack evolution laws is that localized information about the loading (stress) and damage (crack length) is required. In structural health monitoring applications, it is not feasible to instrument every potential crack initiation region to collect this localized information. The component-level damage evolution regression models developed here only require global measurements that quantify the damage and loading at the level of the component rather than at the site of damage. This paper develops damage evolution regression models for an automotive sway bar link undergoing axial fatigue loading with two different damage mechanisms at a weldment and at an electrical discharge machining notch. Restoring force diagrams are used to calculate the load indicators as damage progresses and transmissibility functions are used to calculate the damage indicator during tests to failure. A component-level load intensity factor (ΔK) is calculated during these tests so that the rate of damage accumulation can be used to predict the growth of damage and ultimate failure. [DOI: 10.1115/1.2793137]

Keywords: damage identification, prognosis, load identification, restoring forces, empirical damage evolution models

1 Introduction

In order to predict the life of structural components, damage evolution laws are required. Phenomenological laws such as Paris law (Paris et al. [1,2]) for cracks are a natural choice for ferrous materials because this law has many forms to suit different conditions at the crack tip under fatigue loading. One of the drawbacks of Paris-like crack evolution laws is that localized information about the loading (stress) and damage (crack length) is required. In structural health monitoring applications, it is not feasible to instrument every potential crack initiation region to collect this localized information. A component-level damage evolution regression model would be useful if the model required only global measurements that quantify the damage and loading at the level of the component rather than at the site of damage. Many structural health monitoring algorithms could then be applied to extract the damage indicator and the load indicator. This paper develops a component-level damage evolution regression model and then applies it to predict the growth of damage in a ferrous sway bar link at a weldment and within an electrical discharge machining (EDM) notch.

Damage causes changes in the internal loading of components in mechanical systems, and it is important to track such changes in loading to predict the growth of damage. In this paper, restoring force curves are used to characterize the internal loading on components of vehicle suspension systems and to track changes in the loads with damage for predictive purposes. Restoring forces were introduced by Masri et al. [3–6] for nonparametric identification of nonlinear systems. The nonlinear characteristics (restoring

force maps) were expressed in terms of orthogonal functions for system identification. Surace et al. [7] used restoring forces to characterize the dynamic properties of automotive dampers. Haroon et al. [8,9] utilized restoring forces for nonlinear characterization and system identification of mechanical systems in the absence of an input measurement. In this work, the authors noted that only response acceleration measurements are needed to generate restoring force curves, which is an essential property for analyzing operating data in structural components. Haroon and Adams [10] showed that restoring forces can be used to characterize and quantify the changes in the internal component loads with the onset and progression of damage. This feature has been utilized in this paper to quantify internal component loads. This information is then combined with a damage index based on transmissibility functions to develop damage evolution models for mechanical damage prognosis. Zimmerman et al. [11], Schultz et al. [12], James et al. [13], and Zhang et al. [14] showed that transmissibility measurements are useful for detecting and locating damage. Worden [15] and Johnson and Adams [16] developed transmissibility-based damage detection indices and used them to successfully detect damage in numerous applications.

In the following sections, the framework for the damage growth regression models is presented and the models are developed for different crack damage mechanisms in automotive sway bar links.

2 Framework

2.1 Load Identification

2.1.1 Restoring Forces. The restoring force is an internal force that opposes the motion of an inertial element within a system. The left hand side of Newton's second law for a body with constant mass m and acceleration vector $\mathbf{a}$ contains the stiffness and damping restoring forces, $\Sigma \mathbf{F} = m\mathbf{a}$.

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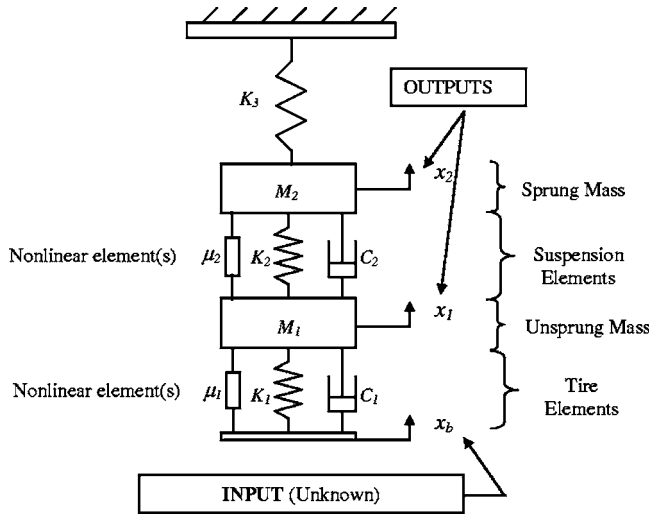


Fig. 1 Two DOF quarter car model

Restoring force techniques only require that the output accelerations of a component be measured; therefore, restoring forces can be identified for systems of components in practical applications without using sophisticated sensing arrays. Consider the two degree-of-freedom quarter car model shown in Fig. 1. The equations of motion for the sprung mass, M_2 , can be rearranged to give the following expression for the restoring force in the suspension:

$$M_2 \ddot{x}_2 = -C_2(\dot{x}_2 - \dot{x}_1) - K_2(x_2 - x_1) - K_3 x_2 + N_1[x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t)] \quad (1)$$

where $x_k(t)$ are the displacements of the unsprung and sprung masses, M_k , C_2 is the suspension viscous damping coefficient, K_k are the stiffness in the suspension and vehicle body, and $N_1[x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t)]$ denotes the nonlinear forces in the suspension.

The plot between the acceleration of the sprung mass and the relative velocity between the sprung mass and the unsprung mass allows the damping restoring force, or the component force dependent on the velocity, in the suspension to be characterized. Similarly, the plot between the acceleration of the sprung mass and the relative displacement between the sprung mass and the unsprung mass allows the stiffness restoring force, or the component force dependent on the displacement, in the suspension to be characterized. (Note that the restoring forces obtained by plotting acceleration versus displacement or velocity are scaled by the mass, e.g., M_2 in Eq. (1)). The frequency characteristics of the internal forces can be observed by plotting the restoring forces at different frequencies. Restoring force plots can be generated for any two response locations by using similar two degree-of-freedom models.

Restoring forces provide two pieces of information about internal system loads:

- (1) The area entrained by the restoring force curves is determined in part by the system parameters (mass, stiffness, and damping) and is proportional to the magnitude of the internal loads.
- (2) The shape of the restoring force curves identifies the nature of the internal loads, linear or nonlinear. Nonlinear restoring forces can be identified using the measured shapes, e.g., Coulomb friction, cubic stiffness, etc.

Feature (1) can be illustrated by analyzing a single degree-of-freedom (SDOF) system (Fig. 2) in a manner similar to the phase-plane analysis described by Stites et al. [17]. These authors showed that the phase-plane plot for a forced SDOF system can

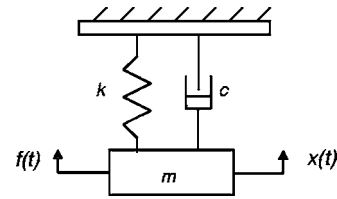


Fig. 2 SDOF linear system

be represented by an equation for an ellipse. Consider the equation of motion for the system in Fig. 2 when the system is excited by a simple harmonic forcing function

$$m\ddot{x} + c\dot{x} + kx = f(t) \quad (2)$$

where m is the mass, c is the damping, k is the stiffness, and $f(t)$ is the harmonic forcing function. The authors showed that the steady-state phase-plane plot can be written in the following form of the equation for an ellipse:

$$\frac{\dot{x}^2}{1/m} + \frac{x^2}{1/k} = R \quad (3)$$

where R is a constant that depends on the system damping, input amplitude, and input frequency. Consequently, in a linear system subjected to a constant forcing amplitude and frequency, any change in the geometry of the phase-plane plot can be attributed to changes in the system parameters (mass, stiffness, and damping).

The restoring force plots can be described in a similar manner. Like the phase-plane plot, restoring force plots are also the regression of a higher order variable on a lower order variable (acceleration on velocity or acceleration on displacement). Consider once again the SDOF system in Fig. 2. If a time derivative of Eq. (2) is taken, $\ddot{x}$ can be expressed as the time derivative of $\dot{x}$ using the chain rule as shown in the second equation below:

$$m\ddot{x} + c\dot{x} + kx = \dot{f}(t) \quad (4)$$

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx} \frac{dx}{dt} = \dot{x} \frac{d\dot{x}}{dx} \quad (5)$$

Equation (4) can be integrated over one complete cycle of motion of the mass m and arranged as shown below, where the period of oscillation is $2\pi/\omega$ and Z is a constant of integration:

$$m\dot{x}^2 + kx^2 = 2 \left[\int_{\text{cycle}} \dot{f}(t) dx - \int_{\text{cycle}} c\dot{x} dx + Z \right] \quad (6)$$

For an input harmonic force with constant amplitude F and frequency ω ,

$$f(t) = F \sin(\omega t) \quad (7)$$

the steady-state system response takes the form

$$x(t) = A \sin(\omega t) + B \cos(\omega t) \quad (8)$$

where A and B are constant amplitudes.

The integrals in Eq. (6) can be evaluated by substituting Eqs. (7) and (8) with appropriate derivatives and using the fact that $\dot{x}$ is equivalent to $(dx/dt)dt$. The results are

$$\int_0^{2\pi/\omega} \dot{f}(t) dx = \int_0^{2\pi/\omega} \dot{f}(t) \dot{x} dt = -\omega^2 F B \pi \quad (9)$$

$$\int_0^{2\pi/\omega} c\dot{x} dx = \int_0^{2\pi/\omega} c\dot{x}^2 dt = c\omega^3 (A^2 + B^2) \pi \quad (10)$$

Now, Eq. (6) simplifies to

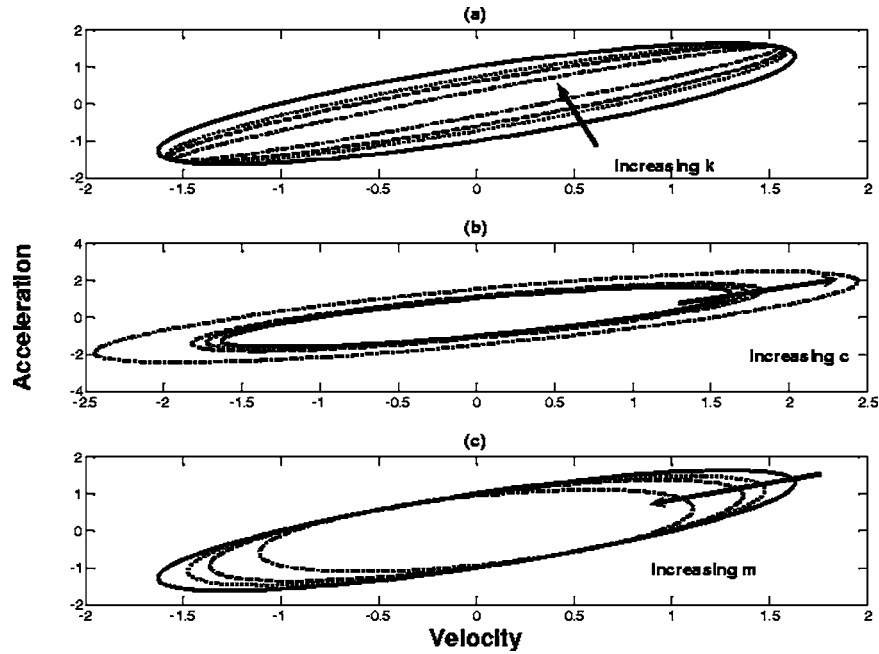


Fig. 3 Change in area of acceleration-velocity ellipse with system parameters: (a) stiffness, (b) damping, and (c) mass

$$\frac{\ddot{x}^2}{1/m} + \frac{\dot{x}^2}{1/k} = 2[-\omega^2 FB\pi + c\omega^3(A^2 + B^2)\pi + Z] = R$$

$$\frac{\ddot{x}^2}{R/m} + \frac{\dot{x}^2}{R/k} = 1 \quad (11)$$

This equation describes an ellipse. For a constant amplitude and frequency forcing function, changes in the system properties (mass, stiffness, and damping) will cause a change in the major and minor axes, and, consequently, the area of the ellipse formed by regressing $\ddot{x}$ on $\dot{x}$. Figure 3 shows the changes in the ellipse of Eq. (11) with stiffness, damping, and mass changes for a constant amplitude and frequency harmonic forcing function. This example illustrates that the restoring force plots are sensitive to changes in mechanical parameters (mass, stiffness, and damping). Because the area of the restoring force curve is proportional to the magnitude of the internal load, the area of a restoring force curve can be estimated to quantify changes in internal loads with damage.

The frequency dependent nature of restoring forces requires that inputs be narrowband in nature so that changes in the restoring force characteristics can be observed at discrete frequencies; hence, slow sine sweeps are used to generate the restoring force curves. Cafferty et al. [18] showed that the restoring forces can be generated using random excitation, but these restoring forces are corrupted by small stochastic components. Acceleration measurements are the most convenient measurements to make in experimental data analysis and can also be integrated to estimate velocity and displacement time histories; therefore, restoring force curves are especially appropriate for experimental purposes. Note, however, that the static (dc) components of the velocity and displacement time histories are lost in the integration process; consequently, certain types of nonlinear internal forces such as quadratic stiffness nonlinearities, which produce steady streaming (i.e., a dc response), may be difficult to identify.

2.2 Damage Indicator. A damage indicator based on transmissibility functions is used in this work. This damage indicator is related to fatigue crack geometry (length, depth) but is more practical to estimate than the crack parameters even in a system of components. Transmissibility functions are like frequency re-

sponse functions (FRFs), with the only difference being that transmissibility functions are frequency dependent ratios between two inputs rather than the ratio between an input and an output. Hence, transmissibility functions are functions of only the zeros, and not the poles, of the system, and therefore, contain information about localized regions of components. The transmissibility is defined as

$$T_{ij}(\omega) = \frac{X_i(\omega)}{X_j(\omega)} \quad (12)$$

where T is the transmissibility between acceleration, X , at measurement degrees of freedom (DOFs) i and j . Thus, acceleration response measurements can be used to directly assess information about the local mechanical changes in a system of components. No measurement of the input is required for computing the transmissibility, which can be applied to experimental and operational data for diagnosis and prognosis of mechanical damage. Broadband random inputs are used to generate the transmissibility functions.

A damage indicator based on the natural log of the transmissibility across the link is used to track the growth of damage (Johnson [19]):

$$DI_k = \frac{\sum_{i=a}^b |\text{Re}(\ln(T_k(\omega_i))) - \text{Re}(\ln(T_{k-1}(\omega_i)))|}{N_f} \quad (13)$$

where T_k is the transmissibility at the k th measurement, T_{k-1} is the previous measurement, $\ln$ is the natural log, ω is the frequency, i is the index that determines the range of frequencies (a to b) over which the change is summed, and N_f is the total number of frequencies. The advantage of this damage indicator is that it gives the same value of the change in the function irrespective of which measurement DOF is in the numerator.

2.3 Loads+Damage=Prognosis. Load identification is essential for prognostics (damage prediction). Information about the current status of damage is not sufficient for accurate prediction of future damage levels. Internal loads are determined by the external loads along with the system parameters. Damage causes



Fig. 4 Fatigue test setup

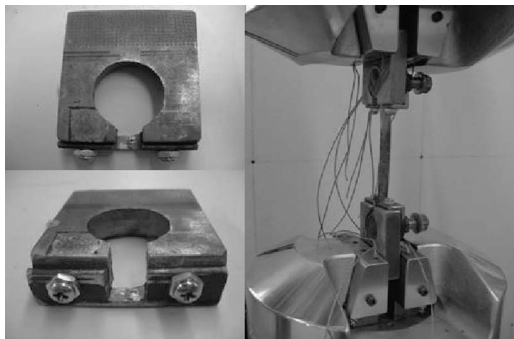


Fig. 5 Fixture for placing link under test in fatigue machine grips

changes in the parameters of the system, which consequently affects the internal load distribution. Identical sets of external loads will result in different internal loads for undamaged and damaged systems. Loads and damage estimates can be used separately for trend-based prognostics; however, if both pieces of information are used together, then empirical relationships can be defined for objective damage prognosis. These relationships can be thought of in terms of the crack power laws like the Paris law of crack growth,

$$\frac{da}{dN} = E(Y\Delta\sigma\sqrt{\pi a})^m \quad (14)$$

where a is half the crack length, N is the number of cycles, σ is the applied stress, E and m are parameters that depend on the material, environment, frequency, temperature and stress ratio, and Y is a dimensionless parameter that depends on both the specimen and crack geometry. In Eq. (14),

$$\Delta K = (Y\Delta\sigma\sqrt{\pi a}) \quad (15)$$

is the range of the stress intensity factor that is dependent in part on the crack length, local stress distribution, and component geometry.



Fig. 6 Accelerometers attached to the ends of sway bar link



Fig. 7 Weld location on sway bar link



Fig. 8 Initial circumferential crack in sway bar link under cyclic loading

This law is an empirical law that indicates that the growth of the crack is dependent on the current state of the crack and the current load distribution. In other words, the damage and load together determine how the damage grows. Similar relations can be determined for mechanical system prognosis. Information about the current status of damage (Eq. (13)) and the change in loads accompanying damage (restoring force areas) can be used together to develop empirical damage evolution models for damage prognosis. This relation to predict damage growth can be expressed as

$$\frac{d(\Delta T)}{dN} = C|\Delta T|^m \times |\Delta(RF_{\text{area}})|^n \quad (16)$$

In this equation, ΔT represents the change in the transmissibility across the link estimated by the damage indicator DI_k , ΔRF_{area}

represents the changing load (change in restoring force area), and C , m , and n are some system constants that depend on material properties, material geometry, boundary conditions, loading, and damage mechanism. Developing such empirical relations for different mechanical damage mechanisms will improve the process of predicting damage growth and ultimately failure.

3 Experimental Setup

An MTS[®] 810 material test system was used to perform tension-tension fatigue tests on the front sway bar link of an Isuzu impulse. A picture of the experimental setup is shown in Fig. 4. A fixture was designed to hold the link in the grips of the fatigue machine (Fig. 5). Two single axis accelerometers with a nominal

Table 1 Parameters for sine sweep and random inputs; Link 1

Input type	Amplitude (mm)	Frequency content (Hz)	Sampling frequency (Hz)	Test length (s)
Sine sweep	0.1	0–15	2000	100
Random	0.08 rms	0–30	2000	100

sensitivity of 1000 mV/g were attached to the two ends of the link to measure axial acceleration (Fig. 6). An initial tensile load was applied to the link and then the link was subjected to a cyclic

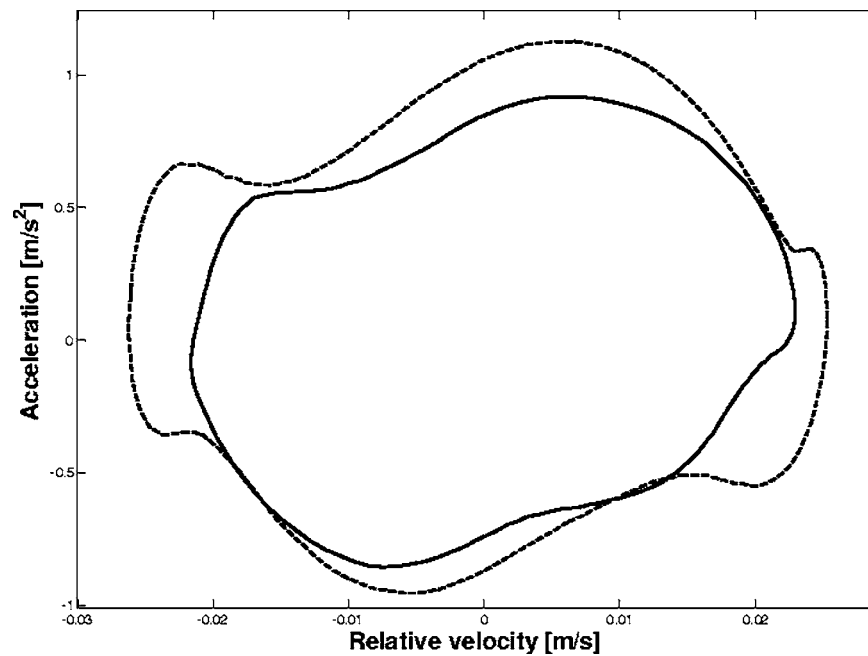


Fig. 9 Change in restoring force with the appearance of the initial circumferential crack in link: undamaged (—) and initial crack (---)

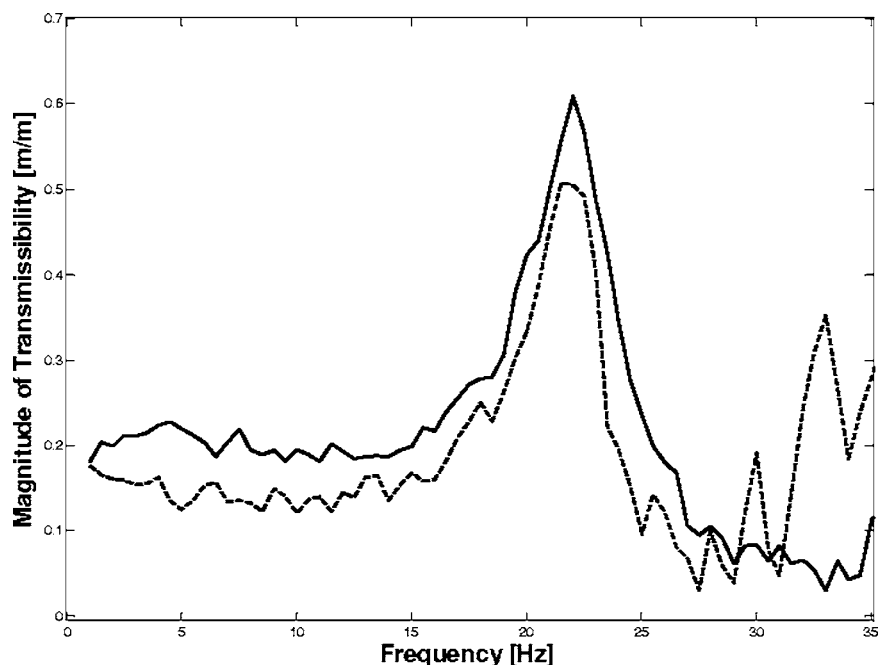


Fig. 10 Change in transmissibility with the appearance of the initial circumferential crack in link: undamaged (—) and initial crack (---)

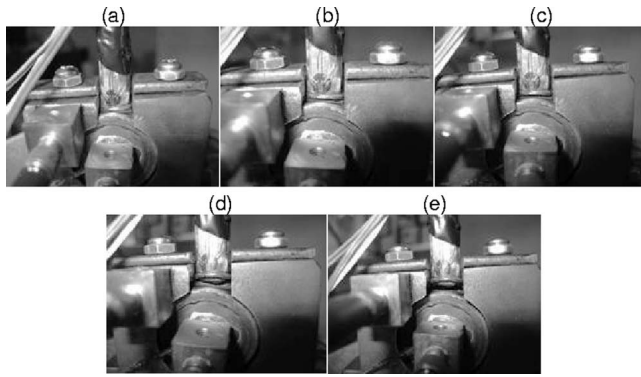


Fig. 11 Appearance and progressive growth of circumferential crack to failure in sway bar link under tension-tension fatigue loading: (a)–(e)



Fig. 12 Fatigue failure of sway bar link under constant amplitude cyclic loading

load at 5 Hz, which can be thought of as a durability cycle. Sine sweep and random input characterization tests were run intermittently to estimate and track the load and damage indicators.

4 Empirical Damage Growth Models

4.1 Fatigue Crack at Weld. The front sway bar link was subjected to an initial tensile load of 6000 N. Sine sweep and random inputs were used to characterize the link and the associated test parameters are given in Table 1.

It was observed that as expected the weakest parts of the link are the weld locations where the main rod is attached to the ends (Fig. 7). The cyclic loading led to the appearance of a circumferential crack at the lower weld location (input side). The load and damage indicators were estimated throughout the initiation and propagation of the crack to failure and the empirical damage propagation model (Eq. (17)) was developed.

The link was subjected to blocks of 2500 cycles of a 0.15 mm amplitude cyclic load at a frequency of 5 Hz with sine sweep and random tests conducted between the cyclic loading blocks. The cyclic load corresponded to a module load range of about 6000 N. The loading blocks of testing were continued as the circumferential crack initiated and propagated to failure.

After 5000 cycles of cyclic loading, a visible circumferential crack appeared at the lower weld (Fig. 8). The changes in the velocity-dependent restoring force (14.5 Hz) and the transmissibility are shown in Figs. 9 and 10. (The velocity-dependent internal load shows primarily hysteresis.) There is a clear increase in damping restoring force area (or load) and a decrease in transmissibility, especially between 5 Hz and 25 Hz. The damping restoring force curve area was estimated to track the change in load and the damage index based on the natural log of the transmissibility (5–15 Hz) (Eq. (13)) across the link was used to track the growth of damage.

The block cyclic loading tests were continued until complete failure and the load and damage indicators were estimated throughout the tests. Figure 11 shows the progressive growth of the circumferential crack to failure in pictures, and Fig. 12 shows

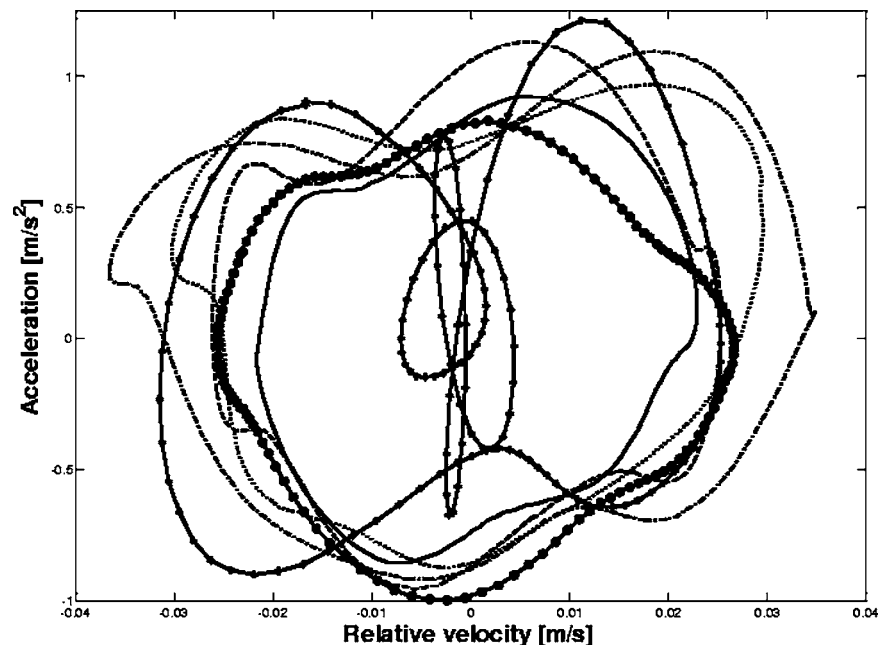


Fig. 13 Change in velocity restoring force with appearance and progressive growth of circumferential crack to failure in sway bar link under tension-tension fatigue loading: undamaged (—), initial crack (---), progression 1 (·····), progression 2 (-.-.-), just before failure (-+--+), and after failure (-°-°-)

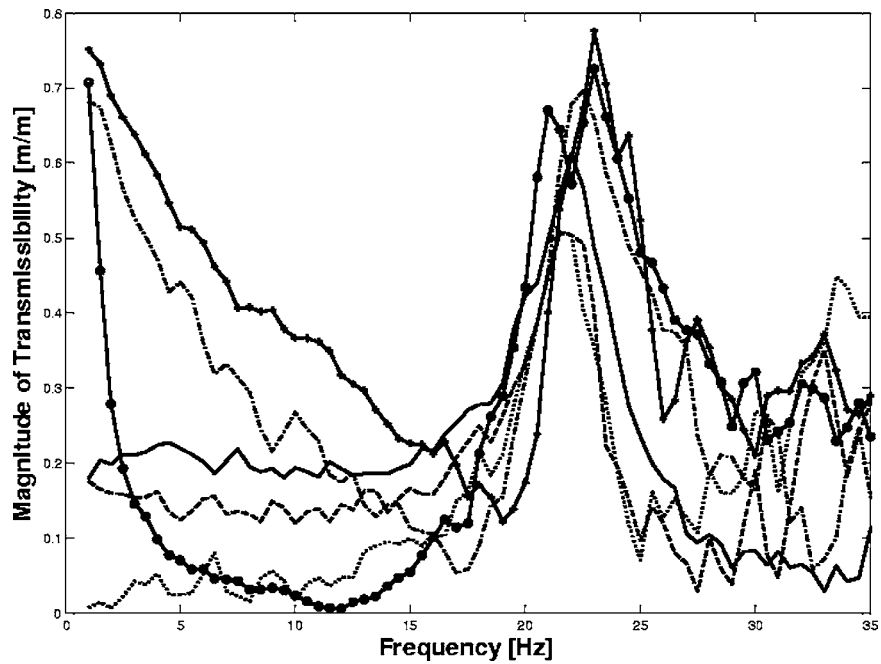


Fig. 14 Change in transmissibility with appearance and progressive growth of circumferential crack to failure in sway bar link under tension-tension fatigue loading: undamaged (—), initial crack (---), progression 1 (·····), progression 2 (-.-.-), just before failure (-+--+), and after failure (°°°°)

the fracture surfaces of the cracked portions of the link that are similar to typical fatigue failures in steel components. Figures 13 and 14 show the progressive changes in the restoring forces (14.5 Hz) and the transmissibility across the link as the crack grows. The velocity restoring force shows an increasing area with growth of the crack and also shows an indication of the approaching failure. Immediately before complete failure, the restoring force becomes multiply connected (Fig. 13 (-+--+)). The

reason for this can be understood by looking at the acceleration response of the lower (input) end of the link as the restoring force curve changes to multiply connected (Fig. 15). As Fig. 15 shows, the frequency content of the acceleration response increases as the crack grows and is highest just before failure. As the crack grows, there is increasing contact between the cracked surfaces of the link that causes rattling. This rattling results in the increased frequency content seen in Fig. 15. It is also interesting to note that

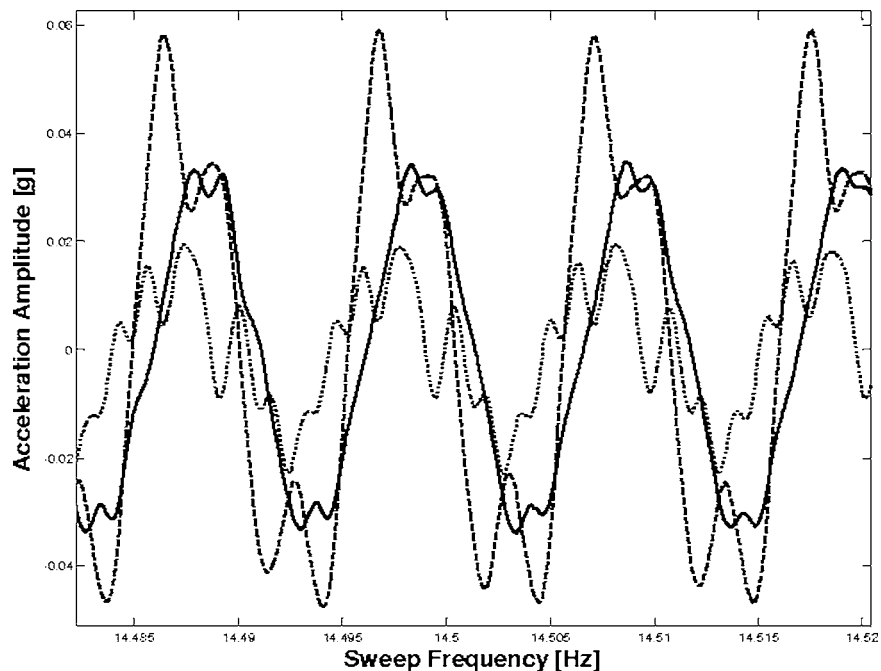


Fig. 15 Increase in the frequency content of the link response as the fatigue crack grows toward failure: undamaged link (—), partially grown crack (---), and just before failure (·····)

Table 2 Changing velocity restoring force areas and transmissibility index with the appearance and progress of circumferential crack in sway bar link

	Undamaged	Crack not visible 2500 cycles	Crack visible 5000 cycles	Progression 1 10,000 cycles	Progression 2 12,500 cycles	Progression 3 15,000 cycles	Just before failure 17,500 cycles
Restoring force area (m ² /s ³)	0.054	0.0605	0.0692	0.0755	0.0776	0.092	0.0840
DI _k	—	0.0608	0.2844	0.2650	0.8613	1.6123	0.4549

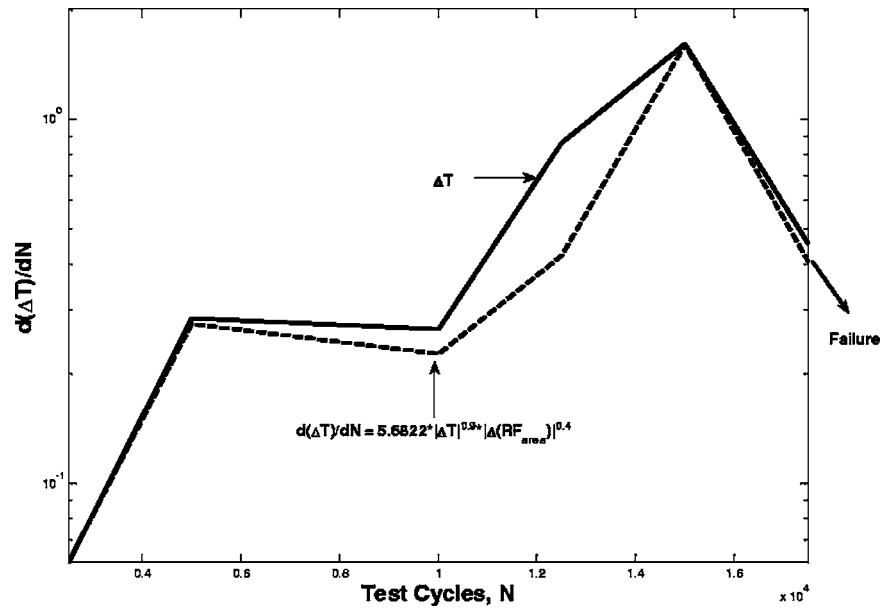


Fig. 16 Empirical regression model relating estimated change in transmissibility to change in restoring force area for experimental circumferential crack damage in sway bar link: damage indicator (—) and damage model (---)

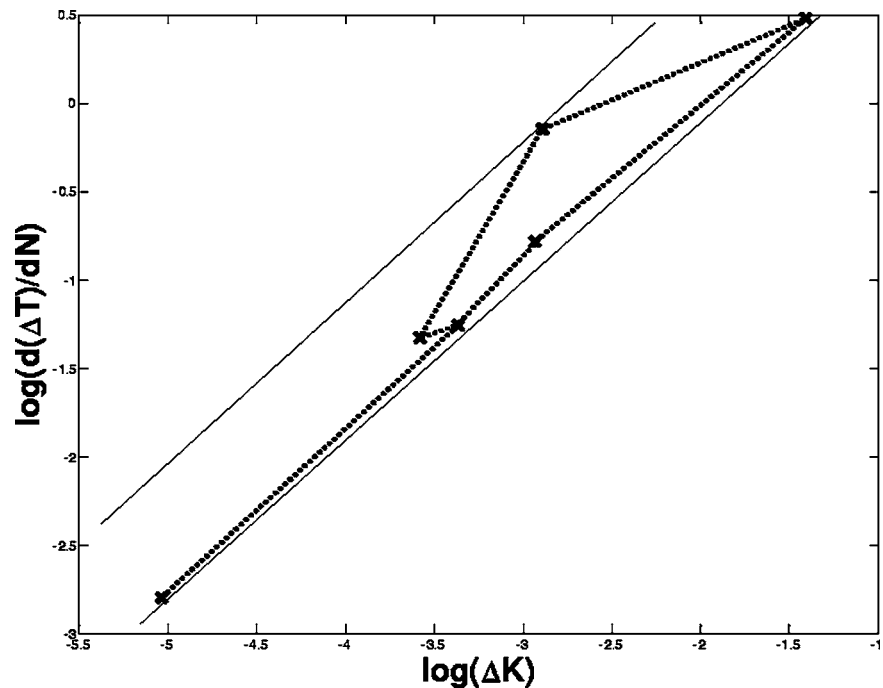


Fig. 17 Correlation of damage growth rate (change in transmissibility) with applied load intensity factor demonstrating analogy of developed damage growth model to Paris crack law; experimental circumferential fatigue crack damage in sway bar link

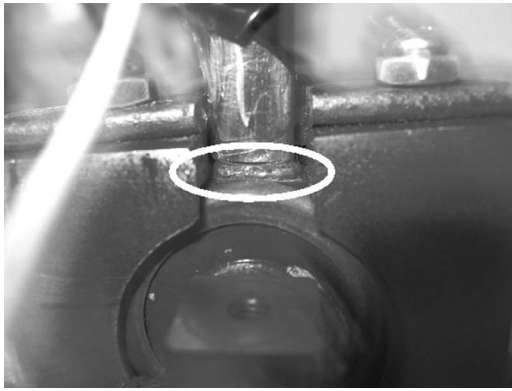


Fig. 18 EDM crack in front sway bar link

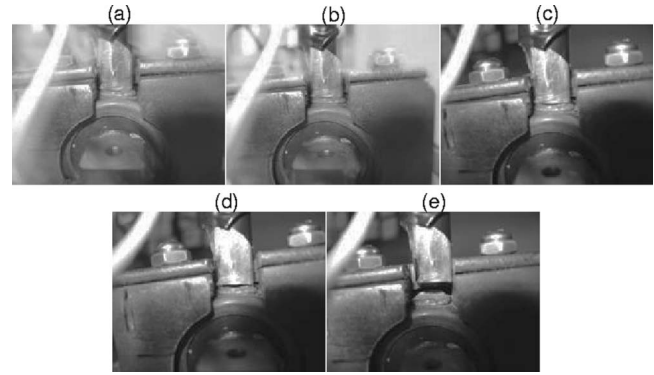


Fig. 19 Progressive growth of EDM crack to failure in sway bar link under tension-tension fatigue loading: (a)–(e)

after complete failure, the restoring force seems to become simply connected again. There is actually no direct transmission of motion through the link, but rather motion to the top accelerometer is transmitted through the structure of the fatigue test system. This transition of the restoring force is an indicator of the approach and occurrence of failure of the link. Table 2 lists the areas of the velocity restoring forces with the progress of the crack along with the damage index in Eq. (13) for the frequency range from 5 Hz to 15 Hz.

The two indicators in Table 2 were used to develop the empirical regression model (Fig. 16) for predicting the rate of change of transmissibility with damage:

$$\frac{d(\Delta T)}{dN} = C|\Delta T|^m \times |\Delta(RF_{\text{area}})|^n \quad (17)$$

where $m=0.9$, $n=0.4$, and $C=5.6822$. The empirical model (right hand side of Eq. (17)) should produce the rate of change of the damage indicator, $\Delta T=DI_k$, in this case, the real part of the natural log of the transmissibility measurement. Figure 16 shows that the empirical model (dashed line) follows the same trend as the damage indicator (solid line); hence, this empirical damage growth

model predicts the growth of damage and the associated damage indicator. It should be noted that the x axis of Fig. 16 starts at 2500 cycles. The reason for this is that Eq. (17) gives the rate of change of transmissibility with number of cycles and the second measurement was made after 2500 cycles of load.

It was stated earlier that the empirical damage growth models (Eq. (16)) are similar to the crack growth laws, e.g., Paris law (Eq. (14)). Paris et al. [1,2] developed the power law in Eq. (14) based on the common observation of the linear relation between the plots of crack growth rate against the range of stress intensity factor on log-log scales for a number of metal alloys. Along similar lines, the developed empirical damage growth model (Eq. (17)) can be written as

$$\frac{d(\Delta T)}{dN} = C[|\Delta T| \times |\Delta(RF_{\text{area}})|^{n/m}]^m = C\Delta K^m \quad (18)$$

where

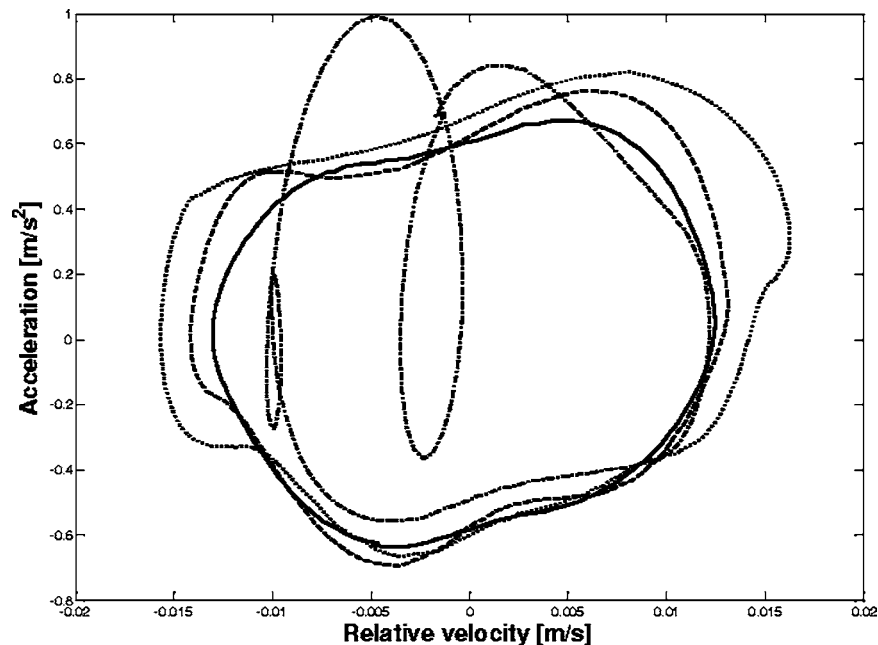


Fig. 20 Change in velocity restoring force with progressive growth of EDM crack to failure in sway bar link under tension-tension fatigue loading: initial EDM crack (—), progression 1 (---), progression 2 (· · · ·), and just before failure (-.-.-)

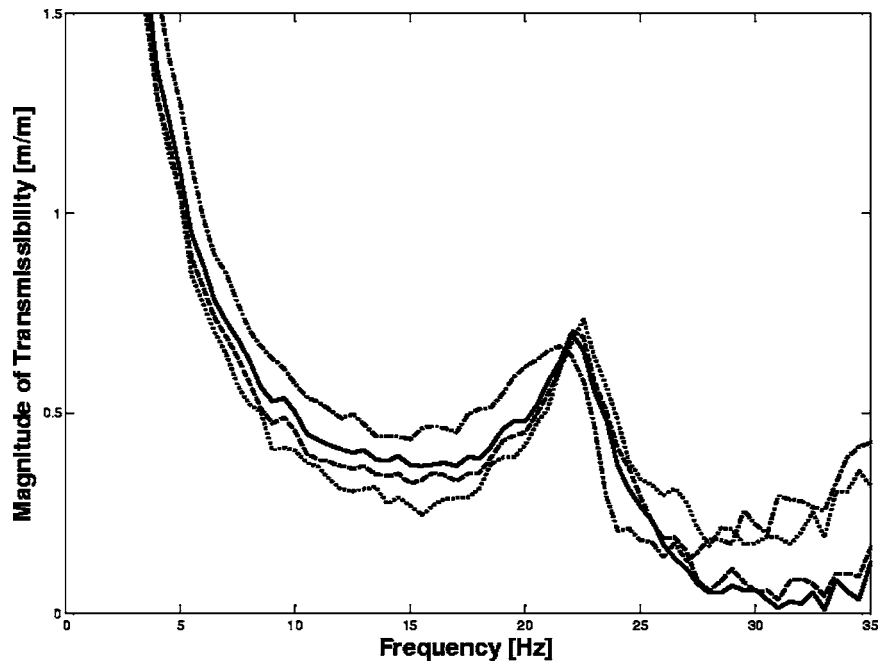


Fig. 21 Change in transmissibility with progressive growth of EDM crack to failure in sway bar link under tension-tension fatigue loading: initial EDM crack (—), progression 1 (---), progression 2 (····), and just before failure (-.-.-)

$$\Delta K = [|\Delta T| \times |\Delta(RF_{area})|^{n/m}] \quad (19)$$

can be thought of as the load intensity factor, which is a function of both the load and damage states. Taking the log of Eq. (18) gives

$$\log\left(\frac{d(\Delta T)}{dN}\right) = m \log(\Delta K) + \log C \quad (20)$$

Equation (20) implies that the plot between the change in the damage indicator and the load intensity factor on log-log scales should be a straight line. Figure 17 shows this plot for the sway

bar link result presented earlier. There is scatter in the data that is expected due to the well-known variability in fatigue behavior (Grandt [20]), but the general trend is linear. This result simply shows that the empirical damage growth models developed in this paper are analogous to crack growth power laws with an important difference: Only global measurements of load and damage at the component level are needed to generate the models in Eq. (16).

4.2 Electrical Discharge Machining Notch. EDM was used to create a notch in the front sway bar link located toward the

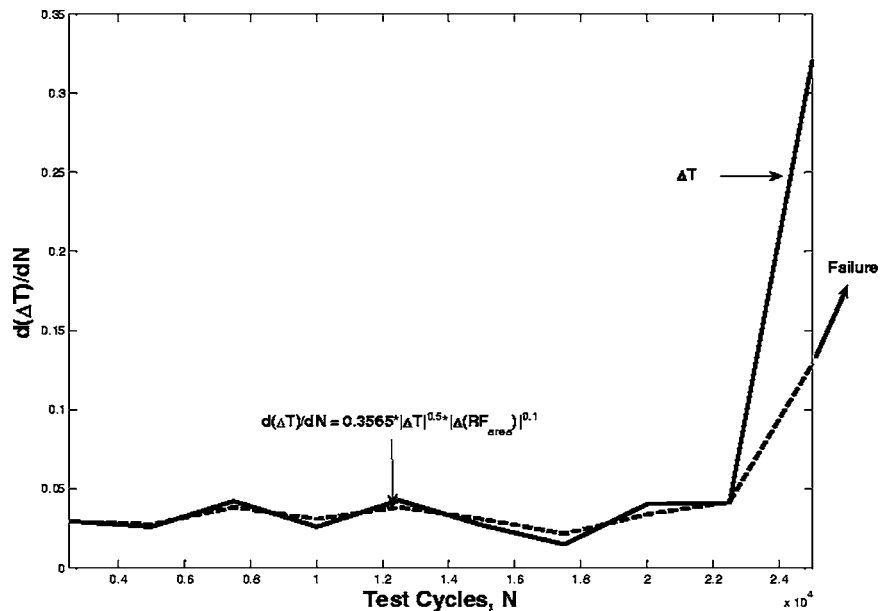


Fig. 22 Empirical regression model relating estimated change in transmissibility to change in restoring force area for experimental EDM crack damage in sway bar link: damage indicator (—) and damage model (---)

Table 3 Parameters for sine sweep and random inputs; EDM notch link

Input type	Amplitude (mm)	Frequency content (Hz)	Sampling frequency (Hz)	Test length (s)
Sine sweep	0.05	0–15	2000	100
Random	0.05 rms	0–30	2000	100

center of the bar just inside of the weld at the lower end. The notch was 1/5000th of an inch deep and 3/1000th of an inch wide (Fig. 18). This link was fatigue tested by subjecting it to an initial tensile load of about 7000 N. Sine sweep and random inputs were used to characterize the growth of damage in the link and the associated test parameters are given in Table 3.

The link was subjected to blocks of 2500 cycles of a 0.15 mm amplitude cyclic load at a frequency of 5 Hz with sine sweep and random tests conducted between the cyclic loading blocks. The cyclic load corresponded to a module load range of about 7500 N. The loading blocks were continued during the initiation and growth of the crack in the notch to failure. Figure 19 shows the progressive growth of the crack to failure in pictures. Figures 20 and 21 show the progressive changes in the restoring forces and the transmissibility across the link as the crack grows. The change in the velocity restoring force area with the growth of the crack is similar to the first case of the circumferential fatigue crack. Table 4 lists the changing areas of the velocity restoring forces with the progressive growth of the crack along with the damage index in Eq. (13) for the frequency range of 5–15 Hz.

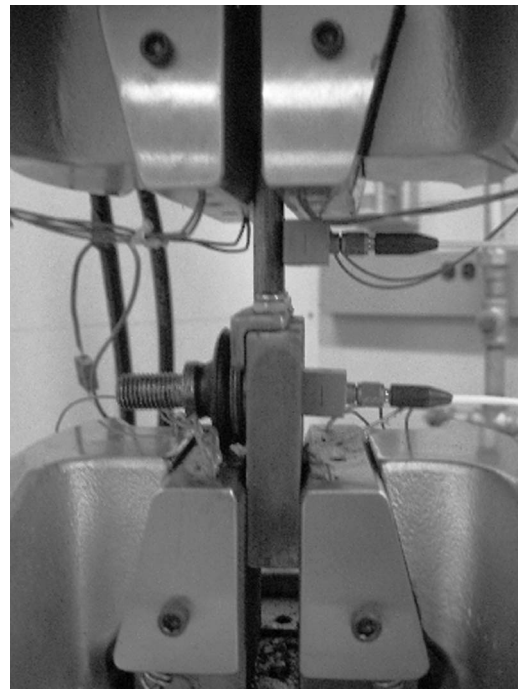
The two indicators in Table 4 were used to develop an empirical regression model (Fig. 22) for predicting the rate of change of transmissibility with loading and damage:

$$\frac{d(\Delta T)}{dN} = C|\Delta T|^m \times |\Delta(RF_{\text{area}})|^n \quad (21)$$

where $m=0.5$, $n=0.1$, and $C=0.3565$. As with the previous cases, the empirical model (dashed line) follows the same trend as the damage indicator (solid line in Fig. 22). The plot between the change in the damage indicator and the load intensity factor on log-log scales also showed a linear relationship in this case.

There are differences between the models and number of cycles to failure for the two crack mechanisms. Some of the causes for the differences in the models and failure time are given below:

- (1) The initial tensile loads were different—6000 N for the weld crack case and 7000 N for the EDM crack case.
- (2) The load ranges were different—6000 N for the weld crack case and 7500 N for the EDM crack case.
- (3) The crack mechanism was different in the case of the EDM notched link.
 - (i) The crack growth was governed primarily by the material properties of the link itself rather than the weld

**Fig. 23 Fatigue testing of opposite weld of failed sway bar link**

properties, as was the case with the first link. The greater fatigue strength of the link material was the reason for the significantly greater number of cycles until failure, even with a larger cyclic load range.

- (ii) The load redistribution (growth in restoring force area) was also significantly less in the case of the EDM notch damage (Fig. 20), as can be seen in Table 4. This difference affects the constant n , which determines the effect of load redistribution on the growth of damage.

4.3 Validation of Developed Empirical Regression Models.

Crack growth prediction models have been developed for two different crack mechanisms. In order to demonstrate that the models actually predict the growth of damage (in this case, the rate of change of transmissibility with damage), multiple tests were run for the same material properties, geometries, loading, and crack mechanisms. If the crack growth models are the same for such tests, then it can be stated with confidence that such empirical relations can be used for damage prognosis.

A limited number of sway bar links were available for running the fatigue tests. Because the fatigue failure only occurred at the lower (input) weld of the links, the weld on the other side was undamaged. Consequently, the link was flipped, gripped at the top by the broken end (Fig. 23), and then fatigue tested to reuse each link available for testing. One accelerometer was placed on the

Table 4 Changing velocity restoring force areas and transmissibility index with the growth of EDM crack in sway bar link

	Initial EDM crack	After 2500 cycles	After 5000 cycles	After 7500 cycles	After 10,000 cycles	After 12,500 cycles	After 15,000 cycles	After 17,500 cycles	After 20,000 cycles	After 22,500 cycles	After 25,000 cycles—just before failure
Restoring force area (m^2/s^3)	0.0251	0.0258	0.0264	0.0280	0.030	0.0316	0.0333	0.0343	0.0349	0.0391	0.0278
DI_k	—	0.0297	0.0263	0.0421	0.0263	0.0426	0.0273	0.0147	0.0405	0.0412	0.3216

Table 5 Parameters for sine sweep and random input; sway bar link with one failed end

Input type	Amplitude (mm)	Frequency content (Hz)	Sampling frequency (Hz)	Test length (s)
Sine sweep	0.05	0–15	2000	100
Random	0.05 rms	0–30	2000	100

lower end and the other was placed on the central rod, as shown in Fig. 23. Three such front sway bar links were tested by subjecting them to an initial tensile load of about 5500 N. Blocks of 2500 cycles of a 0.10 mm amplitude cyclic load at a frequency of 5 Hz were used to fatigue test the links. This displacement corresponded to a cyclic load range of about 8000 N. Sine sweep and random tests (Table 5) were conducted between the cyclic loading.

Table 6 lists the changing areas of the velocity restoring forces along with the damage index in Eq. (13) for the frequency range of 5–15 Hz. It should be noted that unlike the previous cases, the velocity restoring force area actually decreases as the damage progresses. The boundary condition in this case is different because one end is fixed in the grips of the test system, whereas it was attached via a fixture in the previous cases allowing some motion of the top end. In addition, the dynamics of the top weld

are also missing in the present case.

The two indicators in Table 6 were used to develop an empirical regression model (Fig. 24) for predicting the rate of change of transmissibility with damage:

$$\frac{d(\Delta T)}{dN} = C|\Delta T|^m \times |\Delta(RF_{\text{area}})|^n \quad (22)$$

where $m=1.1$, $n=0.02$, and $C=1.15$. The empirical model (dashed line) follows the same trend as the damage indicator (solid line in Fig. 24). The relationship between damage growth rate and load intensity factor is again linear (Fig. 25). It should be noted that the parameters in Eq. (22) are different from those in Eq. (17) as the dynamics of the two links are different because the upper weld is missing in the latter link and it is gripped at the rod rather than at the end as in the earlier tests.

The tests were repeated for the other two links and the damage growth model of Eq. (22) was used to predict the change in the damage indicator based on transmissibility (Eq. (13)) with the appearance and growth to failure of a fatigue crack in the weld. Figures 26 and 27 show the results. It is clear that for the same component, material, boundary conditions, loading, and damage mechanism, there exists an empirical relationship involving the load and damage indicators that predicts the growth of damage to failure. The only errors in the prognoses in Figs. 26 and 27 are offsets, which can be attributed to the variability in the placement of the links in the test system, variability in the loading, and the fact that all three links had different usage levels; the first two

Table 6 Changing velocity restoring force areas and transmissibility index with the growth of fatigue crack in the weld location of the front sway bar link with one failed end

	Undamaged	After 2500 cycles	After 5000 cycles	After 7500 cycles	After 10,000 cycles	After 12,500 cycles—just before failure
Restoring force area (mm^3/s^2)	0.0136	0.0137	0.0125	0.0107	0.0093	0.0045
DI_k	—	0.864	1.1977	1.0748	0.0768	0.7923

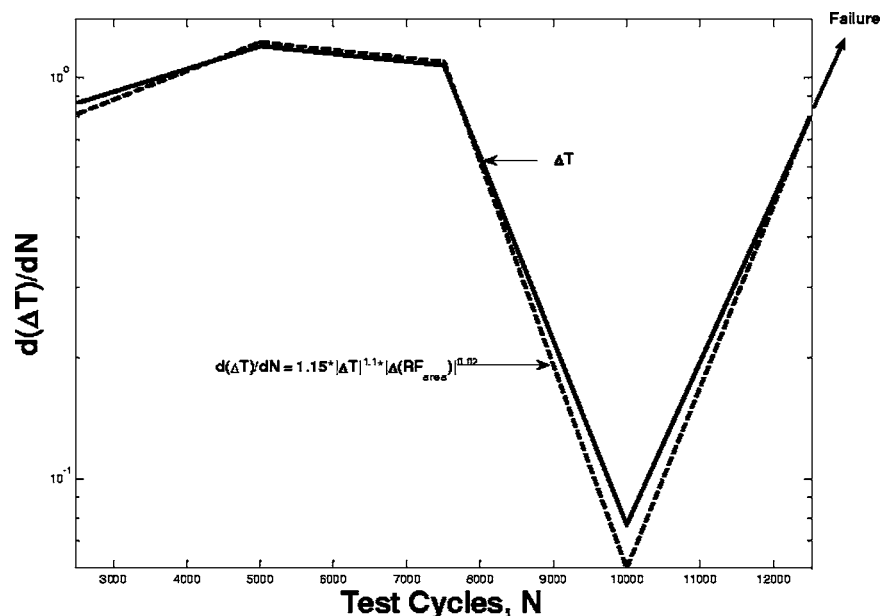


Fig. 24 Empirical regression model relating estimated change in transmissibility to change in restoring force area for experimental fatigue crack damage in sway bar link with one failed end: damage indicator (—) and damage model (---)

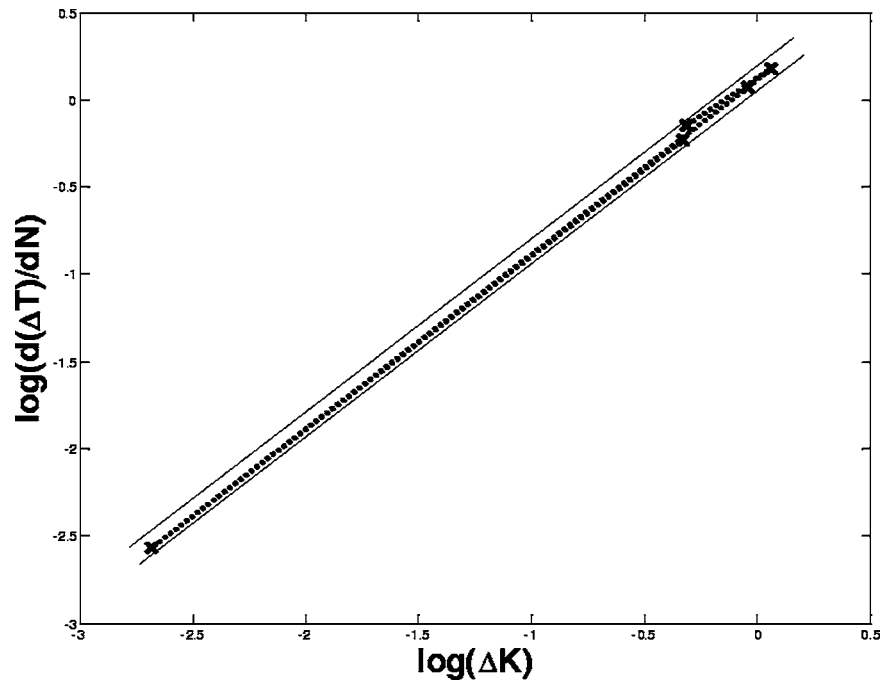


Fig. 25 Correlation of damage growth rate (change in transmissibility) with applied load intensity factor demonstrating analogy of developed damage growth model to Paris crack law; experimental circumferential crack damage in sway bar link with one failed end

links were 14 years old and had been extensively tested in full-vehicle tests and the last link was new and had never been tested prior to the fatigue testing. This difference in usage is also the primary reason that the damage indicator (or damage) grows differently for the three links, but the same empirical model predicts the growth of damage once it has initiated. It should be noted that the small value of n suggests that the load redistribution has relatively little effect on the growth of damage in this case.

Such empirical models can be developed for different mechanical fatigue damage mechanisms, such as bushing degradation,

loose bolts, etc. This extension of the modeling technique is possible because the empirical damage growth models presented in this paper are based on global measurements at the component level and estimates of changes in loads and damage indicators due to damage, and as such are not limited to a particular damage mechanism unlike the various crack laws.

5 Conclusions

It was shown that mechanical damage growth prognosis can be achieved by combining loads (restoring force areas) and damage

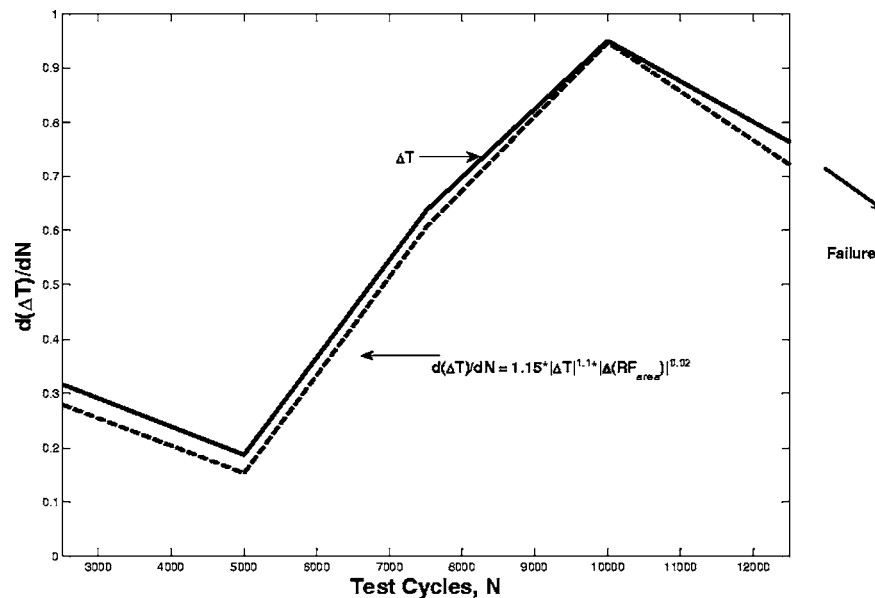


Fig. 26 Empirical regression model in Eq. (22) predicting growth of experimental fatigue crack damage in second sway bar link with one failed end: damage indicator (—) and damage model (---)

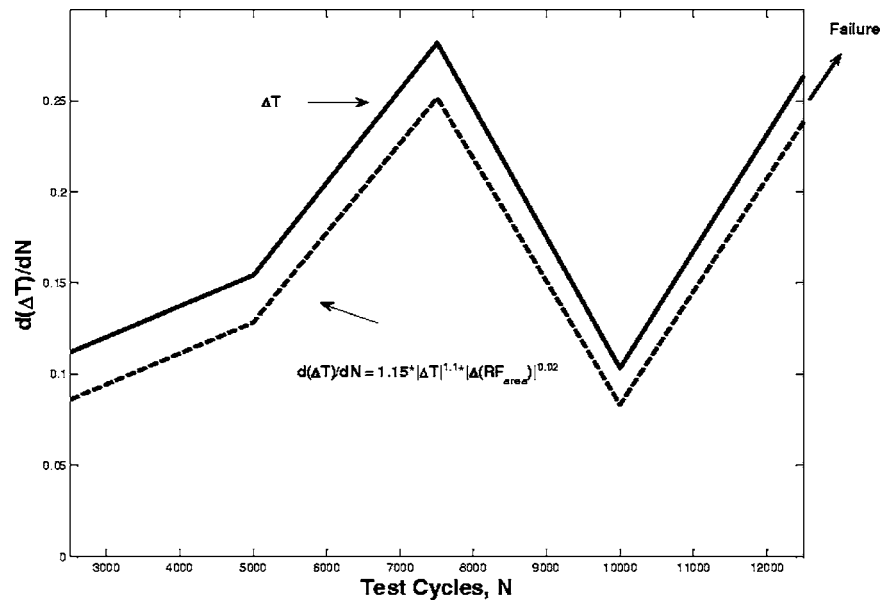


Fig. 27 Empirical regression model in Eq. (22) predicting growth of experimental fatigue crack damage in third sway bar link with one failed end: damage indicator (—) and damage model (---)

(transmissibility) indicators to develop empirical damage evolution models similar to the Paris law for crack growth. Such models can be developed for specific mechanical system/component, loading, boundary conditions, geometry, and damage mechanism combinations. Fatigue tests to failure were run on multiple automotive sway bar links and damage models were developed for different crack mechanisms. One crack mechanism was a fatigue crack initiated in a weld through fatigue testing and the second mechanism was a seeded crack via an EDM notch. Multiple links were tested under the same conditions of loading, geometry, boundary conditions, and damage mechanism to demonstrate the existence of one damage law for a specific combination of the previously mentioned variables. The analogy to crack power laws was further strengthened by the straight line relationship between the damage growth rate and the load intensity factor on log-log scales.

Nomenclature

a	= half crack length
A, B	= constant amplitudes of discrete system response
C, m, n	= coefficients of empirical damage evolution model
DI_k	= damage index based on real part of natural log of transmissibility
F_s	= sampling frequency
$N[x(t), \dot{x}_1(t)]$	= nonlinear forces in suspension
N	= number of cycles
N_f	= number of frequencies
$T_{ij}(\omega)$	= transmissibility function between degrees of freedom i and j
$x_k(t)$	= displacement of degree of freedom k
$\dot{x}_k(t)$	= velocity of degree of freedom k
$\ddot{x}_k(t)$	= acceleration of degree of freedom k
$X(\omega)$	= Fourier transform of response at input frequency ω
Y	= dimensionless Paris law parameter
Z	= constant of integration
σ	= applied stress
ΔT	= change in transmissibility

ΔRF_{area} = change in restoring force area
 $\Delta(\omega)$ = characteristic polynomial of a discrete MDOF system
 ω = cyclic frequency

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Dynamic Crack Extension Along the Interface of Materials That Differ in Thermal Properties: Convection and Thermal Relaxation

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Moving surface loads cause crack extension at a constant subcritical speed between perfectly bonded materials. The materials differ only in thermal properties and are governed by coupled thermoelastic equations that admit as special cases Fourier heat conduction and thermal relaxation with one or two relaxation times. Convection from the crack surfaces is allowed and for the latter two models is itself influenced by thermal relaxation. A dynamic steady state of plane strain is assumed. Fourier heat conduction is shown to dominate away from the crack edge at low speeds; solution behavior at the crack edge at high speeds depends upon the particular thermal model. Thermal mismatch is seen to cause solution behavior similar to that for the isothermal bimaterial, and so insight into the case of general material mismatch is provided.

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Keywords: thermal mismatch, dynamic interface crack, thermal relaxation, Fourier heat conduction, convection, Biot number, singular integral equations

1 Introduction

Joined dissimilar elastic materials occur as a geological formation [1] and as a structural element [2]. Study of interface cracks [3–5] is important in assessing behavior of such material systems. Interface crack studies such as these are sufficient for isothermal materials in equilibrium. If rapid crack extension occurs and coupled thermoelasticity [6] governs, however, material dissimilarity is characterized not only by mass density ρ , shear modulus μ , and Poisson's ratio ν but also by thermal conductivity k , coefficient of volumetric expansion α_v , and specific heat at constant volume c_v . Indeed, if thermal relaxation [7] occurs, then thermal relaxation times(s) τ arise and may also differ.

The effects of thermal parameters were illustrated by Ref. [8] by considering two perfectly bonded half-spaces whose properties (ρ, μ, ν) were identical, but whose thermal properties (k, α_v, c_v, τ) differed. While in a hypothetical limit case, it is known that two materials may indeed differ more in some properties than in others [9]. For generality, a coupled thermoelasticity formulation that included Fourier and thermal relaxation models as special cases was employed. Crack surface convection was assumed to be negligible, however. Thus, although solution response was found to be similar to that for the isothermal bimaterial, crack surface temperatures were (asymptotically) the same, except when a model had two relaxation times.

When bonding fails, the resulting crack surfaces will likely [10] exhibit small breaks and crystal misalignments in a region of microscale order. That is, a de facto thin layer exists on the surfaces that, while not influencing the bulk properties noted above, can modify surface heat flow. As discussed in Ref. [11] surface convection in the presence of a surface layer can be characterized by a Biot number; it depends on layer thickness and (effective) layer

conduction that follows from its nature and that of the external medium. This approach has been adopted [12] for the problem of sliding by a rough indenter.

The present work, therefore, models crack surface heat convection by assigning each crack face a property that is related to a Biot number, the external medium being the atmosphere in the crack gap. The effect of thermal relaxation on convection is included, and heat flow from one half-space to the other across both the interface bond and crack gap-surface layer system is continuous, i.e., neither is a heat source or sink. As in Ref. [8] the half-spaces are originally joined only over half of their interface and are at rest at uniform temperature. Moving loads are then applied on the separated faces, causing debonding. The loads vary neither with time nor in the direction parallel to the crack edge. Therefore, a dynamic steady state of plane strain arises in which loads and interface crack edge move at the same subcritical constant speed.

The study begins with consideration of the related general problem of a growing semi-infinite interface separation zone defined by discontinuities in displacement and temperature. The governing equations include the special cases of Fourier heat conduction [6] and thermal relaxation models with, respectively, one [13] and two [14] thermal relaxation times. Asymptotic forms that are most valid (a) away from the crack edge extending at low speed or (b) near the crack edge extending at high speed are extracted and inverted analytically. The inversions are used to solve exactly the interface crack problem for each of the three models. The solutions indicate that thermal mismatch alone can produce solution response similar to that for the isothermal bimaterial interface crack. Moreover, it is found that Fourier heat conduction dominates the former case (a), but that behavior in (b) is very sensitive to the particular model employed. Finally, crack face convection—especially that influenced by thermal relaxation—leads to a difference in temperature of the faces. The solution process is similar to those used in dynamic steady state studies of thermoelastic sliding contact [15,16] and debonding of a thermoelastic layer from a rigid substrate [17]. Insight arising from

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the study of Stoneley signals due to point sources at the interface of perfectly bonded thermoelastic half-spaces [18,19] is also employed.

2 Governing Equations for General Problem

Two isotropic, homogeneous linearly thermoelastic half-spaces, denoted as Solid 1 and Solid 2, occupy regions $y > 0$ and $y < 0$, respectively, where (x, y, z) are Cartesian coordinates. Properties (μ, ν, ρ) for the two solids are identical, but the thermal properties $(k_i, c_{vi}, \alpha_{vi}, \tau_i)$, $i = 1, 2$ differ. The solids are perfectly bonded for $x > 0$ and initially at rest at a common uniform (absolute) temperature T_0 . Shear and normal forces are then applied to the half-space faces in the region $x < 0$ of separation and moved toward the edge ($x=0$) of the bonded region at constant subcritical speed v . This action causes the separation zone to extend in the positive x direction. The loads vary neither with time nor z , so that a dynamic steady state of plane strain ensues for which the separation zone also extends with speed v . Therefore, by attaching the Cartesian coordinates to the moving zone edge, the time derivative assumes the form $-v\partial_x$, where (∂_x, ∂_y) signify (x, y) differentiation and (x, y) are the independent variables. The governing coupled equations of thermoelasticity for $y \neq 0$ can then be written as

$$(\nabla^2 - c^2 \partial_x^2)(u_{xi}, u_{yi}) + (\partial_x, \partial_y)(m\Delta_i - \alpha_{vi} D_i^{\text{II}} \theta_i) = 0 \quad (1a)$$

$$h_i \nabla^2 \theta_i + c \partial_x \left(\frac{\varepsilon_i}{\alpha_{vi}} D_i \Delta_i + D_i^{\text{I}} \theta_i \right) = 0 \quad (1b)$$

$$\nabla^2 = \partial_x^2 + \partial_y^2 \quad \Delta_i = \partial_x u_{xi} + \partial_y u_{yi} \quad (1c)$$

$$\begin{bmatrix} \sigma_{xi} \\ \sigma_{yi} \\ \sigma_{zi} \end{bmatrix} = \mu \begin{bmatrix} a & a-2 & 1 \\ a-2 & a & 1 \\ a-2 & a-2 & 1 \end{bmatrix} \begin{bmatrix} \partial_x u_{xi} \\ \partial_y u_{yi} \\ -\alpha_{vi} D_i^{\text{II}} \theta_i \end{bmatrix} \quad \sigma_{xyi} = \mu (\partial_x u_{yi} + \partial_y u_{xi}) \quad (1d)$$

Equations (1a) and (1b) partly uncoupled to give

$$(a \nabla^2 - c^2 \partial_x^2) \Delta_i - \alpha_{vi} \nabla^2 D_i^{\text{II}} \theta_i = 0 \quad (2a)$$

$$(\nabla^2 - c^2 \partial_x^2)(\partial_y u_{xi} - \partial_x u_{yi}) = 0 \quad (2b)$$

In Eqs. (1) and (2) $i = (1, 2)$, $(\sigma_{xi}, \sigma_{yi}, \sigma_{zi}, \sigma_{xyi})$ are the nonzero stresses, ∇^2 is the Laplacian in (x, y) , and $(u_{xi}, u_{yi}, \theta_i, \Delta_i)$ are, respectively, the (x, y) components of displacement, change in absolute temperature, and dilatation. The D operators are given by

$$\text{F: } D_i = D_i^{\text{I}} = D_i^{\text{II}} = 1 \quad (3a)$$

$$\text{I: } D_i = D_i^{\text{I}} = 1 - ch_i^{\text{I}} \partial_x \quad D_i^{\text{II}} = 1 \quad (3b)$$

$$\text{II: } D_i^{\text{I}} = 1 - ch_i^{\text{I}} \partial_x \quad D_i = D_i^{\text{II}} = 1 - ch_i^{\text{II}} \partial_x \quad (3c)$$

Here, F denotes the Fourier model [6], and I and II denote, respectively, the single-relaxation time model [13] and double-relaxation time model [14]. Equations (1)–(3) also exhibit parameters

$$m = \frac{1}{1-2\nu} \quad a = 1 + m \quad c = \frac{\nu}{v_r} \quad v_r = \sqrt{\frac{\mu}{\rho}} \quad (4a)$$

$$\varepsilon_i = \frac{\mu T_0}{\rho c_{vi}} \alpha_{vi}^2 \quad h_i = \frac{k_i}{c_{vi} \sqrt{\mu \rho}} \quad (h_i^{\text{I}}, h_i^{\text{II}}) = v_r (\tau_i^{\text{I}}, \tau_i^{\text{II}}) \quad (4b)$$

In particular, v_r is the rotational wave speed in both solids, and ε_i and h_i are the dimensionless coupling constant and characteristic length for coupled thermoelasticity.

Parameters $h_i^{\text{I}}, h_i^{\text{II}}$ are additional characteristic lengths that arise because of the thermal relaxation times $\tau_i^{\text{I}}, \tau_i^{\text{II}}$. Data [6,9,20,21] indicate that $\tau_i^{\text{I}} > \tau_i^{\text{II}}$ for Model II and that

$$v_r \approx O(10^3) \text{ m/s} \quad h_i \approx O(10^{-9}) \text{ m} \quad \varepsilon_i \approx O(10^{-2}) \quad (\tau_i^{\text{I}}, \tau_i^{\text{II}}) \approx O(10^{-13}) \text{ s} \quad (5)$$

In light of Eq. (4b), therefore, $h_i \gg h_i^{\text{I}} > h_i^{\text{II}}$.

Along the interface $y=0$ of Solids 1 and 2, the matching conditions

$$\sigma_{xy1} - \sigma_{xy2} = 0 \quad \sigma_{y1} - \sigma_{y2} = 0 \quad k_1 \partial_y \theta_1 - k_2 \partial_y \theta_2 = 0 \quad (6a)$$

$$u_{x1} - u_{x2} = U(x) \quad u_{y1} - u_{y2} = V(x), \quad \theta_1 - \theta_2 = \Theta(x) \quad (6b)$$

Functions U, V, Θ represent discontinuities in displacement and temperature in the extending separation zone, and so vanish identically for $x > 0$. The last condition in Eq. (6a) signifies that, as noted above, neither the interface bond nor the crack gap-surface layer system is a heat source or sink. The functions are also assumed to be continuous almost everywhere for $x < 0$ and, in particular, to vanish for $x \rightarrow (0-, -\infty)$. The fields u_{x1}, u_{y1}, θ_1 and u_{x2}, u_{y2}, θ_2 themselves should be continuous and bounded above as $\sqrt{x^2 + y^2} \rightarrow \infty$ for $y > 0$ and $y < 0$, respectively. In the Appendix, the exact solution to this problem in integral transform space is presented. This is used to treat the interface crack extension problem, as outlined below.

2.1 Crack Extension: Far Field. Consider the zone of separation to be a crack driven by moving shear and normal line loads of infinite extent parallel to the z axis. Thus, U, V, Θ are unknown but additional conditions are now imposed for $y=0, x < 0$:

$$(\sigma_{xy2}, \sigma_{y2}) = -(P_S, P_N) \delta(x+L) \quad (k_1 \partial_y + \beta_1 D_1^{\text{I}}) \theta_1 = (k_2 \partial_y + \beta_2 D_2^{\text{I}}) \theta_2 \quad (7a)$$

$$\text{F: } D_i^{\text{I}} = 1 \quad (\text{I, II}): D_i^{\text{I}} = 1 - ch_i^{\text{I}} \partial_x \quad i = (1, 2) \quad (7b)$$

The last condition in Eq. (7a) reflects the assumption that crack surface heat convection is also subject to thermal relaxation. Constants β_1, β_2 are the convection parameters for the crack surfaces ($y=0-, 0+$). It is noted that if the crack surface layer thickness t_1, t_2 were specified, then $\beta_1 t_1 / k_1, \beta_2 t_2 / k_2$ would be Biot numbers [11]. In Eq. (7a), P_S, P_N are force magnitudes and L is the fixed distance between load and crack edge maintained in the dynamic steady state. An asymptotic solution to this problem valid for $|x| \gg h^* = \max(h_1, h_2)$ is obtained by using forms of Eqs. (A2)–(A10) in the Appendix valid for $|h^* p| \ll 1, 0 \leq c \ll c$. In particular, Eq. (A2c) behaves as

$$\eta_i^+ \approx \sqrt{\frac{a_i^e}{-ch_i p}} \quad \eta_i^- \approx \sqrt{\frac{a}{a_i^e}} \quad (d_i, d_i^{\text{I}}, d_i^{\text{II}}) \approx 1 \quad a_i^e = a + \varepsilon_i \quad (8)$$

In view of Eq. (5), it is noted that $a_i^e > a > 1$. Use of Eq. (8) in Eq. (A4a) and its counterpart for the transforms of $\sigma_{xy2}, \sigma_{y2}$ gives forms that yield analytical results upon substitution into the inversion integral in Eq. (A1). Substitution of these into Eq. (7) gives for $x < 0$ the set of partly coupled singular integral equations for $dU/dx, dV/dx, \Theta$:

$$\frac{R_U^e}{2BS_e \pi} (vp) \int_{-\infty}^0 \frac{dU}{dt} \frac{dt}{t-x} + \frac{KD_e}{S_e} \frac{dV}{dx} + \frac{2\Omega_s^e \lambda_{12}}{S_e \sqrt{\pi c}} \int_x^0 \frac{\Theta dt}{\sqrt{t-x}} = -\frac{P_S}{\mu} \delta(x+L) \quad (9a)$$

$$-\frac{KD_e}{S_e} \frac{dU}{dx} + \frac{R_V^e}{2S_e \pi} (vp) \int_{-\infty}^0 \frac{dV}{dt} \frac{dt}{t-x} - \frac{K\Omega_s^e \lambda_{12}}{S_e \sqrt{\pi c}} \int_{-\infty}^x \frac{\Theta dt}{\sqrt{x-t}} = -\frac{P_N}{\mu} \delta(x+L) \quad (9b)$$

$$(k_1\lambda_1\beta_2 + k_2\lambda_2\beta_1)\Theta = 0 \quad (9c)$$

In Eq. (9), (vp) signifies Cauchy principal value integration, and

$$R_U^\varepsilon = A_2^\varepsilon R_1^\varepsilon + A_1^\varepsilon R_2^\varepsilon \quad R_V^\varepsilon = R_1^\varepsilon + R_2^\varepsilon \quad S_\varepsilon = A_1^\varepsilon + A_2^\varepsilon \quad D_\varepsilon = A_1^\varepsilon - A_2^\varepsilon \quad (10a)$$

$$\Omega_S^\varepsilon = k_2 \frac{\alpha_{v1} h_1}{a_1^\varepsilon} + k_1 \frac{\alpha_{v2} h_2}{a_2^\varepsilon} \quad \Omega_N^\varepsilon = k_2 A_2^\varepsilon \frac{\alpha_{v1} h_1}{a_1^\varepsilon} - k_1 A_1^\varepsilon \frac{\alpha_{v2} h_2}{a_2^\varepsilon}$$

$$\lambda_{12} = \frac{\lambda_1 \lambda_2}{k_1 \lambda_1 + k_2 \lambda_2} \quad (10b)$$

$$R_i^\varepsilon = 4A_i^\varepsilon B - K^2 \quad A_i^\varepsilon = \sqrt{1 - \frac{c^2}{a_i^\varepsilon}} \quad \lambda_i = \sqrt{\frac{a_i^\varepsilon}{h_i a}} \quad i = (1, 2) \quad (10c)$$

Terms $R_U^\varepsilon, R_V^\varepsilon$ are functions of the Rayleigh type [20] and vanish for

$$c = \pm (c_U^\varepsilon, c_V^\varepsilon) \quad 0 < (c_U^\varepsilon, c_V^\varepsilon) < 1$$

Equation (6) thus becomes

$$c^* = \min(c_U^\varepsilon, c_V^\varepsilon) \quad (11)$$

The solution to Eq. (9c) is

$$\Theta = 0 \quad (12)$$

Equations (9a) and (9b) are now coupled only in $(dU/dx, dV/dx)$ and, after Refs. [15–17], its eigenfunctions are $(-x)^\Gamma$, where

$$\Gamma = -\frac{1}{2} \pm i\omega \quad \pi\omega = \tanh^{-1} \frac{2KD_\varepsilon \sqrt{B}}{\sqrt{R_U^\varepsilon R_V^\varepsilon}} \quad (13)$$

It can be shown that ω is real for $0 < c < c^*$. Solutions to Eqs. (9a) and (9b) are then

$$\frac{dU}{dx} = -\frac{2P}{\mu} S_\varepsilon \left[\frac{\sqrt{R_U^\varepsilon}}{2KD_\varepsilon \sqrt{B}} \sinh^2 \pi\omega \sin \phi \delta(x+L) + \sqrt{\frac{B}{R_U^\varepsilon}} \frac{\cosh^2 \pi\omega}{\pi(x+L)} \sqrt{\frac{L}{|x|}} \cos \Psi \right] \quad (14a)$$

$$\frac{dV}{dx} = \frac{2P}{\mu} S_\varepsilon \left[\frac{\sqrt{R_V^\varepsilon}}{2KD_\varepsilon} \sinh^2 \pi\omega \cos \phi \delta(x+L) - \frac{1}{\sqrt{R_V^\varepsilon}} \frac{\cosh^2 \pi\omega}{\pi(x+L)} \sqrt{\frac{L}{|x|}} \sin \Psi \right] \quad (14b)$$

These results in turn give for $y=0, x>0$

$$(\sigma_{xy}^0, \sigma_y^0) = \frac{P}{\pi} \sqrt{\frac{L}{x}} \frac{\cosh \pi\omega}{x+L} \left(\sqrt{\frac{R_U^\varepsilon}{B}} \cos \Psi, \sqrt{R_V^\varepsilon} \sin \Psi \right) \quad (15a)$$

$$\theta_0 = \frac{2c^2 KP \lambda_{12}}{\mu R_V^\varepsilon} \sqrt{\frac{BD_\varepsilon}{R_U^\varepsilon}} \left(\frac{k_1 \varepsilon_1}{\lambda_2 a_1^\varepsilon} \frac{\alpha_{v2}}{\alpha_{v1}} + \frac{k_2 \varepsilon_2}{\lambda_1 a_2^\varepsilon} \frac{\alpha_{v1}}{\alpha_{v2}} \right) \sqrt{\frac{L}{|x|}} \frac{\cosh \pi\omega \sin \Psi}{\pi(x+L)} \quad (15b)$$

In Eqs. (14) and (15), superscript 0 signifies evaluation for $y=0$, and

$$P = \sqrt{\frac{1}{R_V^\varepsilon P_N^2} + \frac{B}{R_U^\varepsilon P_S^2}} \quad \phi = \tan^{-1} \frac{P_N}{P_S} \sqrt{\frac{R_U^\varepsilon}{BR_V^\varepsilon}} \quad \Psi = \phi + \omega \ln \frac{L}{|x|} \quad (16)$$

These asymptotic analytic results are independent of thermal relaxation parameters (h_i^I, h_i^{II}) , i.e., Fourier heat flow governs. Equations (14) and (15) also exhibit, via the logarithmic term in Ψ , the rapid oscillations in traction and crack surface opening common to interface cracks in isothermal equilibrium studies [3–5].

2.2 F-Model Crack Extension: Near Field. A result valid for $|x| \ll h_0 = \min(h_1, h_2)$ is obtained from Eqs. (A2)–(A10) by using asymptotic forms valid for $|h_0 p| \gg 1, 0 \ll c < c^*$. Use of the inversion integral in Eq. (A1) and

$$\eta_i^+ \approx 1 - \frac{\varepsilon_i}{2ch_i p} \quad \eta_i^- \approx \sqrt{\frac{a}{-ch_i p}} \quad (d_i, d_i^I, d_i^{II}) = 1 \quad (17)$$

gives for $x < 0$ equations

$$\frac{R}{2B\pi} (vp) \int_{-\infty}^0 \frac{dU}{dt} \frac{dt}{t-x} - (1-A) \frac{k_2 \alpha_{v1} + k_1 \alpha_{v2}}{c^2(k_1 + k_2)\pi} (vp) \int_{-\infty}^0 \frac{\Theta dt}{t-x}$$

$$= -\frac{P_S}{\mu} \delta(x+L) \quad (18a)$$

$$\frac{R}{2A\pi} (vp) \int_{-\infty}^0 \frac{dV}{dt} \frac{dt}{t-x} + \frac{K(1-A)}{2A} \frac{k_2 \alpha_{v1} - k_1 \alpha_{v2}}{c^2(k_1 + k_2)} \Theta = -\frac{P_N}{\mu} \delta(x+L) \quad (18b)$$

$$(k_1\beta_2 + k_2\beta_1)\Theta = C_0 \quad (18c)$$

In this case, C_0 is an arbitrary real constant, and parameters

$$R = 4AB - K^2 \quad A = \sqrt{1 - \frac{c^2}{a}} \quad (19)$$

Equation (18) admits the three identical eigenfunctions $1/\sqrt{-x}$. Use of these in Eq. (18) gives $dU/dx, dV/dx$ that are unbounded as $x \rightarrow -\infty$ unless $C_0 \equiv 0$. Thus, Eq. (12) holds again and Eqs. (18a) and (18b) give for $x < 0$:

$$\left(\frac{dU}{dx}, \frac{dV}{dx} \right) = - \left(B \frac{P_S}{\mu}, A \frac{P_N}{\mu} \right) \frac{2\sqrt{L}}{\pi R} \frac{1}{(x+L)\sqrt{-x}} \quad (20)$$

For $y=0, x>0$ quantities,

$$(\sigma_{xy}^0, \sigma_y^0) = (P_S, P_N) \frac{\sqrt{L}}{\pi \sqrt{x} x + L} \quad (21a)$$

$$\theta_0 = \frac{2c^3 KB}{\pi RA} \frac{1-A}{k_1 + k_2} \left(\frac{k_1 \varepsilon_1}{\alpha_{v1} h_1} + \frac{k_2 \varepsilon_2}{\alpha_{v2} h_2} \right) \frac{P_N}{\mu} \tan^{-1} \sqrt{\frac{L}{x}} \quad (21b)$$

The single Rayleigh function R vanishes for $c = \pm c_R, 0 < c_R < 1$, so that Eq. (A10) now gives

$$c^* = c_R \quad (22)$$

Equations (20) and (21) indicate that near-field solution behavior for the Fourier model is essentially identical to that for crack extension in a single material. That is, thermal mismatch by itself did not give rise to the oscillatory behavior seen in Eqs. (14) and (15). As in that case, there is no discontinuity in temperature between the two crack faces.

2.3 Model I Crack Extension: Near Field. A result valid for $|x| \ll h_0 = \min(h_1, h_2, h_1^I, h_2^I)$ can be obtained from Eqs. (A2)–(A10) by considering asymptotic forms valid for $|h_0 p| \gg 1, 0 \ll c < c^*$. Use of

$$2\eta_i^{\pm} \approx \sqrt{(1 + \sqrt{al_i^2})^2 + \varepsilon_i l_i^2} \pm \sqrt{(1 - \sqrt{al_i^2})^2 + \varepsilon_i l_i^2} \quad l_i^1 = \frac{h_i^1}{h_i} \quad (23a)$$

$$(d_i, d_i^1) \approx -ch_i^1 p \quad d_i^1 = 1 \quad (23b)$$

in Eqs. (A1) and (7) gives for $x < 0$ the coupled equations

$$\begin{aligned} & \frac{R_U}{2c^2 B S \pi} (vp) \int_{-\infty}^0 \frac{dU}{dt} \frac{dt}{t-x} + \frac{KD}{c^2 S} \frac{dV}{dx} - \frac{2\Omega_S}{aS\pi} (vp) \int_{-\infty}^0 \frac{\Theta dt}{t-x} \\ & = -\frac{P_S}{\mu} \delta(x+L) \end{aligned} \quad (24a)$$

$$-\frac{KD}{c^2 S} \frac{dU}{dx} + \frac{R_V}{2c^2 S \pi} (vp) \int_{-\infty}^0 \frac{dV}{dt} \frac{dt}{t-x} + \frac{K\Omega_N}{aS} \Theta = -\frac{P_N}{\mu} \delta(x+L) \quad (24b)$$

$$\begin{aligned} & \beta_{12} \frac{2c^4 \Omega_U}{aS} \frac{dU}{dx} - \beta_{12} \frac{c^4 K \Omega_V}{aS\pi} (vp) \int_{-\infty}^0 \frac{dV}{dt} \frac{dt}{t-x} + \left(\beta_{12} \frac{c^2 \Omega_T}{S} \right. \\ & \left. + \beta_1 h_1^1 \right) \Theta = 0 \end{aligned} \quad (24c)$$

In Eq. (24) parameters

$$S = (k_1 A_1^+ A_1^- + k_2 A_2^+ A_2^-) S_1 S_2 + k_1 P_1 Q_2 + k_2 P_2 Q_1 + \frac{c^4}{a^2} S_{12} \quad (25a)$$

$$D = k_1 A_1^+ A_1^- - k_2 A_2^+ A_2^- + \frac{c^4}{a^2} D_{12} \quad (25b)$$

$$\begin{aligned} R_U &= k_1 A_1^+ A_1^- S_1 (8BQ_2 - K^2 S_2) + k_2 A_2^+ A_2^- S_2 (8BQ_1 - K^2 S_1) \\ &+ K^2 \frac{c^4}{a^2} S_{12} \end{aligned} \quad (25c)$$

$$\begin{aligned} \frac{1}{2} R_V &= 2BS_1 S_2 (k_1 A_1^+ A_1^- + k_2 A_2^+ A_2^-) - K^2 [k_1 P_1 (2BQ_2 - S_2) \\ &+ k_2 P_2 (2BQ_1 - S_1)] - 2B \frac{c^4}{a^2} S_{12} \end{aligned} \quad (25d)$$

$$\Omega_S = k_1 A_1^+ A_1^- S_1 \alpha_{v2} + k_2 A_2^+ A_2^- S_2 \alpha_{v1} \quad \Omega_N = k_1 P_1 \alpha_{v2} - k_2 P_2 \alpha_{v1} \quad (25e)$$

$$\Omega_U = \frac{k_1 \varepsilon_1 l_1^1}{\alpha_{v1}} Q_2 - \frac{k_2 \varepsilon_2 l_2^1}{\alpha_{v2}} Q_1 \quad \Omega_V = \frac{k_1 \varepsilon_1 l_1^1}{\alpha_{v1}} S_2 + \frac{k_2 \varepsilon_2 l_2^1}{\alpha_{v2}} S_1 \quad (25f)$$

$$\Omega_T = k_1 S_2 (S_1 A_1^+ A_1^- + A_2^+ P_1) + k_2 \varepsilon_2 l_2^1 \frac{c^4 \alpha_{v1}}{a^2 \alpha_{v2}} \quad (25g)$$

In Eqs. (24) and (25),

$$S_{12} = k_1 \varepsilon_1 l_1^1 \frac{\alpha_{v2}}{\alpha_{v1}} + k_2 \varepsilon_2 l_2^1 \frac{\alpha_{v1}}{\alpha_{v2}} \quad D_{12} = k_1 \varepsilon_1 l_1^1 \frac{\alpha_{v2}}{\alpha_{v1}} - k_2 \varepsilon_2 l_2^1 \frac{\alpha_{v1}}{\alpha_{v2}} \quad (26a)$$

$$\beta_{12} = \beta_2 h_2^1 - \beta_1 h_1^1 \quad (26b)$$

$$S_i = A_i^+ + A_i^- \quad P_i = S_i^2 - Q_i \quad Q_i = A_i^+ A_i^- + A^2 \quad i = (1, 2) \quad (26c)$$

The term A is defined in Eq. (19). Coefficients in Eq. (24) correspond to those in Eqs. (9) and (18); their different form arises due

to cancellation in their ratios of a common factor $(A_1^+ - A_1^-)(A_2^+ - A_2^-)$. So Rayleigh functions $(R_U, R_V) = 0$ at, respectively, $c = \pm (c_U^1, c_V^1)$, where $0 < (c_U^1, c_V^1) < c_m = \min(1, \sqrt{a}/\eta_1^+, \sqrt{a}/\eta_2^+)$. Function S is of the Stoneley type [1] and vanishes $c = \pm c_{12}^1$, $0 < c_{12}^1 < c_m$ when $S(\pm c_m) < 0$. Thus, Eqs. (A10) gives

$$S(\pm c_m) < 0: c^* = \min(c_{12}^1, c_U^1, c_V^1) \quad (27a)$$

$$S(\pm c_m) \geq 0: c^* = \min(c_U^1, c_V^1) \quad (27b)$$

Equations (24a)–(24c) fully couple $(dU/dx, dV/dx, \Theta)$ and the eigenfunctions are $(-x)^\Gamma$, where

$$\Gamma = \left(0, -\frac{1}{2} \pm i\omega \right) \quad \pi\omega = \tanh^{-1} K \sqrt{\frac{M_S D}{M_C}} \quad (28)$$

Use of this result gives for $x < 0$ the solutions

$$\begin{aligned} \frac{dU}{dx} &= -\frac{c^2 S P}{2\mu} \frac{M_U \sqrt{D}}{\sqrt{M_S M_C}} \left[\sinh \pi\omega \sin \phi \delta(x+L) \right. \\ &+ \left. \sqrt{\frac{L}{|x|}} \frac{\cosh \pi\omega \cos \Psi}{\pi(x+L)} \right] - 2\beta_{12} \frac{c^8 K S P_S}{\mu a^2 M_C} \Omega_S \Omega_V \delta(x+L) \end{aligned} \quad (29a)$$

$$\frac{dV}{dx} = c^2 S \frac{P}{\mu} \left[\sinh \pi\omega \cos \phi \delta(x+L) - \sqrt{\frac{L}{|x|}} \frac{\cosh \pi\omega \sin \Psi}{\pi(x+L)} \right] \quad (29b)$$

$$\begin{aligned} \Theta &= -\beta_{12} \frac{c^6 S P}{a\mu} \frac{C_U \sqrt{D}}{\sqrt{M_S M_C}} \left[\sinh \pi\omega \sin \phi \delta(x+L) \right. \\ &+ \left. \sqrt{\frac{L}{|x|}} \frac{\cosh \pi\omega \cos \Psi}{\pi(x+L)} \right] - \beta_{12} \frac{c^6 K S R_U \Omega_V P_S}{2\mu a B M_C} \delta(x+L) \end{aligned} \quad (29c)$$

Results for $y=0$, $x > 0$ are

$$\begin{aligned} & (\sigma_{xy}^0, \sigma_y^0, \theta_0) \\ & = \frac{P}{\pi} \sqrt{\frac{L}{x}} \frac{1}{x+L} \left[\sqrt{\frac{M_C D}{M_S}} \cos \Psi, \frac{R_V}{2} \sin \Psi, \frac{c^6}{a} K \Omega_V \sin \Psi \right] \end{aligned} \quad (30)$$

In Eqs. (29) and (30) parameters

$$\begin{aligned} P &= \sqrt{\left(\frac{2Q_0 P_S}{M_C} \cosh \pi\omega \right)^2 + \left(\frac{P_N}{KD} \sinh \pi\omega \right)^2} \quad \phi \\ &= \tan^{-1} \frac{2Q_0 \sqrt{D} P_S}{\sqrt{M_S M_C} P_N} \end{aligned} \quad (31a)$$

$$\Psi = \phi + \omega \ln \frac{L}{|x|} \quad (31b)$$

In Eqs. (29)–(31) parameters

$$\begin{aligned} M_C &= R_V Q_0 - \beta_{12} \frac{c^6}{a^2} K^2 \Omega_V Q \quad M_S = \beta_{12} c^2 \left(2 \frac{c^4}{a^2} \Omega_U \Omega_N + \Omega_T D \right) \\ &+ \beta_1 h_1^1 S D \end{aligned} \quad (32a)$$

$$\begin{aligned} M_U &= \beta_{12} c^2 \left(2 \frac{c^4}{a^2} K^2 \Omega_V \Omega_N + R_V \Omega_T \right) + \beta_1 h_1^1 R_V S \quad C_U = K^2 \Omega_V D \\ &- R_V \Omega_U \end{aligned} \quad (32b)$$

$$Q = 2D\Omega_S - \frac{R_U}{2B}\Omega_N \quad Q_0 = \beta_{12}c^2 \left(\frac{R_U}{4B}\Omega_T + 2\frac{c^4}{a^2}\Omega_U\Omega_S \right) + \beta_1 h_1^I \frac{R_US}{4B} \quad (32c)$$

Equation (28) gives real ω when $0 < c < c^*$. Equations (29) and (30) are clearly influenced by thermal relaxation, and the rapid oscillations seen in Eqs. (14) and (15) arise again. However, when $P_S \neq 0$ value $\Gamma=0$ now gives terms that are both nonsingular and nonoscillatory. Moreover, discontinuity in temperature of the crack faces occurs.

2.4 Model II Crack Extension: Near Field. A result valid for $|x| \leq h_0 = \min(h_1, h_2, h_1^I, h_2^I, h_1^{II}, h_2^{II})$ can be obtained from Eqs. (A2)–(A10) by considering asymptotic forms valid for $|h_0| \gg 1$, $0 \leq c < c^*$. Use of

$$2\eta_i^+ \approx \sqrt{(1 + \sqrt{a l_i^I})^2 + \varepsilon_i l_i^I} \pm \sqrt{(1 - \sqrt{a l_i^I})^2 + \varepsilon_i l_i^I} \quad l_i^{II} = \frac{h_i^{II}}{h_i} \quad (33a)$$

$$d_i^I \approx -ch_i^I p \quad (d_i, d_i^{II}) \approx -ch_i^{II} p \quad (33b)$$

in Eqs. (A1) and (7) gives for $x < 0$ the coupled equations

$$\begin{aligned} & \frac{R_U}{2c^2 B S \pi} (vp) \int_{-\infty}^0 \frac{dU}{dt} \frac{dt}{t-x} + \frac{KD}{c^2 S} \frac{dV}{dx} - \frac{2c\Omega_S}{aS\pi} (vp) \int_{-\infty}^0 \frac{d\Theta}{dt} \frac{dt}{t-x} \\ & = -\frac{P_S}{\mu} \delta(x+L) \end{aligned} \quad (34a)$$

$$-\frac{KD}{c^2 S} \frac{dU}{dx} + \frac{R_V}{2c^2 S \pi} (vp) \int_{-\infty}^0 \frac{dV}{dt} \frac{dt}{t-x} + \frac{cK\Omega_N}{aS} \frac{d\Theta}{dx} = -\frac{P_N}{\mu} \delta(x+L) \quad (34b)$$

$$\begin{aligned} & \beta_{12} \frac{2\Omega_U}{cS} \frac{dU}{dx} - \beta_{12} \frac{K\Omega_V}{cS\pi} (vp) \int_{-\infty}^0 \frac{dV}{dt} \frac{dt}{t-x} + \left(\beta_{12} \frac{c^2\Omega_T}{S} + \beta_1 h_1^I \right) \frac{d\Theta}{dx} \\ & = 0 \end{aligned} \quad (34c)$$

In Eq. (34) coefficients $S, D, R_U, R_V, \Omega_S, \Omega_N$ are again defined by Eqs. (25) and (26b), but Eq. (33a) holds instead of Eq. (23a) and Eq. (26a) is replaced by

$$S_{12} = \frac{k_1 \varepsilon_1}{\alpha_{v1} h_1} \alpha_{v2} h_2^{II} + \frac{k_2 \varepsilon_2}{\alpha_{v2} h_2} \alpha_{v1} h_1^{II} \quad D_{12} = \frac{k_1 \varepsilon_1}{\alpha_{v1} h_1} \alpha_{v2} h_2^{II} - \frac{k_2 \varepsilon_2}{\alpha_{v2} h_2} \alpha_{v1} h_1^{II} \quad (35)$$

In addition, Eqs. (25f) and (25g) for $\Omega_U, \Omega_V, \Omega_T$ are replaced by

$$\Omega_U = \frac{k_1 \varepsilon_1 Q_2}{\alpha_{v1} h_1} - \frac{k_2 \varepsilon_2 Q_1}{\alpha_{v2} h_2} \quad \Omega_V = \frac{k_1 \varepsilon_1 S_2}{\alpha_{v1} h_1} + \frac{k_2 \varepsilon_2 S_1}{\alpha_{v2} h_2} \quad (36a)$$

$$\Omega_T = k_1 S_2 (S_1 A_1^+ A_1^- + A_2^+ P_1) + k_2 \varepsilon_2 \frac{c^4 \alpha_{v1} h_1^{II}}{a^2 \alpha_{v2} h_2} \quad (36b)$$

Rayleigh functions $(R_U, R_V)=0$ for $c = \pm(c_U^I, c_V^I)$, respectively, where $0 < (c_U^I, c_V^I) < c_m = \min(1, \sqrt{a/\eta_1^+}, \sqrt{a/\eta_2^+})$, and Stoneley function $S=0$ for $c = \pm c_{12}^{II}$ ($0 < c_{12}^{II} < c_m$) when $S(\pm c_m) < 0$. Therefore, Eq. (A10) gives

$$S(\pm c_m) < 0: c^* = \min(c_{12}^{II}, c_U^I, c_V^I) \quad (37a)$$

$$S(\pm c_m) \geq 0: c^* = \min(c_U^I, c_V^I) \quad (37b)$$

Equations (31a) and (31b) fully couple $dU/dx, dV/dx, d\Theta/dx$, three eigenfunctions $(-x)^\Gamma$ are again defined by Eq. (28), and solutions for $x < 0$ are

$$\begin{aligned} \frac{dU}{dx} = & -\frac{c^2 SP}{2\mu} \frac{M_U \sqrt{D}}{\sqrt{M_S M_C}} \left[\sinh \pi \omega \sin \phi \delta(x+L) \right. \\ & \left. + \sqrt{\frac{L}{|x|}} \frac{\cosh \pi \omega \cos \Psi}{\pi(x+L)} \right] - 2\beta_{12} \frac{c^4 K S P_S}{\mu a M_C} \Omega_S \Omega_V \delta(x+L) \end{aligned} \quad (38a)$$

$$\frac{dV}{dx} = cS \frac{P}{\mu} \left[\sinh \pi \omega \cos \phi \delta(x+L) - \sqrt{\frac{L}{|x|}} \frac{\cosh \pi \omega \sin \Psi}{\pi(x+L)} \right] \quad (38b)$$

$$\begin{aligned} \frac{d\Theta}{dx} = & -\beta_{12} \frac{cSP}{\mu} \frac{C_U \sqrt{D}}{\sqrt{M_S M_C}} \left[\sinh \pi \omega \sin \phi \delta(x+L) \right. \\ & \left. + \sqrt{\frac{L}{|x|}} \frac{\cosh \pi \omega \cos \Psi}{\pi(x+L)} \right] - \beta_{12} \frac{cSR_U K \Omega_V P_S}{2\mu B M_C} \delta(x+L) \end{aligned} \quad (38c)$$

For $y=0, x>0$ results are

$$(\sigma_{xy}^0, \sigma_y^0) = \frac{P}{\pi} \sqrt{\frac{L}{x}} \frac{1}{x+L} \left[\sqrt{\frac{M_C D}{M_S}} \cos \Psi, \frac{R_V}{2} \sin \Psi \right] \quad (39a)$$

$$\theta_0 = -cK\Omega_V \frac{P\sqrt{L}}{\mu\pi} \int_x^\infty \frac{\sin \Psi dt}{\sqrt{t(t+L)}} \quad (39b)$$

Equations (28) and (31) for (P, ϕ, Ψ, ω) still hold, but Eq. (32) is replaced by

$$\begin{aligned} M_C = & R_V Q_0 - \beta_{12} \frac{c^2}{a} K^2 \Omega_V Q \quad M_S = \beta_{12} c^2 \left(\frac{2}{a} \Omega_U \Omega_N + \Omega_T D \right) \\ & + \beta_1 h_1^I S D \end{aligned} \quad (40a)$$

$$\begin{aligned} M_U = & \beta_{12} c^2 \left(\frac{2}{a} K^2 \Omega_V \Omega_N + R_V \Omega_T \right) + \beta_1 h_1^I R_V S \quad C_U = K^2 \Omega_V D \\ & - R_V \Omega_U \end{aligned} \quad (40b)$$

$$Q = 2D\Omega_S - \frac{R_U}{2B}\Omega_N \quad Q_0 = \beta_{12}c^2 \left(\frac{2}{a}\Omega_U\Omega_S + \frac{R_U\Omega_T}{4B} \right) + \beta_1 h_1^I \frac{R_US}{4B} \quad (40c)$$

Yet again, Eq. (28) gives real ω for $0 < c < c^*$. Equations (38) and (39) exhibit the oscillations also seen in Eqs. (14), (15), (29), and (30). As in the Model I result, additional terms appear, which exhibit neither singular nor oscillatory behavior, and a jump in the crack face temperatures exists. The temperature quantities (Θ, θ_0) in Eqs. (29c) and (30) are singular at the crack edge. The integral of Eq. (38c), however, is finite for $x \rightarrow 0^-$ and for $x \rightarrow 0^+$ the integral in Eq. (39b) gives the finite result

$$\frac{\pi}{\sqrt{L}} \frac{\sin \phi}{\cosh \pi \omega}$$

3 Some General Observations

It is seen that the three thermal models (F, I, II) give rise to four asymptotic results. These results arise from the solution to sets of three singular integral equations for the crack separation gradients $(dU/dx, dV/dx)$ and either the temperature difference Θ between the two crack faces, or its gradient $d\Theta/dx$. Results (12) and (13) of Sec. 2.1 show that Fourier conduction governs the dynamic steady state behavior in the crack plane for low crack speeds ($0 < c \ll c^*$) except near the crack edge ($|x/h^*| \approx 1$). The three equations (19) fully couple only $(dU/dx, dV/dx)$, thereby giving rise to two complex conjugate eigenfunctions, and oscillatory singular

behavior of the interface stresses and temperature change, c.f. the isothermal equilibrium interface crack. Moreover, no difference in the temperature change on the two crack faces occurs.

In view of Eq. (5) the Sec. 2.1 results are the most robust, but numerical studies in thermal fluids [22] show that the effects of thermal relaxation are most pronounced near a concentrated source. The results of Sec. 2.2–2.4 valid near the crack edge ($|x/h^*| \ll 1$) for high crack speeds ($0 \ll c < c^*$) seem to bear this out. In particular, coupling produces three eigenfunctions. The Fourier Model F results (20) and (21) in Sec. 2.2 produce the degenerate case of three identical real eigenfunctions, yet maintain the asymptotically negligible difference in the temperature of the two crack faces seen in the far field. However, interface stresses are singular but nonoscillatory, and interface temperature change is both nonoscillatory and finite at the crack edge. The nonoscillatory behavior shows that thermal mismatch effects are dominated by the elastically identical nature of the two half-spaces.

Section 2.3 results (29) and (30) for Model I (one relaxation time for each half-space) differ from those for Sec. 2.2. The three eigenfunctions are distinct, with one being unity and the other two being complex conjugates. Therefore, so long as shear loading is present, both stress and temperature along the interface exhibit terms that are singular and oscillatory, and terms that are neither. Moreover, a difference in crack face temperatures occurs. Section 2.4 results (38) and (39) for Model II (two relaxation times for each half-space) exhibit behavior to that for Sec. 2.3. However, both differences in crack face temperature and interface temperature are finite at the crack edge.

Summary

The results presented indicate that for low ($0 < c \ll c^*$) extension speeds, thermal mismatch by itself gives rise to solution behavior near, but not at, the edge of an extending interface crack similar to that seen for the isothermal bimaterial crack. Fourier heat conduction dominates, whether thermal relaxation exists or not, and the temperature change generated on the two crack faces is (asymptotically) identical.

For high ($0 \ll c < c^*$) extension speeds, thermal mismatch when thermal relaxation is present gives rise to solution behavior very near the crack edge that is, again, similar to that for the isothermal bimaterial analysis. Moreover, a difference in crack face temperature occurs. In the single-relaxation time model [13], interface temperature change is singular at the crack edge, but the change is finite for the double-relaxation time model [14].

Only for the near-field Fourier model [6] is thermal mismatch somewhat of a negligible effect, and solution behavior is similar to that for crack growth in a purely isotropic material. Model behavior is closer to the far-field result in that difference in the temperature of the crack faces is negligible. It is closer to that for double-relaxation time in that change in interface temperature is finite at the crack edge.

These observations have analogies in the transient response to an interface thermomechanical source of perfectly bonded materials that differ only in thermal property [19]. In particular, Stoneley signals for times after loading that exceed thermoelastic characteristic times are dominated by Fourier heat conduction. For times that are smaller than the characteristic times, Stoneley signals for the Fourier, single- and double-relaxation time models exhibit distinctive behavior.

The present work considered crack faces governed by convection and thermal relaxation. When convection is negligible, results in Ref. [8] showed that, save for the double-relaxation time model at high speeds, (asymptotic) difference in crack face temperature is also negligible. Both these and the present effort show that debonding of materials that differ only in thermal properties can produce behavior similar to that for isothermal bimaterials. Both efforts are phenomenological in that distinctive model behavior is illustrated through mathematical results. They can therefore pro-

vide a basis for calculations that would illustrate the relative importance of these behaviors and thus, perhaps, the validity of the models.

Appendix: General Transform Solution

After Refs. [23,24], the bilateral Laplace transform and inverse operations are introduced:

$$\hat{f} = \int_{-\infty}^{\infty} f(x) \exp(-px) dx \quad f(x) = \frac{1}{2\pi i} \int \hat{f} \exp(px) dp \quad (A1)$$

Here transform variable p is imaginary in the transform integral, and integration in the inversion integral is over a Bromwich contour. Application of the transform integral to Eq. (2) gives the eigenfunctions and eigenvalues

$$\exp(-p\xi A_i^\pm |y|) \quad \exp(-p\xi B |y|) \quad \xi = \frac{\sqrt{-p}}{\sqrt{p}} \quad (A2a)$$

$$A_i^\pm = \sqrt{1 - c_{i\pm}^2} \quad B = \sqrt{1 - c^2} \quad c_i^\pm = \frac{1}{\sqrt{a}} \eta_i^\pm c \quad (A2b)$$

$$2\eta_i^\pm = \sqrt{\left[1 + \sqrt{\frac{-ad_i^\pm}{ch_i p}}\right]^2 - \frac{\varepsilon_i d_i^{\pm II}}{ch_i p}} \pm \sqrt{\left[1 - \sqrt{\frac{-ad_i^\pm}{ch_i p}}\right]^2 - \frac{\varepsilon_i d_i^{\pm II}}{ch_i p}} \quad (A2c)$$

In Eq. (A2) $i=(1,2)$, and for the Fourier Model F and relaxation Models I and II,

$$\text{F: } d_i^\pm = d_i^{\pm II} = 1 \quad (A3a)$$

$$\text{I: } d_i^\pm = d_i^{\pm II} = 1 - ch_i^\pm p \quad (A3b)$$

$$\text{II: } d_i^\pm = 1 - ch_i^\pm p \quad d_i^{\pm II} = 1 - ch_i^{\pm II} p \quad (A3c)$$

Application of the transform integral to Eqs. (1), (3), (4), and (6) in light of Eq. (A2) then gives the displacement and temperature change transforms for the general problem. For present purposes, it is sufficient to display the results for solid 2 ($y < 0$):

$$\begin{bmatrix} \hat{u}_{x2} \\ \xi \hat{u}_{y2} \\ \hat{\theta}_2 \end{bmatrix} = \frac{1}{S} \begin{bmatrix} 1 & 1 & B \\ A_2^+ & A_2^- & 1 \\ -\Omega_2^+ & -\Omega_2^- & 0 \end{bmatrix} \begin{bmatrix} C_+ \exp(p\xi A_2^+ y) \\ C_- \exp(p\xi A_2^- y) \\ C_B \exp(p\xi B y) \end{bmatrix} \quad (A4a)$$

$$\begin{bmatrix} C_+ \\ C_- \\ C_B \end{bmatrix} = \begin{bmatrix} 2U_+ & 2KV_+ & \Theta_+ \\ 2U_- & 2KV_- & \Theta_- \\ -KS/2B & S & 0 \end{bmatrix} \begin{bmatrix} \hat{U} \\ \xi \hat{V} \\ \hat{\Theta}/c^2 \end{bmatrix} \quad (A4b)$$

$$(\hat{U}, \hat{V}, \hat{\Theta}) = \int_{-\infty}^0 (U, V, \Theta) \exp(-pt) dt \quad (A4c)$$

In Eq. (A4b) matrix coefficients

$$U_+ = \Omega_1^+ \Omega_2^- A_1^- (k_1 A_1^+ + k_2 A_2^-) - \Omega_1^- \Omega_2^- A_1^+ (k_1 A_1^- + k_2 A_2^-) + k_1 \Omega_1^+ \Omega_2^- A_1^- (A_1^+ - A_1^-) \quad (A5a)$$

$$V_+ = \Omega_1^- \Omega_2^- (k_1 A_1^- + k_2 A_2^-) - \Omega_1^+ \Omega_2^- (k_1 A_1^+ + k_2 A_2^-) + k_1 \Omega_1^+ \Omega_2^- (A_1^+ - A_1^-)$$

$$\Theta_+ = k_1 \Omega_1^+ A_1^+ (A_1^- + A_2^-) - k_1 \Omega_1^- A_1^- (A_1^+ + A_2^-) - k_2 \Omega_2^- A_2^- (A_1^+ - A_1^-)$$

$$U_- = \Omega_1^- \Omega_2^+ A_1^+ (k_1 A_1^- + k_2 A_2^+) - \Omega_1^+ \Omega_2^+ A_1^- (k_1 A_1^+ + k_2 A_2^+) - k_1 \Omega_1^+ \Omega_2^+ A_1^+ (A_1^- - A_1^+)$$

$$V_- = \Omega_1^+ \Omega_2^+ (k_1 A_1^+ + k_2 A_2^+) - \Omega_1^- \Omega_2^+ (k_1 A_1^- + k_2 A_2^+) - k_1 \Omega_1^+ \Omega_2^+ (A_1^+ - A_1^-) \quad (A5b)$$

$$\Theta_- = k_1 \Omega_1^- A_1^+ (A_1^+ + A_2^+) - k_1 \Omega_1^+ A_1^- (A_1^- + A_2^-) + k_2 \Omega_2^+ A_2^+ (A_1^+ - A_1^-)$$

$$\begin{aligned} S = & (k_1 \Omega_1^+ \Omega_1^- + k_2 \Omega_2^+ \Omega_2^-) (A_1^+ - A_1^-) (A_2^+ - A_2^-) - \Omega_1^+ \Omega_2^- (k_1 A_1^+ + k_2 A_2^-) \\ & \times (A_1^- + A_2^+) + \Omega_1^- \Omega_2^+ (k_1 A_1^- + k_2 A_2^+) (A_1^+ + A_2^-) + \Omega_1^+ \Omega_2^+ (k_1 A_1^+ \\ & + k_2 A_2^+) (A_1^- + A_2^-) - \Omega_1^- \Omega_2^- (k_1 A_1^- + k_2 A_2^-) (A_1^+ + A_2^+) \end{aligned} \quad (A5c)$$

In Eqs. (A4) and (A5)

$$\Omega_i^\pm = \frac{p q_i^\pm}{\alpha_{vi} d_i^{\text{II}}} \quad q_i^\pm = 1 - \eta_i^{\pm 2} \quad q_i^+ q_i^- = \frac{\varepsilon_i d_i}{ch_i p} \quad K = c^2 - 2 \quad (A6)$$

In Eq. (A6) $i=(1,2)$, and for the Fourier Model F and thermal relaxation Models I and II

$$\text{F: } d_i = d_i^{\text{II}} = 1 \quad (A7a)$$

$$\text{I: } d_i = 1 - ch_i^{\text{I}} p \quad d_i^{\text{II}} = 1 \quad (A7b)$$

$$\text{II: } d_i = d_i^{\text{II}} = 1 - ch_i^{\text{II}} p \quad (A7c)$$

The real parts of the exponential arguments must be non-negative if Eq. (A4a) is to be bounded as $|y| \rightarrow \infty$. Because radical B is positive real for $|c| < 1$, its term satisfies the condition if we require that $\text{Re}(\sqrt{\pm p}) \geq 0$ in the p plane with branch cuts $\text{Im}(p) = 0$, $\text{Re}(p) < 0$ and $\text{Im}(p) = 0$, $\text{Re}(p) > 0$, respectively. The situation is more complicated for the $A_i^\pm$ terms. Nevertheless, it can be shown for real p that branch points $|c| = \sqrt{a}/\eta_i^\pm$ of radical $A_i^\pm$ are real and that

$$0 < \eta_i^- < 1 < \eta_i^+ \quad \frac{\sqrt{a}}{\eta_i^+} < 1 \quad (A8a)$$

$$ch_i p < \frac{m + \varepsilon_i}{\varepsilon_i l_i + m(l_i^{\text{I}} - 1)} \cdot \frac{\sqrt{a}}{\eta_i^-} < 1 \quad (A8b)$$

In Eq. (A8) $i=(1,2)$ and dimensionless parameters (l_i, l_i^{I}) are defined as

$$\text{F: } l_i = l_i^{\text{I}} = 0 \quad (A9a)$$

$$\text{I: } l_i = l_i^{\text{I}} = \frac{h_i^{\text{I}}}{h_i} \quad (A9b)$$

$$\text{II: } l_i = \frac{h_i^{\text{II}}}{h_i} \quad l_i^{\text{I}} = \frac{h_i^{\text{I}}}{h_i} \quad (A9c)$$

Thus for real p , positive real arguments of Eq. (A4a) are guaranteed when

$$c < c^* = \min \left(1, \frac{\sqrt{a}}{\eta_1^+}, \frac{\sqrt{a}}{\eta_2^+} \right) \quad (A10)$$

For the moment, therefore, Eq. (A10) defines a subcritical separation zone extension rate.

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Analytical Solutions for a Family of Gaussian Impinging Jets

Various types of impinging jet flows are analytically modeled using inviscid free Gaussian jet solutions superimposed with experimentally fitted boundary layer models. Improved (more robust) and simplified solutions to existing models are defined. Velocity profiles, surface pressure distributions, and streamline plots are calculated for circular, plane, and annular impinging jets. The models show excellent agreement with existing experimental results in both laminar and turbulent conditions and for different Reynolds numbers.
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1 Introduction

Impinging jets have extensive practical applications ranging from industrial processing, such as mixing, heating/cooling, or drying, to environmental flows related to ventilation or the simulation of downburst winds. The mathematical modeling of the flow field has therefore both fundamental and practical importance.

A three region modeling method for round impinging jets was put forward and the flow field of the wall jet region was broadly investigated [1–3]. The normalized radial velocity profile derived by Glauert [1] and verified by Bakke [2] is often cited in many references. However, it cannot be used to predict the flow field in the impinging jet region.

Yih [4] obtained a solution of steady rotational flow equations of an inviscid fluid describing rapidly varying flows such as flows from a two-dimensional channel or a circular tube toward a sink. A similar solution for an impinging jet was found by Phares et al. [5]. They derived a double layer summarized infinite series solution that includes surface integrals to be computed numerically. The solution can be considered as an analytical-numerical mixed solution.

In 1998, Lee et al. [6] derived a pure analytical solution for a simple Gaussian impinging jet and applied the solution to the flow field research for helicopter vanes.

Herein, we improve the analytical model of Lee et al. and extend it to the general case of a family of Gaussian impinging jets. The extended solution is more adequate for modeling an extended set of engineering problems involving round impinging jets [7]. A direct solution for the steady rotational flow equations subject to the inhomogeneous boundary conditions is obtained. An approximate method for computing the oscillatory series of the solution is proposed and proved suitable for general oscillatory series. Based on this, a complete flow field of the Gaussian impinging jets can be readily computed. The approximate solution for the oscillatory series accelerates convergence and adds robustness to the model when compared to previous ones.

2 Formulation

2.1 Governing Equations and Boundary Conditions. In accordance with the formulation of Lee et al. [6], all lengths are nondimensionalized with jet radius R^* (or half-width of the slot B_0^* for the plane jet) and all velocities are nondimensionalized with the maximum influx velocity w_m . Flow field pressures are nondimensionalized by the maximum dynamic pressure at the jet centerline. To simplify notations, the dimensional variables are marked with *. The dimensionless variables are given as

$$r = \frac{r^*}{R^*} \quad z = \frac{z^*}{R^*} \quad \text{for round jets and} \quad (1)$$

$$x = \frac{x^*}{B_0^*} \quad z = \frac{z^*}{B_0^*} \quad \text{for plane jets}$$

$$u = \frac{u^*}{w_m} \quad w = \frac{w^*}{w_m} \quad p = \frac{2p^*}{\rho w_m^2} \quad (2)$$

The governing equations written in stream function and vorticity form are

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} = -r^2 \Omega(r, z) \quad (3)$$

$$\frac{\partial \psi}{\partial r} = rw \quad \frac{\partial \psi}{\partial z} = -ru \quad (4)$$

$$\Omega = \frac{1}{r} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \right) \quad (5)$$

in cylindrical-polar coordinates for the round jet case and

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x^2} = -\Omega(x, z) \quad (6)$$

$$\frac{\partial \psi}{\partial x} = w \quad \frac{\partial \psi}{\partial z} = -u \quad (7)$$

$$\Omega = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \quad (8)$$

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in Cartesian coordinates for the plane jet case.

The boundary conditions can be approximately expressed according to a finite domain [5,6], but here, precise expressions can be used for an infinite domain:

$$\psi = \int_0^r r w_\infty dr \quad \text{at } z = \infty \quad (9)$$

$$\psi = 0 \quad \text{at } z = 0 \quad (10)$$

$$\psi = 0 \quad \text{at } r = 0 \quad (11)$$

$$\frac{\partial \psi}{\partial r} = 0 \quad \text{at } r = \infty \quad (12)$$

for the round jet case and

$$\psi = \int_0^x w_\infty dx \quad \text{at } z = \infty \quad (13)$$

$$\psi = 0 \quad \text{at } z = 0 \quad \text{and at } x = 0 \quad (14)$$

$$\frac{\partial \psi}{\partial x} = 0 \quad \text{or } \psi = \int_0^z -u_\infty dz \quad \text{at } x = \infty \quad (15)$$

for the plane jet case.

2.2 Influx Velocity Profiles. Several empirical expressions are available to describe the jet influx velocity profile as a function of the nozzle shape, the spacing between the jet outlet and the impinging surface, and the flow state. The following Gaussian jet profile:

$$-w_\infty = e^{-r^2/k} \quad (16)$$

is chosen because Eq. (9) has an exact solution with this influx velocity profile and the spacing distance (H^*) can be related to the final solution as well.

Firstly, in order to find a relation between Gaussian jets and a jet from a round pipe, their flow rates can be compared as follows.

Considering an influx flowing out from a pipe with $r=1$ and the uniform velocity distribution (step profile, $w_\infty=-1$ for $r \leq 1$, $w_\infty=0$ elsewhere):

$$Q_0 = \int_0^1 -2\pi r dr = -\pi \quad (17)$$

Using the influx profile of Eq. (16) and integrating from $r=0$ to ∞ , the flow rate corresponding to the Gaussian jet is obtained:

$$Q = \int_0^\infty w_\infty 2\pi r dr = \int_0^\infty -e^{-r^2/k} 2\pi r dr = -k\pi \quad (18)$$

Obviously, the Gaussian influx velocity profile (16) provides k times the unit flow rate (kQ_0).

Secondly, a turbulent jet produces flow entrainment into the shear layer, hence increasing its flow rate as it moves downward. In order to take into account the entrainment effects into the inviscid model, the relation between the flow rate multiplier (k) and the distance from the jet to the impinging plate (H^*) should be specified.

According to Abramovich's experiments for a free round turbulent jet (Ref. [8], pp. 20–26) the parameter k and the distance from the jet outlet to a downstream flow station (H^*) can be related by

$$k = \frac{Q}{Q_0} = 2.2 \left(a_1 \frac{H^*}{R^*} + b_1 \right) \quad (19)$$

where a_1 is a jet shape coefficient, typically taking $a_1 = 0.06$ – 0.08 ; b_1 is an empirical constant chosen as $b_1 = 0.294$; H^*

is the axial distance from the jet exit.

Hence, in order to include the entrainment effects, a flow rate Q ($=kQ_0$) can be used as the influx. The parameter k can be regarded as a linkage between the infinite and finite domains and introduced into the model.

For the same reason, the following influx profile is chosen for the plane jet case:

$$-w_\infty = (1 + 2x/k)e^{-2x/k} \quad (20)$$

and an empirical expression for k is given by (Ref. [8], pp. 27–28)

$$k = \frac{Q}{Q_0} = 1.2 \sqrt{\frac{a_2 H^*}{B_0^*} + b_2} \quad (21)$$

where a_2 is the jet shape coefficient, $a_2 = 0.10$ – 0.11 ; b_2 is an empirical constant, $b_2 = 0.41$.

2.3 Analytical Solutions for a Family of Gaussian Impinging Jets. An infinite series solution is obtained for the steady axial-symmetric flow (Eq. (3)) given the inhomogeneous mixed boundary conditions (Eqs. (9)–(12)) with the influx velocity profile (Eq. (16)).

(a) The stream function is

$$\psi = -\frac{k}{2}(1 - e^{-r^2/k}) + r^2 e^{-r^2/k} \sum_{n=1}^{\infty} c_n L_n^1 \left(\frac{2r^2}{k} \right) e^{-\sqrt{8n/k}z} \quad (22)$$

with

$$c_n = \frac{\int_0^\infty (k/2)(1 - e^{-r^2/k}) L_n^1(2r^2/k) e^{-r^2/k} r dr}{\int_0^\infty r^2 e^{-2r^2/k} [L_n^1(2r^2/k)]^2 r dr} = \frac{(-1)^n}{n \times n!} \quad (23)$$

where $L_n^1(x)$ are associated Laguerre polynomials.

As r and z are small, the solution shows that $\psi \sim z r^2$.

The velocity solutions are as follows:

(b) The radial velocity is

$$u(r, z) = r e^{-r^2/k} \sum_{n=1}^{\infty} c_n L_n^1 \left(\frac{2r^2}{k} \right) \sqrt{8n/k} e^{-\sqrt{8n/k}z} \quad (24)$$

(c) The axial velocity is

$$w(r, z) = -e^{-r^2/k} + 2 \left(1 - \frac{r^2}{k} \right) e^{-r^2/k} \sum_{n=1}^{\infty} c_n L_n^1 \left(\frac{2r^2}{k} \right) e^{-\sqrt{8n/k}z} + r e^{-r^2/k} \sum_{n=2}^{\infty} c_n e^{-\sqrt{8n/k}z} \frac{\partial L_n^1(2r^2/k)}{\partial r} \quad (25)$$

For the particular cases of the plate surface ($z=0$) and the jet centerline ($r=0$), the associated velocities are obtained as

$$u(r, 0) = r e^{-r^2/k} \sum_{n=1}^{\infty} c_n L_n^1 \left(\frac{2r^2}{k} \right) \sqrt{\frac{8n}{k}} \quad (26)$$

$$w(0, z) = -1 + 2 \sum_{n=1}^{\infty} (-1)^{n-1} e^{-\sqrt{8n/k}z} \quad (27)$$

Lee et al. [6] provided a derivation for their solution of a simple Gaussian jet ($k=1$ in Eq. (16)) impinging onto a plate. In the following, we extend the result to the general case (k chosen arbitrarily).

Derivation. First, using Eq. (16) as the influx condition at $z = \infty$, the influx stream function can be expressed by

$$\psi_\infty = \int_0^r r(-e^{-r^2/k})dr = -\frac{k}{2}(1 - e^{-r^2/k}) \quad (28)$$

The initial azimuthal vorticity distribution is defined by Eq. (5) as

$$\Omega = \Omega_\infty = -\frac{1}{r} \frac{\partial w_\infty}{\partial r} = -\frac{2}{k} e^{-r^2/k} \quad (29)$$

The conservation laws for axial-symmetric flow require Ω to be constant on streamlines [6], so that

$$\Omega = H(\psi) \quad (30)$$

The stream function ψ still satisfies Eq. (3) but now takes the special form

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} = -r^2 H(\psi) \quad (31)$$

The azimuthal vorticity in the undisturbed jet at upstream infinity is given by Eq. (29). Making use of Eqs. (28) and (30), we have

$$H(\psi_\infty) = -\frac{4}{k^2} \left(\psi_\infty + \frac{k}{2} \right) \quad (32)$$

$H(\psi_\infty)$ is invariant throughout the flow field. Hence, Eq. (31) simplifies to a linear partial differential equation:

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} = \frac{4r^2}{k^2} \left(\psi + \frac{k}{2} \right) \quad (33)$$

Letting $\Psi = \psi + k/2$, Eq. (33) yields

$$\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi}{\partial r} = \frac{4r^2}{k^2} \Psi \quad (34)$$

A particular solution of this equation for ψ is given by Eq. (28). The homogeneous equation can be solved by the method of separation of variables.

Letting $\Psi = F(r) \cdot G(z)$, then

$$F \frac{\partial^2 G}{\partial z^2} + G \frac{\partial^2 F}{\partial r^2} - \frac{G}{r} \frac{\partial F}{\partial r} - \frac{4r^2}{k^2} FG = 0 \quad (35)$$

The above equation can be rewritten into

$$\frac{1}{F} \frac{\partial^2 F}{\partial r^2} - \frac{1}{rF} \frac{\partial F}{\partial r} - \frac{4r^2}{k^2} = -\frac{1}{G} \frac{\partial^2 G}{\partial z^2} = \alpha \quad (36)$$

This leads to the following group equations:

$$G'' + \alpha G = 0 \quad (37)$$

$$F'' - \frac{1}{r} F' - \left(\frac{4r^2}{k^2} + \alpha \right) F = 0 \quad (38)$$

Letting $t = 2r^2/k$ and $\alpha = -8n/k$ in Eq. (38), we have

$$tF_t'' + \left(n - \frac{t}{4} \right) F_t' = 0 \quad (39)$$

Further, let $F_t = E_t(t/2)e^{-t/2}$. Then Eq. (39) becomes

$$tE_t'' + (2-t)E_t' + (n-1)E_t = 0 \quad (40)$$

Equation (40) is a standard Laguerre associated differential equation. The solutions of Eq. (40) are given by the associated Laguerre polynomials (Ref. [10], pp. 155–156). By means of backward substitutions, we obtain

$$F_n = L_n^1 \left(\frac{2r^2}{k} \right) \frac{r^2}{k} e^{-r^2/k} \quad \text{for } \alpha = -\frac{8n}{k} \quad n = 1, 2, 3, \dots \quad (41)$$

Substituting this into Eq. (37) yields

$$G_n = a_n e^{-\sqrt{8n/k}z} + b_n e^{\sqrt{8n/k}z} \quad (42)$$

G_n must be a bounded function so that b_n is equal to zero. Combining Eqs. (28), (41), and (42), a complete solution in the form of a series of associated Laguerre polynomials Eq. (22) is obtained (in that equation, $c_n = a_n/k$).

Boundary conditions (9), (11), and (12) are satisfied automatically. For the wall boundary condition (Eq. (10)), setting $z=0$ in Eq. (22), multiplying by $e^{-r^2/k} L_m^1(2r^2/k)$, then integrating from $r=0$ to ∞ , the orthogonality of Laguerre polynomials leads to a set of equations and yields the coefficients c_n (Eq. (23)).

Finally, substituting Eq. (22) into Eq. (4) yields the radial and axial velocities (Eqs. (24) and (25)), as we claimed.

Setting $k=1$, Eq. (22) becomes the simple Gaussian impinging jet solution obtained by Lee et al. [6], which is a particular case in the solution family. Note that their expression for coefficients $c_n = (-1)^n/(n+1)$ is not correct (most likely a print error), conducting to a divergent solution.

2.4 Analytical Solutions for Plane Impinging Jets. For the plane impinging jets, a solution similar to Eq. (22) exists as well. A simpler analytical solution is found and presented herein:

(a) Stream function:

$$\psi = \frac{1}{k} [k - (k+x)e^{-2x/k}] [(k+z)e^{-2z/k} - k] \quad (43)$$

It is easy to see that $\psi \sim xz$ as x and z are small.

(b) Velocity in the x direction:

$$u(x, z) = \left[1 - \left(1 + \frac{x}{k} \right) e^{-2x/k} \right] \left(1 + \frac{2z}{k} \right) e^{-2z/k} \quad (44)$$

(c) Velocity in the z direction:

$$w(x, z) = \left[\left(1 + \frac{z}{k} \right) e^{-2z/k} - 1 \right] \left(1 + \frac{2x}{k} \right) e^{-2x/k} \quad (45)$$

For the particular cases of the impinging plate surface and the jet centerline, we have the associated velocities expressed as

$$u(x, 0) = 1 - \left(1 + \frac{x}{k} \right) e^{-2x/k} \quad (46)$$

$$w(0, z) = \left(1 + \frac{z}{k} \right) e^{-2z/k} - 1 \quad (47)$$

Derivation. Let $\psi = CX(x) \cdot Z(z)$. Then integrating Eq. (20) from 0 to x , we obtain

$$X(x) = X_{z=\infty} = \int_0^x w dx = k - (k+x)e^{-2x/k} \quad (48)$$

Due to symmetry,

$$Z(z) = Z_{x=\infty} = - \int_0^z u dz = (k+z)e^{-2z/k} - k \quad (49)$$

Substituting the above results into the stream function, we obtain

$$\psi = C [k - (k+x)e^{-2x/k}] [(k+z)e^{-2z/k} - k] \quad (50)$$

Note that the velocity at the infinite centerline should reach unity, i.e.,

$$-w(z=\infty, x=0) = 1 \quad (51)$$

This leads to $C=1/k$, and Eq. (43) is obtained.

The above solution can be verified to satisfy both the differential equations (6)–(8) and the boundary conditions (13)–(15) by direct substitution.

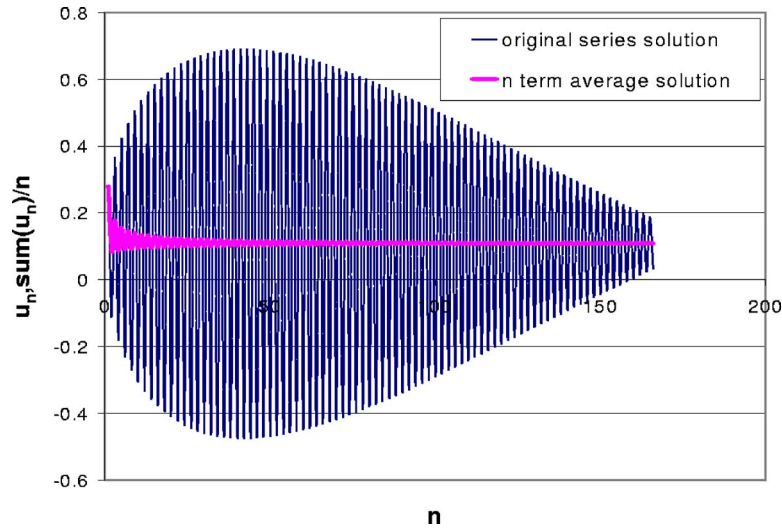


Fig. 1 Comparison of front n term average value series

3 Implementation for Round Impinging Jets

It is necessary to develop a method to obtain a stable convergent result for the flow field parameters from Eqs. (22), (24), and (25), because all of the parameters are expressed by oscillatory series. A front n term averaged method is put forward and found efficient for the flow field computations. The method is based on the following theorem.

THEOREM. For a given convergent oscillatory series, the average values of the first n terms constitute a series that will converge to the same value as that of the original series and the first n term average value series has a faster convergence speed.

The detailed proof of the above theorem is given in the Appendix.

Lee et al. [6] showed that the solution of a simple Gaussian impinging jet converges with terms decaying as $n^{-1/4}$. The present method produces a much faster converging rate of n^{-1} . For example, if we search for solutions of accuracy of 1%, we need 10^8 terms in the original series while we need 100 terms only in the front n term average series. Moreover, as it will be demonstrated later, even if the original series is not convergent, the average value series can still provide a “convergent value.”

Figure 1 illustrates this method with a comparison between the original velocity series solution and its front n term average value solution. Accordingly, the maximum relative error of the front n term average value series is less than $1/n$.

Fortunately, all parameters, including the average value series, can be expressed by recurrence formulas, which allow us to obtain the desired accuracy. Here are the recurrence formulas used in the implementation (Eq. (53) can be found in a mathematical handbook, see Ref. [9], p. 156):

$$c_{n+1} = -c_n \frac{n}{(n+1)^2} \quad (52)$$

$$L_1^1 = -1 \quad L_2^1 = 2 \left(\frac{2r^2}{k} - 2 \right)$$

$$L_n^1 = \frac{n}{n-1} \left\{ \left[2(n-1) - \frac{2r^2}{k} \right] L_{n-1}^1 - (n-1)^2 L_{n-2}^1 \right\} \quad n = 3, 4, \dots \quad (53)$$

$$\frac{\partial L_1^1}{\partial r} = 0 \quad \frac{\partial L_2^1}{\partial r} = \frac{8r}{k}$$

$$\frac{\partial L_n^1}{\partial r} = \frac{n}{n-1} \left\{ \left[2(n-1) - \frac{2r^2}{k} \right] \frac{\partial L_{n-1}^1}{\partial r} - (n-1)^2 \frac{\partial L_{n-2}^1}{\partial r} \right\} - \frac{4nr}{(n-1)k} L_{n-1}^1 \quad n = 3, 4, \dots \quad (54)$$

The origin $O(0,0)$ is a singular point for Eq. (3). The solution displays this feature from Eq. (27), since the axial velocity is not convergent at the origin:

$$w(0, 0) = -1 + 2 - 2 + \dots, \quad \text{i.e.,} \quad w_1(0, 0) = 1 \quad w_2(0, 0) = -1, \dots \quad w_n(0, 0) = (-1)^{n-1} \quad (55)$$

With the aid of the front n term averaged method, a convergent value $w(0,0)=0$ is obtained, which is its physical velocity value.

For any other points in the domain, stable convergent flow field parameters can be obtained using the above method.

4 Boundary Layer Approximation

Comparing the solution of plane stagnation point flow and that of axial-symmetric stagnation flow (Ref. [10], pp. 152–159), one can see that the boundary layer thickness of a plane stagnation flow is 1.2 times of that of the axial symmetric:

$$\delta_{\text{plane}}^* = 1.2 \delta_{\text{asym}}^* \quad (56)$$

The above expression can be used to determine the boundary layer depth for plane jet impingement as long as the boundary layer depth for the round jet case has been determined or vice versa.

Based on impinging jet experiments [7], the following empirical expressions were obtained to describe the dimensionless boundary layer depth for the round impinging jet:

$$\delta = \frac{4.5}{\sqrt{\text{Re}_j}} \sqrt{\frac{r}{u_s}} \quad \text{for local } \text{Re}_j < 2.5 \times 10^4 \quad (57)$$

$$\frac{\sqrt{r}}{\sqrt{\delta}} = -1.95 \ln \left(\frac{\Delta}{0.16 R^*} + \frac{278}{\text{Re}_j u_s r \sqrt{\delta}} \right) \quad \text{for } \text{Re}_j > 2.5 \times 10^5 \quad (58)$$

where Re_j is the jet Reynolds number, $\text{Re}_j = w_{\text{jet}}^* R^* / 2\nu$; Re_l is the local Reynolds number, $\text{Re}_l = u_s^* r^* / \nu$; Δ is the roughness height (in m); ν is the kinematical viscosity (in m^2/s); u_s^* is the inviscid surface velocity (in m/s).

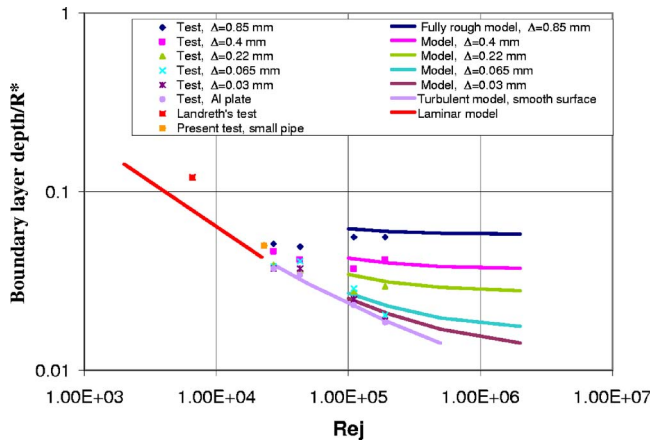


Fig. 2 Comparison of the impinging jet boundary layer depths between empirical models and roughness test data (Δ is the roughness height of the sandpaper)

Note that r , δ , and u_s above are dimensionless variables.

Figure 2 shows a comparison of the empirical expressions (57) and (58) with the boundary layer depth experimental data. When the Reynolds number is less than a critical value ($Re_j \approx 2.5 \times 10^4$), the flow is Reynolds number dependent. Above one order of the critical value, an asymptotic state is reached where the boundary layer increases with the surface roughness.

The displacement depths can be estimated by (Ref. [11], pp. 30–36)

$$\delta_{\text{disp}} = \delta/3 \quad \text{for a laminar flat plate boundary layer} \quad (59)$$

and

$$\delta_{\text{disp}} = \delta/7 \quad \text{for a turbulent flat plate boundary layer} \quad (60)$$

The radial velocity profiles of the impinging jet for the combined model (inviscid solution plus the boundary layer approximation) are presented in the following two subsections.

4.1 Laminar Boundary Layer Case. We divide the z domain into three regions ($[0 \sim \delta]$, $[\delta \sim 4\delta/3]$, and $[4\delta/3 \sim \infty]$) and obtain the combined velocity profiles by the following steps:

- We displace the inviscid velocity profile up a distance of δ_{disp} from the ground surface.
- In the outer region, we use the displaced inviscid velocity profile:

$$u(r, z) \leftarrow u(r, z - \delta/3) \quad \text{for } z \geq 4\delta/3 \quad (61)$$

- In the region of $z = [\delta \sim 4\delta/3]$, we use a parabolic equation to smooth the curve to the maximum value of u . The parabolic equation is subjected to the following three constraints:

$$\begin{aligned} u(r, 4\delta/3) &\leftarrow u(r, \delta) & \partial u / \partial z(r, 4\delta/3) &\leftarrow \partial u / \partial z(r, \delta) \\ \partial u / \partial z(r, \delta) &= 0 \end{aligned} \quad (62)$$

This yields

$$u(r, z) = A_1 z^2 + B_1 z + C_1 \quad (63)$$

with

$$\begin{aligned} A_1 &= \frac{3u'_z(r, \delta)}{2\delta} & B_1 &= -3u'_z(r, \delta) \\ C_1 &= u(r, \delta) + \frac{4}{3}\delta u'_z(r, \delta) \end{aligned} \quad (64)$$

and the maximum velocity

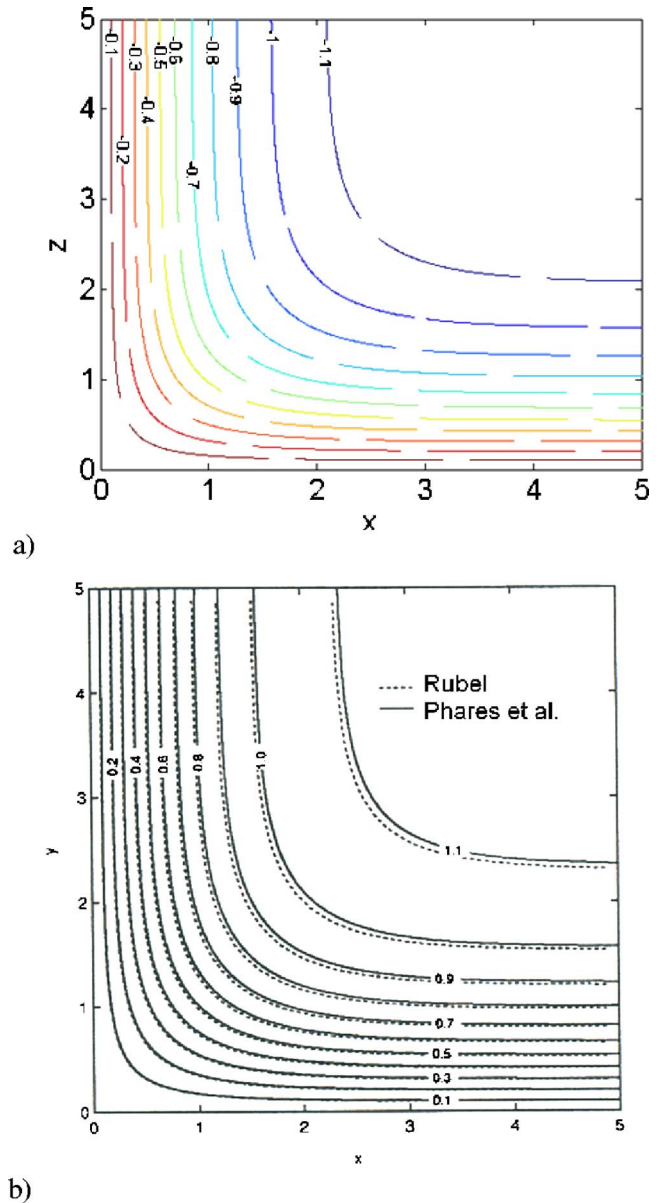


Fig. 3 Streamlines of a plane impinging jet model: (a) present model ($-\psi$ corresponds to Rubel's ψ value); (b) calculated by Rubel and Phares et al., cited from Ref. [5]

$$u_{\text{max}}(r, \delta) = u(r, \delta) - \frac{1}{6}\delta u'_z(r, \delta) \approx \frac{1}{2} \left[u \left(r, \frac{2}{3}\delta \right) + u(r, \delta) \right] \quad (65)$$

where

$$u'_z(r, z) = -re^{-r^2/k} \sum_{n=1}^{\infty} c_n \frac{8n}{k} L_n^1 \left(\frac{2r^2}{k} \right) e^{-(\sqrt{8n/k})z} \quad (66)$$

- Within the boundary layer ($z < \delta$), we use the polynomial approximation of Homman's boundary layer velocity profile (data from Ref. [10], p. 156, Tables 3–4):

$$\begin{aligned} u(r, \eta) &= (-1.43\eta^4 + 5.39\eta^3 - 7.45\eta^2 \\ &\quad + 4.49\eta)u_{\text{max}}(r, \delta) \quad \text{for } \eta \leq 1 \end{aligned} \quad (67)$$

where $\eta = z/\delta$.

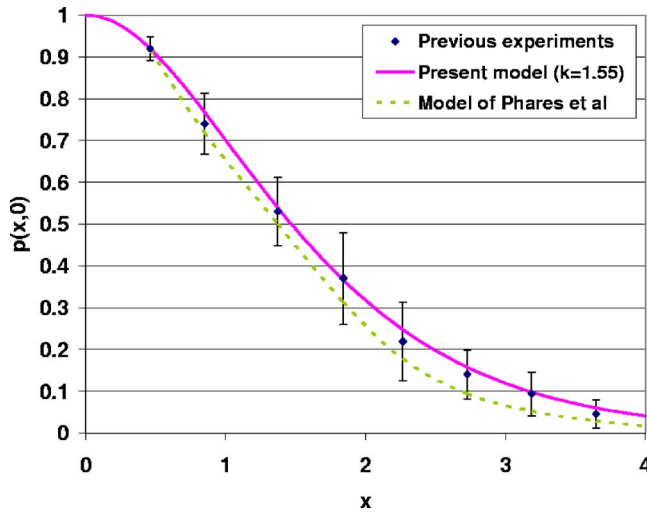


Fig. 4 Comparison of surface pressure distribution (the error bar is for the experiments)

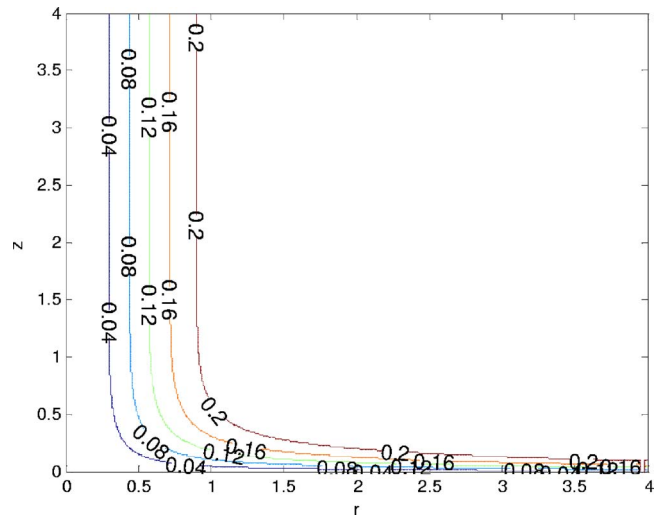


Fig. 5 Laminar circular impinging jet streamlines ($-\psi$ is used)

4.2 Turbulent Boundary Layer Case. Similar to the laminar case, we divide the z domain into three parts ($[0 \sim \delta]$, $[\delta \sim 8\delta/7]$ and $[8\delta/7 \sim \infty]$) and obtain the radial velocity profiles by the following steps:

- We displace the inviscid velocity profile up a distance of δ_{disp} from the ground surface.
- In the outer region, $z=[8\delta/7 \sim \infty]$, we use the displaced inviscid velocity profile:

$$u(r, z) \leftarrow u(r, z - \delta/7) \quad \text{for } 8\delta/7 \leq z \quad (68)$$

- In the region of $z=[\delta \sim 8\delta/7]$, we use a parabolic equation to smooth the curve to the maximum value of u . The parabolic equation is subjected to the following three constraints:

$$\begin{aligned} u(r, 8\delta/7) &\leftarrow u(r, \delta) & \partial u / \partial z(r, 8\delta/7) &\leftarrow \partial u / \partial z(r, \delta) \\ \partial u / \partial z(r, \delta) &= 0 \end{aligned} \quad (69)$$

This yields

$$u(r, z) = A_2 z^2 + B_2 z + C_2 \quad (70)$$

with

$$\begin{aligned} A_2 &= 7u'_z(r, \delta)/2\delta & B_2 &= -7u'_z(r, \delta) \\ C_2 &= u(r, \delta) + \frac{24}{7}\delta u'_z(r, \delta) \end{aligned} \quad (71)$$

and the maximum velocity

$$u_{\text{max}}(r, \delta) = u(r, \delta) - \frac{1}{14}\delta u'_z(r, \delta) \approx \frac{1}{2} \left[u\left(r, \frac{6}{7}\delta\right) + u(r, \delta) \right] \quad (72)$$

- Within the boundary layer ($z \leq \delta$), we use Eq. (67) or any similar expressions for turbulent boundary layers.

5 Results and Comparisons

Example 1. Plane impinging jets. Equation (27) can be used to calculate the values of stream functions with different k . Figure 3(a) is an example of the mapped streamlines in a 5×5 domain (set $k=1.16$), which matches the numerical and analytical-numerical mixed results, see Fig. 3(b), cited from Ref. [5].

According to the definition (see Eq. (2)), the normalized pressures on the surface of the impinging jet plate can be determined from Bernoulli's equation, i.e.,

$$p(x, 0) = 1 - u^2(x, 0) \quad (73)$$

where $u(x, 0)$ is the surface radial velocity from the inviscid model.

Phares et al. [5] compared the surface pressure distribution for a plane impinging jet with previous experiments including these

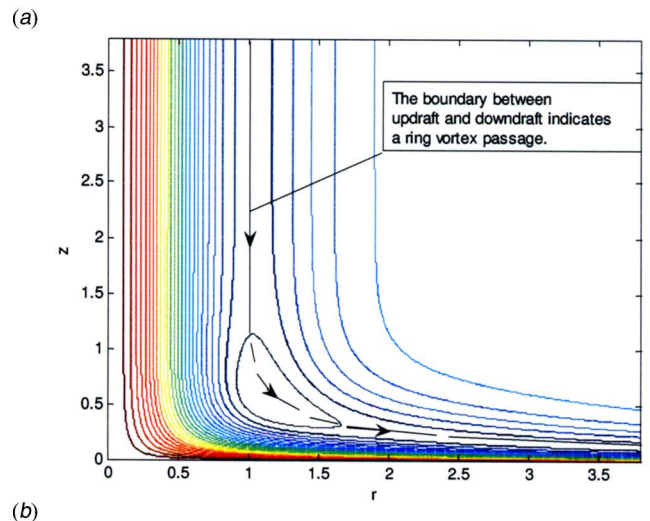
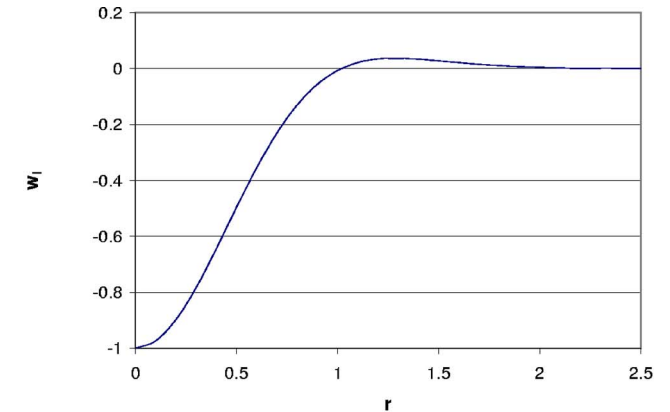


Fig. 6 An axis symmetric impinging jet combined model with updraft and downdraft: (a) influx profile; (b) contours of stream function

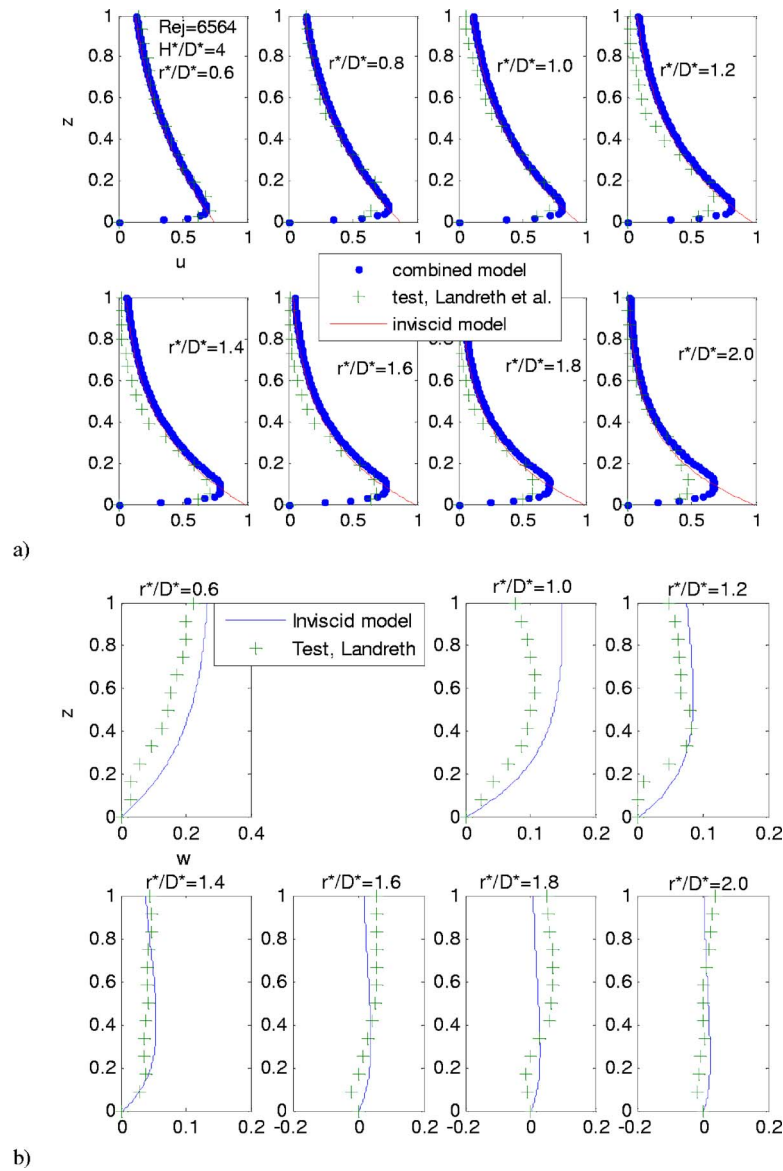


Fig. 7 Velocity comparison (model: $k=2.05$; test: $H^*/D^*=4$, cited from Ref. [17]): (a) radial velocity profiles; (b) axial velocity profiles

by Kumada and Mabuchi [12], and Beltaos and Rajaratnam [13]. The present model shows improved fitting of the experimental data (see Fig. 4). In Fig. 4, $k=1.55$ is chosen for the present model according to the fully developed jet condition of $H^*/B_0^*=12$.

Example 2. Axis symmetric impinging jets.

(a) *Streamlines for a laminar axis symmetric impinging jet.* For a laminar flow in a circular pipe (Hagen-Poiseuille flow), the velocity profile is parabolic about the centerline, and the average velocity is equal to half of the centerline velocity (Ref. [11], p. 116):

$$\bar{u} = \frac{1}{2} u_{\max} \quad (74)$$

where $u_{\max}$ is the centerline velocity. Hence, $k=1/2$ is chosen to model the laminar impinging jet, i.e., the influx velocity into the domain at $z=\infty$ is

$$w_{\infty} = -e^{-2r^2} \quad \text{and} \quad \psi_{\infty} = -\frac{1}{4}(1 - e^{-2r^2}) \quad (75)$$

Figure 5 shows the streamlines of the laminar impinging Gaussian jet based on the analytical model.

(b) *Vortex trace for a turbulent axis symmetric impinging jet.* The trace of vortices in the mixing layer of a turbulent impinging jet is determined using the analytical model. In order to generate a vortex trace using this inviscid model, a mixed influx velocity profile is used with both down flow and up flow in conjunction with the inviscid model. The boundary between the two opposite direction velocities can be considered the trace of the ring vortices. The following influx velocity function has the mixed shape with a velocity value of 0 at $r \approx 1$:

$$w_{\infty} = -2e^{-2r^2} + e^{-4r^2/3} \quad (76)$$

This combined influx velocity profile, shown in Fig. 6(a), corresponds to twice a downdraft of $k=0.5$ and an updraft of $k=0.75$.

The stream function is now expressed by $2\psi(0.5) - \psi(0.75)$, where $\psi(k)$ is defined by Eq. (22). The corresponding streamlines are shown in Fig. 6(b).

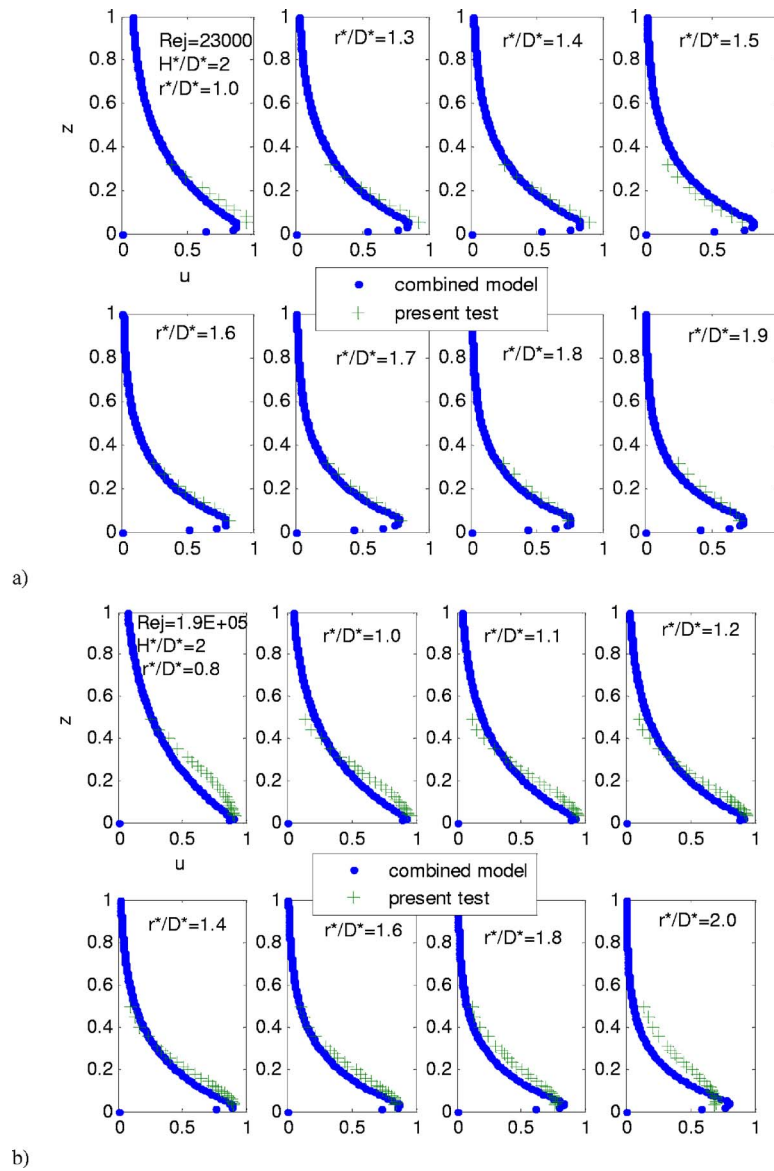


Fig. 8 Radial velocity comparisons for turbulent circular impinging jets: (a) $Re_j = 23000$, $H^*/D^* = 2$; (b) $Re_j = 190000$, $H^*/D^* = 2$.

(c) *Velocity field of experimental axis symmetric impinging jets.* In order to compare the combined model with experiments, the parameter k is determined according to the particular test conditions. Landreth and Adrian [14] presented the velocity profiles of a turbulent circular jet based on PIV measurements ($D^* = 26.9$ mm, $H^*/D^* = 4$, $Re_j = 6564$, fluid: water). Substituting $H^*/R^* = 2H^*/D^* = 8$ into Eq. (15), the parameter k is obtained as

$$k = 2.2(0.08 \times 8 + 0.294) = 2.05 \quad (77)$$

Therefore, the flow parameters of a Gaussian impinging jet of $k = 2.05$ are computed and compared with Landreth and Adrian's experiments.

Figure 7(a) compares the radial velocity profiles for the $H^*/D^* = 4$ case. The inviscid model is modified with a laminar boundary layer approximation described in Sec. 4.1. It is shown that the model compares well against the experiments. Figure 7(b) compares the axial velocity profiles for the same case. It can be observed that for r^*/D^* in $[0.8, 1.4]$, the model and experimental profiles are consistent. For $r^*/D^* > 1.6$, some negative axial velocity values are observed in the near surface experimental results that are not reproduced by the model.

Figure 8(a) compares the radial velocity profiles for another case. The experimental results are from measurements taken with a boundary layer probe in a small impinging jet facility [7] and correspond to $H^*/D^* = 2$ and $Re_j = 23,000$. Again, the inviscid model is modified with a laminar boundary layer approximation and shows good agreement with the experiments.

Figure 8(b) compares the radial velocity profiles from a larger jet facility test for $H^*/D^* = 2$ and $Re_j = 190,000$ [7]. Here, the inviscid model is modified with a turbulent boundary layer approximation described in Sec. 4.2. While, overall, the model agreement with the test data is fair, note the difficulty in reproducing the thinner surface layer at this higher Reynolds number, see also Fig. 2.

(d) *Surface pressures of turbulent axis symmetric impinging jets.* For axis symmetric impinging jet cases, the normalized pressures on the surface of the impinging jet plate can be expressed by

$$p(r, 0) = 1 - u^2(r, 0) \quad (78)$$

where $u(r, 0)$ is the surface radial velocity from the inviscid model.

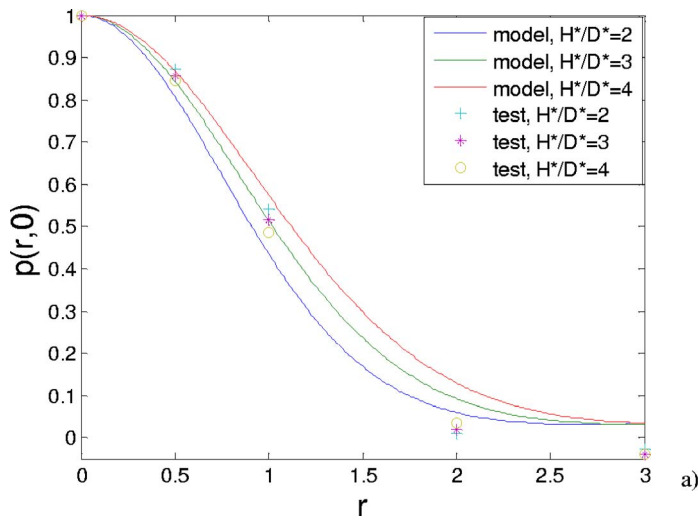


Fig. 9 Comparison of the plate surface pressure distribution

Pressure tests were carried out with a small impinging jet facility ($D^*=38$ mm, $Re_j=23,000$) [7] and compared here with the model predictions in Fig. 9. While the model shows a more pronounced sensitivity to H^*/D^* compared to the experiments, the overall agreement with the experiments is quite good.

Example 3: Annular influx impinging jets. In order to model an annular jet impingement, the influx velocity profile can be expressed by multiple Gaussian jet equations, namely, by

$$-w_\infty = a_3 e^{-r^2/K_1} - b_3 e^{-r^2/K_2} \quad (79)$$

For a certain influx velocity profile, the radius corresponding to the maximum velocity (r_m) and the maximum deficit velocity ($\delta w = w_\infty(r_m) - w_\infty(0)$) are usually specified. Subject to the constraints of $w_\infty(r_m) = -1$ and $dw_\infty/dr|_{r=r_m} = 0$, the parameters in Eq. (79) can be found by solving the following iterations:

$$K_1 = tr_m \quad K_2 = tr_m/2$$

$$t = \frac{r_m}{\ln(2b_3/a_3)} \quad a_3 = \frac{1 - \delta w - e^{2r_m/t}}{1 - e^{r_m/t}} \quad b_3 = a_3 + \delta w - 1 \quad (80)$$

For example, if $\delta w = 0.8$ and $r_m = 0.65$ are set, the following expression based on Eqs. (79) and (80) is found:

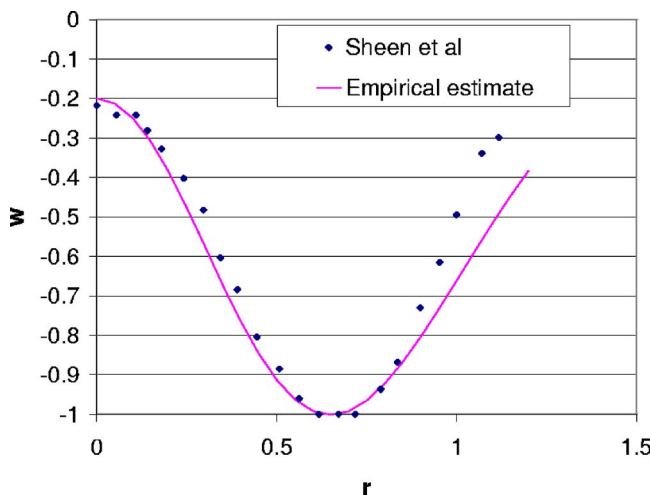
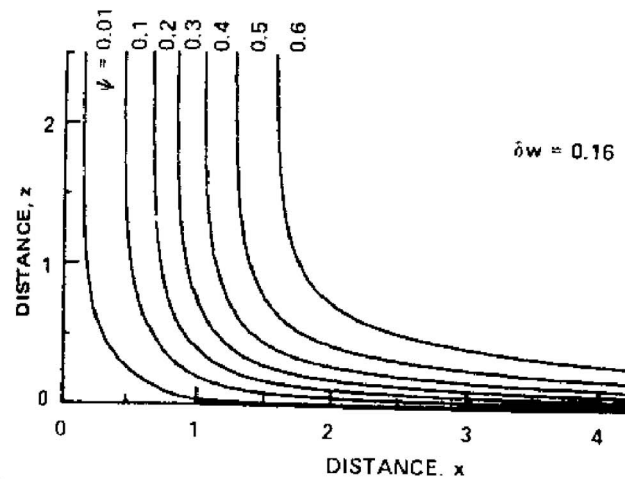
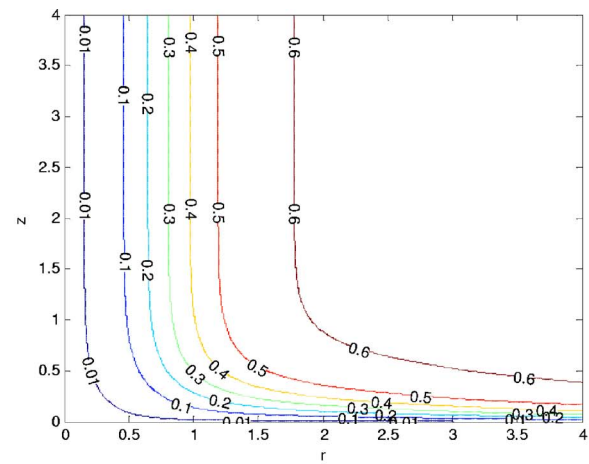


Fig. 10 Particular influx velocity profile of an annular jet



b)

Fig. 11 Annular impinging jet ($r_m=0.35$, $\delta w=0.16$): (a) streamlines by present model ($-\psi$ is used); (b) numerical solution by Rubel [16]

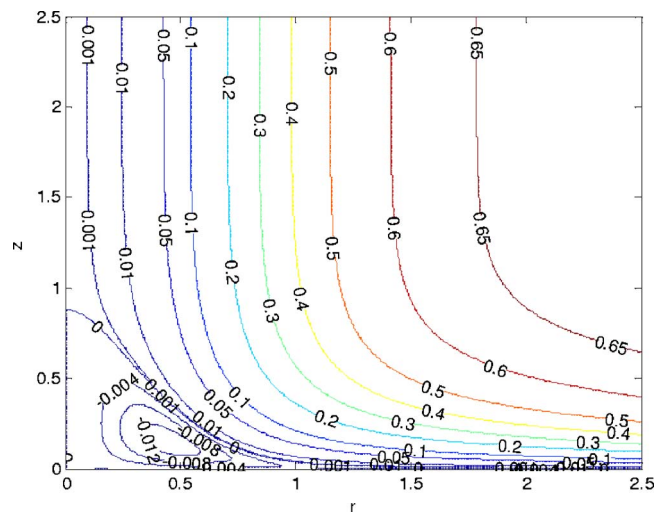


Fig. 12 Streamlines for an annular impinging jet with the influx profile shown in Fig. 10 ($r_m=0.65$, $\delta w=0.8$)

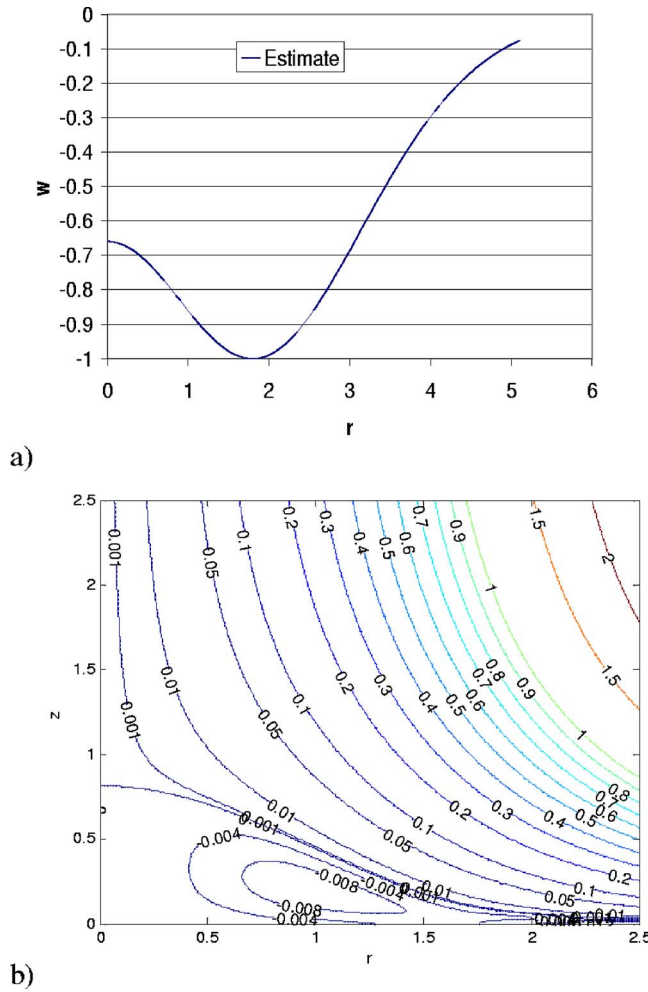


Fig. 13 An annular impinging jet with a large recirculation ring: (a) influx velocity profile estimated from the surface pressure values of Ref. [8]; (b) Streamlines ($-\psi$ is used)

$$-w_{\infty} = 3.79e^{-r^2/0.661} - 3.59e^{-r^2/0.330} \quad (81)$$

This particular expression provides reasonable agreement with the experimental data by Sheen et al. [15], see Fig. 10.

The two terms of Eq. (81) are used to form two Gaussian impinging jet stream functions ($K_1=0.661$, $K_2=0.33$), which, when added together, provide the stream function

$$\psi = 3.59\psi(0.33) - 3.79\psi(0.661) \quad (82)$$

where $\psi(k)$ is expressed by Eq. (22).

(a) *Comparison with previous numerical and mixed models.* The present model is in good agreement with the numerical flow field simulated by Rubel [16] for $r_m=0.35$, $\delta w=0.16$, and $\Omega(0)=0$. In this case, no flow recirculation zone appears, see Fig. 11. The recirculation zone is predicted by Rubel's numerical model only when a nonzero value of $\Omega(0)$ is specified [16]. The present model can also predict a recirculation zone when δw exceeds 0.2 (even up to the maximum 1), in which case, Rubel's method becomes divergent.

Figure 12 shows the contours of the stream function of the annular impinging jet for $r_m=0.65$ and $\delta w=0.8$, which are in good agreement with the analytical-numerical mixed solution by Phares et al. [5]. For this case, Rubel's model [16] has no convergent solution.

(b) *Comparison with the experiment by Donaldson and Snedeker.* Figure 13(a) shows another influx profile estimated from the Bernoulli equation using the experimental impinging plate surface

pressures by Donaldson and Snedeker [17]. Figure 13(b) shows the streamlines generated by the present model using the estimated influx profile. The model predicts a stagnation bubble with the radius comparable to the experiments.

6 Concluding Remarks

An inviscid solution for a family of Gaussian orthogonal impinging jets is derived and it constitutes a base for analytical modeling of the flow field of various real impinging jets. The solution is robust and extends the analytical model by Lee et al. [6]. A first n term averaged method is put forward to speed up the convergence of the oscillatory series, thus simplifying the flow field computations. A new and simpler solution of the plane impinging jets is also obtained.

The inviscid solutions are then combined with laminar and turbulent boundary layer solutions to model real impinging jets. This family of combined inviscid-boundary layer solutions compare well in terms of flow field (streamlines, ring vortex trace, and velocities) and surface pressure field with experiments for both laminar and turbulent boundary layers and different Reynolds numbers. Moreover, an expression for an annular jet profile is derived. The solutions of several annular impinging jets show good agreement with numerical, mixed models, and experimental results.

Acknowledgment

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Nomenclature

- a_1, a_2 = jet shape coefficients
- b_1, b_2 = empirical constants
- B_0^* = half-width of plane jet slot, m
- D^* = diameter of round jet, m
- H^* = distance between jet outlet and impinging plate, m
- k = flow rate multiplier
- p^* = pressure, Pa
- p = dimensionless pressure, $p=2p^*/\rho w_m^2$
- Q = dimensionless flow rate
- R^* = radius of round jet, m
- Re_j = Reynolds number of jet flow based on jet diameter D^* and jet velocity w_m
- Re_l = local Reynolds number based on radial distance r^* and surface velocity u_s^*
- r^*, z^* = radial and axial coordinates, m
- r, z = dimensionless radial and axial coordinates; $r=r^*/R^*, z=z^*/R^*$
- u^*, w^* = radial and axial velocities, m/s
- u_s^* = inviscid velocity on impinging plate surface, m/s
- u, w = dimensionless radial and axial velocities, $u=u^*/w_m, w=w^*/w_m$
- r_m = radius corresponding to maximum velocity
- w_m = maximum axial velocities at jet center line, m/s
- δw = maximum deficit velocity
- x, z = dimensionless plane coordinates; $x=x^*/B_0^*, z=z^*/B_0^*$

Greek Symbols

- ρ = fluid density, kg/m³
- ν = kinematical viscosity, m²/s
- Δ = roughness height of sand paper grain, m

δ^* = thickness of viscous boundary layer, m
 δ = dimensionless thickness of viscous boundary layer
 δ_{disp}^* = displacement thickness of viscous boundary layer, m
 δ_{disp} = dimensionless displacement thickness
 ψ = stream function
 Ω = vorticity function

$$N' = \left(\frac{2NM}{\varepsilon} \right) + 1 \quad (\text{A6})$$

where the notation $[]$ represents the largest integer $\leq 2NM/\varepsilon$. Then, it follows from Eq. (A5) that for $n > N'$,

$$|\bar{S}_n - S| < \frac{NM}{n} + \frac{1}{2}\varepsilon < \frac{NM}{N'} + \frac{1}{2}\varepsilon < \frac{NM}{2NM/\varepsilon} = \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \quad (\text{A7})$$

This proves Part (i).

Note that for Part (i), the reverse is not true, i.e., if $\lim_{n \rightarrow \infty} \bar{S}_n = S$, it is not necessary to have $\lim_{n \rightarrow \infty} S_n = S$. For example, consider the sequence of $S_1 = -0.1$, $S_n = 0.4 + (-1)^n$ for $n \geq 2$: $-0.1, 1.4, -0.6, 1.4, \dots, 0.4 + (-1)^n, \dots$. It is easy to see that

$$\bar{S}_n = \frac{1}{n} \sum_{i=1}^n S_i = 0.4 \pm \frac{1}{2n} \rightarrow 0.4 \quad \text{as } n \rightarrow \infty$$

However, S_n is not convergent.

The above counterexample implies that the rate of convergence of the sequence $\bar{S}_n$ is greater than that of the sequence S_n if S_n is randomly alternating between $+M_1$ and $-M_2$ (M_1, M_2 are positive bounded numbers). It also shows that, conservatively, the rate of convergence of the sequence $\bar{S}_n$ is proportional to n^{-1} (the above counter example is the worst case whose terms reach the two bounded numbers for all $n \geq 2$). However, it is not easy to prove this in general. In the following, we prove Part (ii) under the assumption that S_n is an alternative sequence about its limit, i. e., $(S_n - S)(S_{n+1} - S) < 0$.

Without loss of generality, suppose S_n changes sign from $n=1$ and $|S_n - S|$ is decreasing. Then, it is easy to prove that $\bar{S}_n$ is also an alternative sequence. Further, suppose $S_n - S > 0$ and $S_{n+1} - S < 0$. Similarly, we have $\bar{S}_n - S > 0$, and $\bar{S}_{n+1} - S < 0$. Assume that $S > 0$ (if $S < 0$, we can consider the sequence $-S_n$ and the same argument applies). Then,

$$|S_n - S| - |S_{n+1} - S| = S_n + S_{n+1} - 2S > 0 \Rightarrow S_n + S_{n+1} > 0 \quad (\text{A8})$$

and

$$|\bar{S}_n - S| - |\bar{S}_{n+1} - S| = \bar{S}_n + \bar{S}_{n+1} - 2S > 0 \Rightarrow \bar{S}_n + \bar{S}_{n+1} > 0 \quad (\text{A9})$$

We want to prove that there exists $\bar{N}$ such that

$$F(\bar{S}_n, \bar{S}_{n+1}, S) \equiv |\bar{S}_n - S| - |\bar{S}_{n+1} - S| < |S_n - S| - |S_{n+1} - S| \equiv F(S_n, S_{n+1}, S) \quad (\text{A10})$$

A direct calculation shows that

$$\begin{aligned}
 F(\bar{S}_n, \bar{S}_{n+1}, S) &= |\bar{S}_n - S| - |\bar{S}_{n+1} - S| = \bar{S}_n - S + \bar{S}_{n+1} - S \\
 &= \frac{1}{n}(S_1 + S_2 + \dots + S_n) - S \\
 &\quad + \frac{1}{n+1}(S_1 + S_2 + \dots + S_n + S_{n+1}) - S \\
 &= \frac{1}{n}[(S_1 - S) + (S_2 - S) + \dots + (S_{n-1} - S)] + \frac{1}{n}(S_n - S) \\
 &\quad + \frac{1}{n+1}[(S_1 - S) + (S_2 - S) + \dots + (S_{n-1} - S)] \\
 &\quad + \frac{S_n + S_{n+1} - 2S}{n+1}
 \end{aligned} \quad (\text{A11})$$

Appendix: Proof of Convergence of the Front n Term Average Value Series

Given a series $\sum_{i=1}^{\infty} a_i$, let the partial summation of the series be a sequence:

$$S_n = \sum_{i=1}^n a_i \quad n = 1, 2, \dots \quad (\text{A1})$$

Then take the average of S_n as a new sequence:

$$\bar{S}_n = \frac{1}{n} \sum_{i=1}^n S_i \quad n = 1, 2, \dots \quad (\text{A2})$$

We want to prove that

- (i) if $\lim_{n \rightarrow \infty} S_n = S$, then $\lim_{n \rightarrow \infty} \bar{S}_n = S$;
- (ii) if further $S_i S_{i+1} < 0$, $i = 1, 2, \dots$, then $\text{cov } \bar{S}_n > \text{cov } S_n$.

(Here, cov denotes the rate of convergence.)

Proof. To prove (i), we need to show that

For any $\varepsilon > 0$, there exists an integer N' such that $|\bar{S}_n - S| < \varepsilon \forall n > N'$.

From $\lim_{n \rightarrow \infty} S_n = S$, we know that the sequence is bounded, i.e.,

$$|S_n - S| \leq M \quad \forall n = 1, 2, \dots, (M > 0) \quad (\text{A3})$$

Also, since $\lim_{n \rightarrow \infty} S_n = S$, for any chosen $\varepsilon > 0$, we can find an integer N such that

$$|S_n - S| < \frac{1}{2}\varepsilon \quad \forall n > N \quad (\text{A4})$$

Then, for $n > N$,

$$\begin{aligned}
 |\bar{S}_n - S| &= \left| \frac{1}{n} \sum_{i=1}^n S_i - S \right| \\
 &= \frac{1}{n} |S_1 + S_2 + \dots + S_n - nS| \\
 &= \frac{1}{n} |(S_1 - S) + (S_2 - S) + \dots + (S_n - S)| \\
 &\leq \frac{1}{n} (|S_1 - S| + |S_2 - S| + \dots + |S_n - S|) \\
 &= \frac{1}{n} (|S_1 - S| + |S_2 - S| + \dots + |S_N - S| + |S_{N+1} - S| \\
 &\quad + \dots + |S_n - S|) \\
 &\leq \frac{1}{n} \left[NM + \frac{1}{2}(n - N)\varepsilon \right] \quad (\text{Eqs. (A3) and (A4) are used}) \\
 &= \frac{NM}{n} + \frac{n - N}{2n}\varepsilon < \frac{NM}{n} + \frac{1}{2}\varepsilon
 \end{aligned} \quad (\text{A5})$$

Now, let

There are two cases.

- (a) When n is even, $(S_1 - S) + (S_2 - S) + \cdots + (S_{n-1} - S) < 0$, we have

$$\begin{aligned} F(\bar{S}_n, \bar{S}_{n+1}, S) &< \frac{S_n - S}{n} + \frac{S_n + S_{n+1} - 2S}{n+1} \\ &< S_n + S_{n+1} - 2S \quad \text{for } n > \bar{N} \\ &= |S_n - S| - |S_{n+1} - S| = F(S_n, S_{n+1}, S) \end{aligned} \quad (\text{A12})$$

where $\bar{N}$ is an integer, satisfying

$$(S_{\bar{N}} - S) + (S_{\bar{N}+1} - S) + \frac{S_{\bar{N}+1} - S}{\bar{N} - 2} > 0 \quad (\text{A13})$$

which can be reached since S_n, S are bounded, and $(S_{\bar{N}} - S) + (S_{\bar{N}+1} - S) > 0$ for all n .

- (b) When n is odd, $(S_2 - S) + \cdots + (S_{n-1} - S) < 0$, we have

$$\begin{aligned} F(\bar{S}_n, \bar{S}_{n+1}, S) &< \frac{2n+1}{n(n+1)}(S_1 - S) + \frac{S_n - S}{n} + \frac{S_n + S_{n+1} - 2S}{n+1} \\ &< \frac{2n+2}{n(n+1)}(S_1 - S) + \frac{S_n - S}{n} + \frac{S_n + S_{n+1} - 2S}{n+1} \\ &= \frac{2}{n}(S_1 - S) + \frac{S_n - S}{n} + \frac{S_n + S_{n+1} - 2S}{n+1} \\ &< S_n + S_{n+1} - 2S \quad \text{for } n > \bar{N} \\ &= |S_n - S| - |S_{n+1} - S| \\ &= F(S_n, S_{n+1}, S) \end{aligned} \quad (\text{A14})$$

where $\bar{N}$ is an integer, satisfying

$$(S_{\bar{N}} - S) + (S_{\bar{N}+1} - S) + \frac{S_{\bar{N}+1} - 2S_1 + S}{\bar{N} - 2} > 0 \quad (\text{A15})$$

which is possible since S_n, S are bounded, and $(S_{\bar{N}} - S) + (S_{\bar{N}+1} - S) > 0$ for all n . This finishes the proof for Part (ii).

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Elastic Boundary Layers in Two-Dimensional Isotropic Lattices

The phenomenon of elastic boundary layers under quasistatic loading is investigated using the Floquet–Bloch formalism for two-dimensional, isotropic, periodic lattices. The elastic boundary layer is a region of localized elastic deformation, confined to the free edge of a lattice. Boundary layer phenomena in three isotropic lattice topologies are investigated: the semiregular Kagome lattice, the regular hexagonal lattice, and the regular fully triangulated lattice. The boundary layer depth is on the order of the strut length for the hexagonal and the fully triangulated lattices. For the Kagome lattice, the depth of boundary layer scales inversely with the relative density. Thus, the boundary layer in a Kagome lattice of low relative density spans many cells.

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1 Introduction

Recently, Fleck and Qiu [1] observed elastic boundary layers near the free edge of three isotropic lattices: the regular fully triangulated lattice, the semiregular Kagome lattice, and the regular hexagonal lattice as shown in Fig. 1. They noted that a deep boundary layer exists for the Kagome lattice under remote tension or shear, see Fig. 2. In contrast, the fully triangulated and hexagonal lattices possess boundary layers of depth on the order of one unit-cell size. Fleck and Qiu [1] also calculated the reduction in macroscopic stiffness of a finite-width panel due to the presence of a compliant boundary layer. The drop in stiffness is significant for a Kagome lattice, but not for the fully triangulated and hexagonal lattices.

The boundary layer along the sides of the Kagome lattice provides insight into a paradox in the literature on the effective modulus of triaxial composites. Kueh et al. [2] measured the modulus of carbon-fiber, epoxy-matrix composites with a Kagome weave. They found that the elastic modulus is neither isotropic nor independent of the width of the specimen. Narrow specimens loaded in the direction shown in Fig. 2(a) have a lower modulus than wide specimens, while no such width effect was observed for specimens loaded in the transverse direction. These observations are readily explained in terms of the compliant boundary layer as shown in Fig. 2(a).

The boundary layer phenomenon can be considered to be an analog of St. Venant's edge solutions in the theory of linear elasticity [3]. The boundary layer can also be thought of as an exponentially decaying wave of zero frequency into the medium from the free edge. Thus, a wave propagation technique can be employed to search for these solutions. The Floquet–Bloch technique has already been employed by the authors [4] in order to compute the band structure of lattice materials. In the present study, this formalism is modified to investigate the phenomenon of elastic boundary layers. The waves of interest are of zero frequency and infinite wavelength along the edge. In this study, a general formulation is developed to search for waves of finite frequency and finite wavelength in the x_2 direction, and an exponentially decaying amplitude, with possible oscillation, in the orthogonal x_1 direction, as defined in Fig. 1. This formalism is then specialized to the quasistatic problem of zero frequency and infinite wavelength

along the edge of the lattice. In this limit, the elastic boundary layers shown, for example, in Fig. 2 are the eigensolutions of a quadratic eigenvalue problem (QEP). The eigenvalues give the spatial decay rate of the boundary layer deformation.

The paper is organized as follows. Section 2 gives the formulation of the eigenvalue problem for plane wave propagation at finite frequencies and finite wavelength within a spatially periodic lattice. In Sec. 3, this formalism is used to investigate elastic boundary layers in the three isotropic lattices and the dependence of boundary layer thickness on relative density is computed for the three microstructures. Concluding remarks are given in Sec. 4.

2 Formulation of the Harmonic Wave Propagation Problem

Recall from classical wave theory [5] that a plane wave of frequency ω is represented by the displacement $\mathbf{u}(\mathbf{x}, t)$ of a material point $\mathbf{x}$ at time t as

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_0 \exp[i(\omega t - \mathbf{k} \cdot \mathbf{x})] \quad (1)$$

where $\mathbf{u}_0$ is the wave amplitude and $\mathbf{k}$ is the wave vector. In this section, we consider the problem of finding all possible plane waves which propagate along the free edge of a lattice and decay into the medium. With reference to the coordinate system already introduced in Fig. 1, assume that surface waves with a finite frequency ω travel along the x_2 direction, and decay, with oscillation, in the orthogonal x_1 direction. For this wave, the wave vector $\mathbf{k}$ has a purely real component k_2 along the x_2 direction and a complex component k_1 , with negative imaginary part, along the x_1 direction.

Consider a notional unit cell of any two-dimensional lattice as sketched in Fig. 3. The Cartesian reference frame $(\mathbf{x}_1, \mathbf{x}_2)$ is again employed and the unit base-vector pair $(\mathbf{e}_1, \mathbf{e}_2)$ is introduced. The Euler–Lagrange equations of motion of the unit cell of a spatially periodic lattice panel can be written in the form [4]

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f} \quad (2)$$

where $\mathbf{M}$ and $\mathbf{K}$ are the assembled mass and stiffness matrices of the unit cell obtained by following the usual finite element procedure [6]. The vectors $\mathbf{q}$ and $\mathbf{f}$ denote the displacement degrees of freedom and nodal forces of the unit cell, respectively. Here, the unit cell is discretized into a network of Timoshenko beams. Each beam is assumed to have three degrees of freedom at each end: an axial displacement, a transverse displacement, and a rotation of the beam cross section.

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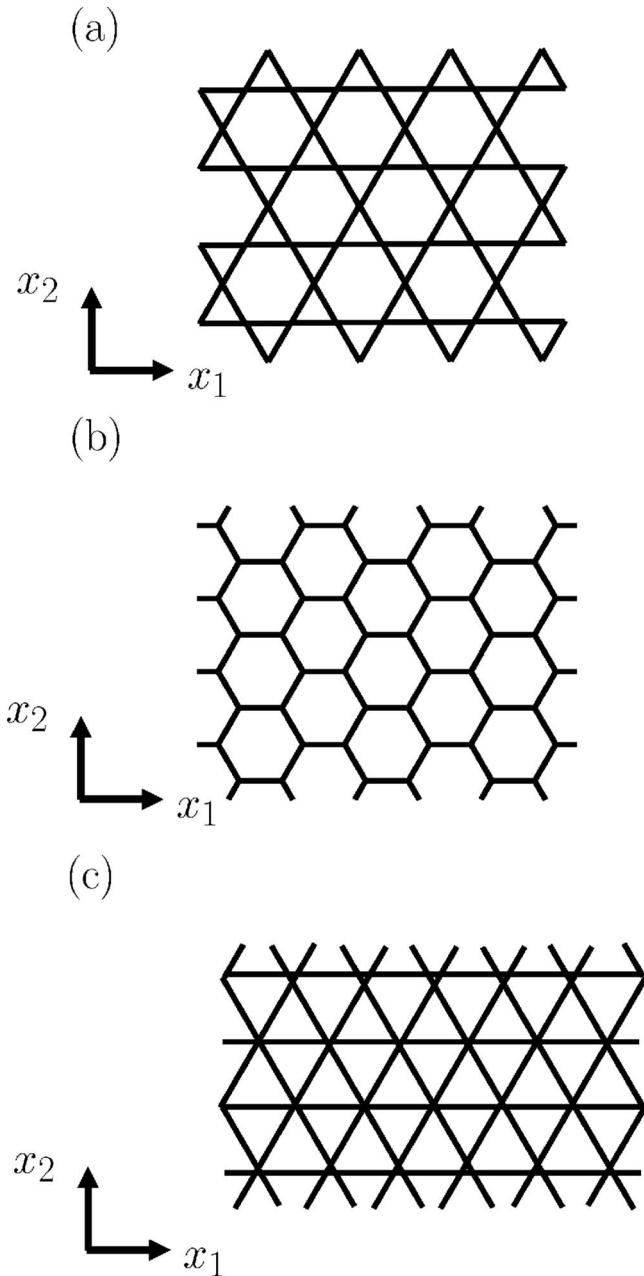


Fig. 1 The two-dimensional isotropic lattices considered in the present study: (a) semiregular Kagome lattice, (b) regular hexagonal lattice, and (c) regular fully triangulated lattice

Assume a time-harmonic displacement solution of the form $\mathbf{q} = \mathbf{q}e^{i\omega t}$. Then Eq. (2) can be simplified to

$$\mathbf{D}\mathbf{q} = \mathbf{f} \quad \text{where } \mathbf{D} \equiv [-\omega^2\mathbf{M} + \mathbf{K}] \quad (3)$$

$\mathbf{D}$ is the *dynamic* stiffness and, in the limit of zero frequency, it reduces to the static stiffness matrix $\mathbf{K}$. It proves convenient to partition the degrees of freedom of the unit cell into distinct groups, as labeled in Fig. 3. For example, $\mathbf{q}_l$ are the generalized displacements of the nodes on the left-hand side of the unit cell. Similarly, $\mathbf{q}_r$, $\mathbf{q}_b$, $\mathbf{q}_t$, and $\mathbf{q}_i$ refer to the displacements of the nodes situated on the right-hand side, bottom, top, and interior, respectively. The equations of motion given in Eq. (3) can now be written in the partitioned form:

$$\begin{bmatrix} \mathbf{D}_{ll} & \mathbf{D}_{lr} & \mathbf{D}_{lb} & \mathbf{D}_{lt} & \mathbf{D}_{li} \\ \mathbf{D}_{rl} & \mathbf{D}_{rr} & \mathbf{D}_{rb} & \mathbf{D}_{rt} & \mathbf{D}_{ri} \\ \mathbf{D}_{bl} & \mathbf{D}_{br} & \mathbf{D}_{bb} & \mathbf{D}_{bt} & \mathbf{D}_{bi} \\ \mathbf{D}_{tl} & \mathbf{D}_{tr} & \mathbf{D}_{tb} & \mathbf{D}_{tt} & \mathbf{D}_{ti} \\ \mathbf{D}_{il} & \mathbf{D}_{ir} & \mathbf{D}_{ib} & \mathbf{D}_{it} & \mathbf{D}_{ii} \end{bmatrix} \begin{bmatrix} \mathbf{q}_l \\ \mathbf{q}_r \\ \mathbf{q}_b \\ \mathbf{q}_t \\ \mathbf{q}_i \end{bmatrix} = \begin{bmatrix} \mathbf{f}_l \\ \mathbf{f}_r \\ \mathbf{f}_b \\ \mathbf{f}_t \\ \mathbf{f}_i \end{bmatrix} \quad (4)$$

Let $q(\mathbf{r}_j)$ denote the displacement of a lattice point j located by the position vector $\mathbf{r}_j$ in the reference unit cell. If a plane wave solution is admitted, then it follows from Eq. (1) that $q(\mathbf{r}_j)$ is of the form

$$q(\mathbf{r}_j) = q_j \exp i(\omega t - i\mathbf{k} \cdot \mathbf{r}_j) \quad (5)$$

where q_j is the amplitude, ω is the circular frequency, and $\mathbf{k}$ is the wave vector of the plane wave. With reference to the chosen unit cell, let the integer pair (n_1, n_2) identify any other cell obtained by n_1 translations along the $\mathbf{x}_1$ direction and n_2 translations along the $\mathbf{x}_2$ direction. Then, the position vector of the point within the cell (n_1, n_2) , corresponding to the $\mathbf{r}_j$ point in the reference unit cell, is given by

$$\mathbf{r} = \mathbf{r}_j + n_1 l_1 \mathbf{e}_1 + n_2 l_2 \mathbf{e}_2 \quad (6)$$

Substitution of Eq. (6) into Eq. (5) gives

$$q(\mathbf{r}) = q(\mathbf{r}_j) e^{-i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_j)} = q(\mathbf{r}_j) e^{-i(k_1 n_1 l_1 + k_2 n_2 l_2)} \quad (7)$$

where l_1 and l_2 are the dimensions of the unit cell of the lattice. This is Bloch's theorem [7–9]. The components (k_1, k_2) of the wave vector $\mathbf{k}$ are expressed in the general form

$$k_1 = \epsilon_1 + i\delta_1 \quad k_2 = \epsilon_2 + i\delta_2 \quad (8)$$

The real part ϵ and the imaginary part δ of the wave components are called the *phase* and *attenuation* constants, respectively. The phase constant is a measure of the phase change, while the imaginary part is a measure of attenuation of the wave as it travels from one unit cell to the next. The waves of interest in the present study propagate along the x_2 direction and exhibit an exponential decay in the x_1 direction. Consequently, the wave-vector components are given by $k_2 = \epsilon_2, k_1 = \epsilon_1 + i\delta_1$, and $\delta_1 < 0$.

The Bloch wave assumption in the x_1 direction provides connections between nodal quantities on the right and left sides of the unit cell:

$$\mathbf{q}_r = \gamma \mathbf{q}_l \quad (9)$$

$$\mathbf{f}_r = -\gamma \mathbf{f}_l \quad (10)$$

where

$$\gamma \equiv \exp(-ik_1 l_1) \quad (11)$$

The aim is to find all possible values of the complex quantity γ for any specified value of the pair (ω, k_2) . We make further use of Bloch's theorem and equilibrium considerations in the x_2 direction in order to obtain a QEP in γ .

Bloch's theorem and the compatibility of displacements in the x_2 direction enforces the following relationship between the displacements of the shared degrees of freedom along the top-bottom interface of two neighboring cells:

$$\mathbf{q}_t = e^{-ik_2 l_2} \mathbf{q}_b \quad (12)$$

Force equilibrium along the edge at the top-bottom interface requires

$$\mathbf{f}_t + e^{-ik_2 l_2} \mathbf{f}_b = \mathbf{0} \quad (13)$$

while equilibrium of the internal nodes implies

$$\mathbf{f}_i = \mathbf{0} \quad (14)$$

Substitution of the expression for $\mathbf{f}_i$ from Eq. (4) into the above equation gives

$$\mathbf{q}_i = -\mathbf{D}_{ii}^{-1} [\mathbf{D}_{il} \mathbf{q}_l + \mathbf{D}_{ir} \mathbf{q}_r + \mathbf{D}_{ib} \mathbf{q}_b + \mathbf{D}_{it} \mathbf{q}_t] \quad (15)$$

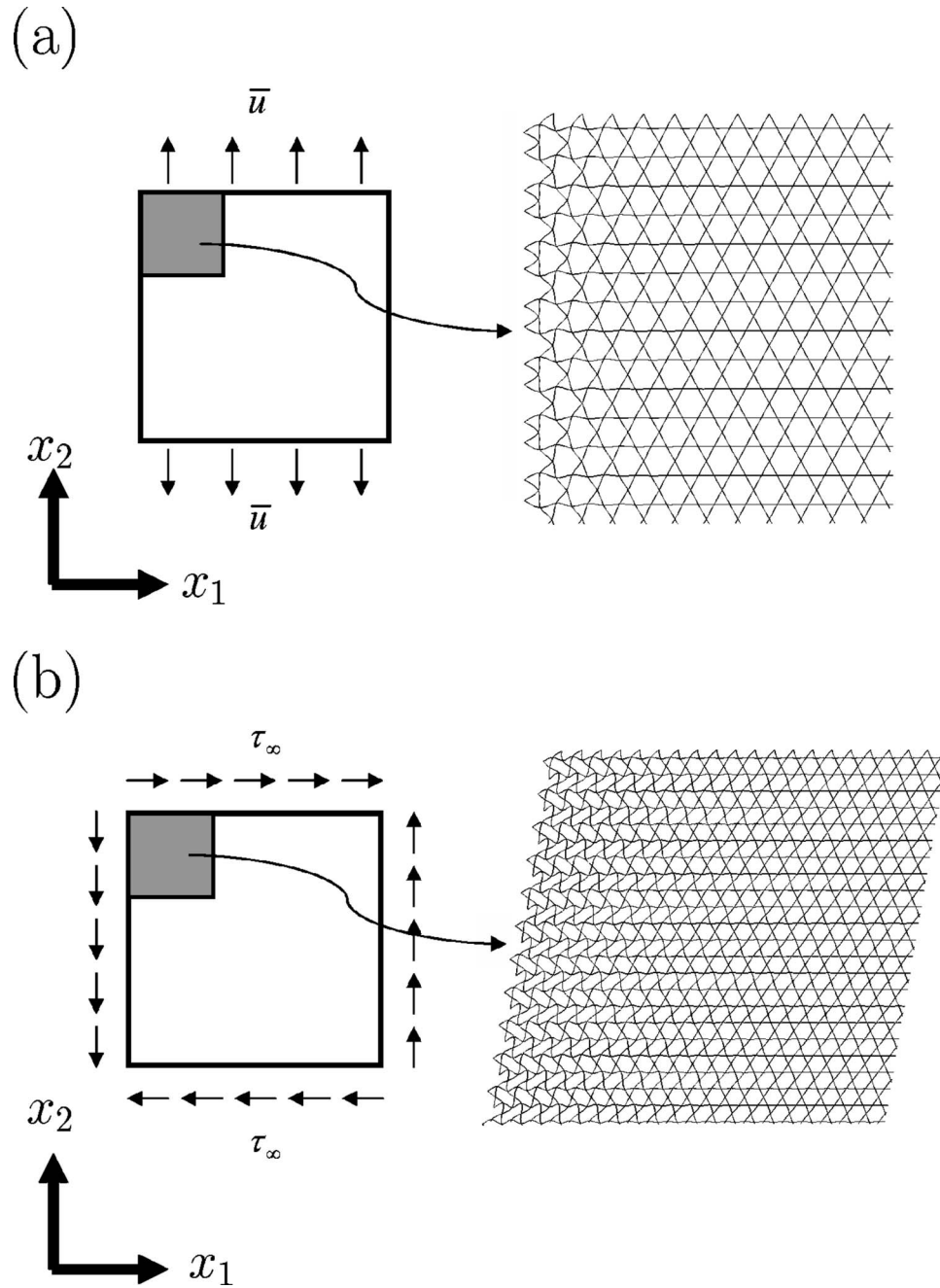


Fig. 2 Deformed mesh of Kagome lattice ($\bar{\rho}=10\%$) revealing a boundary layer at the sides of the specimen for (a) uniaxial tension and (b) simple shear. Based on the work of Fleck and Qiu [1].

Now substitute the expressions for $\mathbf{f}_b$ and $\mathbf{f}_l$ from the third and fourth rows of Eq. (4) into Eq. (13) and use Eqs. (15) and (12) to simplify

$$\mathbf{P}\mathbf{q}_l + \mathbf{Q}\mathbf{q}_r + \mathbf{R}\mathbf{q}_b = \mathbf{0} \quad (16)$$

where the matrices $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$ are defined as

$$\mathbf{P} \equiv \mathbf{D}_{il} + \exp(-ik_2l_2)\mathbf{D}_{bl} - \mathbf{D}_{ii}\mathbf{D}_{ii}^{-1}\mathbf{D}_{il} - \exp(-ik_2l_2)\mathbf{D}_{bi}\mathbf{D}_{ii}^{-1}\mathbf{D}_{il}$$

$$\mathbf{Q} \equiv \mathbf{D}_{ir} + \exp(-ik_2l_2)\mathbf{D}_{br} - \mathbf{D}_{ii}\mathbf{D}_{ii}^{-1}\mathbf{D}_{ir} - \exp(-ik_2l_2)\mathbf{D}_{bi}\mathbf{D}_{ii}^{-1}\mathbf{D}_{ir}$$

$$\begin{aligned} \mathbf{R} \equiv & [\mathbf{D}_{ib} + \exp(-ik_2l_2)\mathbf{D}_{bb} - \mathbf{D}_{ii}\mathbf{D}_{ii}^{-1}\mathbf{D}_{ib} - \exp(-ik_2l_2)\mathbf{D}_{bi}\mathbf{D}_{ii}^{-1}\mathbf{D}_{ib}] \\ & + \exp(-ik_2l_2)[\mathbf{D}_{ir} + \exp(-ik_2l_2)\mathbf{D}_{br} - \mathbf{D}_{ii}\mathbf{D}_{ii}^{-1}\mathbf{D}_{ir} - \exp(-ik_2l_2)\mathbf{D}_{bi}\mathbf{D}_{ii}^{-1}\mathbf{D}_{ir}] \end{aligned} \quad (17)$$

Equation (16) gives the displacement degrees of freedom associated with the bottom nodes $\mathbf{q}_b$ in terms of $\mathbf{q}_l$ and $\mathbf{q}_r$:

$$\mathbf{q}_b = -\mathbf{R}^{-1}[\mathbf{P}\mathbf{q}_l + \mathbf{Q}\mathbf{q}_r] \quad (18)$$

The nodal displacements ($\mathbf{q}_r$, $\mathbf{q}_b$, $\mathbf{q}_l$, $\mathbf{q}_i$) can now be written in terms of $\mathbf{q}_l$ upon using Eqs. (9), (12), (15), and (18). Similarly, the

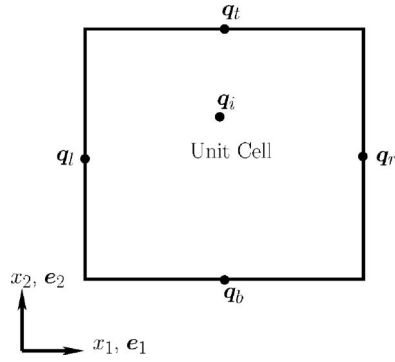


Fig. 3 A unit cell for a two-dimensional periodic structure showing the degrees of freedom shared with the neighboring unit cells and the coordinate system employed

nodal forces ($\mathbf{f}_l, \mathbf{f}_r$) can be written in terms of $\mathbf{q}_l$ via the first two rows of Eq. (4). Now enforce the Bloch wave statement in Eq. (10) to obtain a QEP in γ ,

$$[\gamma^2 \mathbf{A} + \gamma \mathbf{B} + \mathbf{C}] \mathbf{q}_l = \mathbf{0} \quad (19)$$

The matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are defined as

$$\mathbf{A} \equiv \mathbf{D}_{lr} + \mathbf{D}_{lb} \mathbf{Y} + \exp(-ik_2 l_2) \mathbf{D}_{lt} \mathbf{Y} - \mathbf{D}_{li} \mathbf{D}_{ii}^{-1} \mathbf{D}_{ir} - \mathbf{D}_{li} \mathbf{D}_{ii}^{-1} \mathbf{D}_{ib} \mathbf{Y} - \exp(-ik_2 l_2) \mathbf{D}_{li} \mathbf{D}_{ii}^{-1} \mathbf{D}_{it} \mathbf{Y}$$

$$\mathbf{B} \equiv \mathbf{D}_{ll} + \mathbf{D}_{lb} \mathbf{X} + \mathbf{D}_{lt} \mathbf{X} - \mathbf{D}_{li} \mathbf{D}_{ii}^{-1} \mathbf{D}_{il} - \mathbf{D}_{li} \mathbf{D}_{ii}^{-1} \mathbf{D}_{ib} \mathbf{X} - \exp(-ik_2 l_2) \mathbf{D}_{li} \mathbf{D}_{ii}^{-1} \mathbf{D}_{it} \mathbf{X} + \mathbf{D}_{rr} + \mathbf{D}_{rb} \mathbf{Y} + \exp(-ik_2 l_2) \mathbf{D}_{rt} \mathbf{Y} - \mathbf{D}_{ri} \mathbf{D}_{ii}^{-1} \mathbf{D}_{ir} - \mathbf{D}_{ri} \mathbf{D}_{ii}^{-1} \mathbf{D}_{ib} \mathbf{Y} - \exp(-ik_2 l_2) \mathbf{D}_{ri} \mathbf{D}_{ii}^{-1} \mathbf{D}_{it} \mathbf{Y} \quad (20)$$

$$\mathbf{C} \equiv \mathbf{D}_{rl} + \mathbf{D}_{rb} \mathbf{X} + \exp(-ik_2 l_2) \mathbf{D}_{rt} \mathbf{X} - \mathbf{D}_{ri} \mathbf{D}_{ii}^{-1} \mathbf{D}_{il} - \mathbf{D}_{ri} \mathbf{D}_{ii}^{-1} \mathbf{D}_{ib} \mathbf{X} - \exp(-ik_2 l_2) \mathbf{D}_{ri} \mathbf{D}_{ii}^{-1} \mathbf{D}_{it} \mathbf{X}$$

where the matrices $\mathbf{X}$ and $\mathbf{Y}$ are

$$\mathbf{X} \equiv -\mathbf{R}^{-1} \mathbf{P}$$

$$\mathbf{Y} \equiv -\exp(-ik_2 l_2) \mathbf{R}^{-1} \mathbf{Q} \quad (21)$$

The number of eigenvalues is given by twice the dimension N of the displacement vector $\mathbf{q}_l$. The eigenvalues appear as reciprocal complex pairs. For our present purposes of elastic boundary layer analysis, only the exponentially decaying waves are relevant, such that $|\gamma| < 1$. We further limit our attention to the quasistatic eigenstates such that $\omega \equiv 0$ and further specialize the problem to the case where $k_2 \equiv 0$, i.e., there is no variation in the elastic deformation from one unit cell to the other along the x_2 direction.

The joint forces $\tilde{\mathbf{F}}_i, i=1, \dots, N$, associated with each eigenvector are calculated from Eq. (3) for each eigenvector. Note that the N eigenforce vectors are linearly independent since all eigenvalues are nonzero in Eq. (19). A subset of these eigenstates will be matched with particular solutions for a uniform stress state in the lattice in order to generate the boundary layer solutions.

3 Elastic Boundary Layers in the Quasistatic Case

Consider the three topologies shown in Fig. 1. Boundary layers are now obtained for two separate loading cases. For *loading case 1*, consider a lattice subjected to uniaxial stretching in the x_2 direction while the edge of the lattice is traction free (see Fig. 2(a)). For *loading case 2*, consider a lattice subjected to simple shear (see Fig. 2(b)). The unit cells employed are shown in Fig. 4.

In general, the stress state of internal bars is not sustainable at the free surface due to the reduced connectivity at the surface. By St. Venant's principle, an elastic boundary layer develops near the

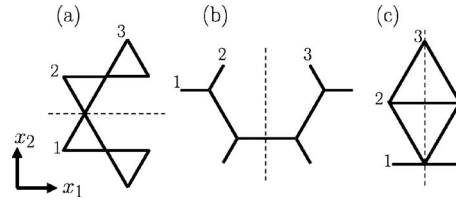


Fig. 4 Unit cells for (a) semiregular Kagome lattice, (b) regular hexagonal lattice, and (c) fully triangulated lattice. The axis of reflective symmetry is shown as a dashed line. Joints on the boundary are labeled numerically.

free edge. This boundary layer provides a smooth transition of stress state from zero traction at the free edge to a uniform stress state within the lattice. The stress state deep in the interior of the lattice can be obtained by considering the equilibrium of a unit cell under the prescribed external macroscopic loading and using the Cauchy–Born hypothesis [10,11]. Denote this uniform solution, valid in the interior of the lattice, as a particular solution. A complementary function is required such that the sum of the complementary and particular solutions match the free surface boundary conditions on the edge, i. e., the generalized force is zero at the joints lying on the free edge. We shall compute the characteristic solutions which constitute the complementary function needed for a given loading condition by adopting the following procedure.

1. Select the unit cell of a lattice and assemble the dynamic stiffness matrix $\mathbf{D}$ by following standard finite element procedures.

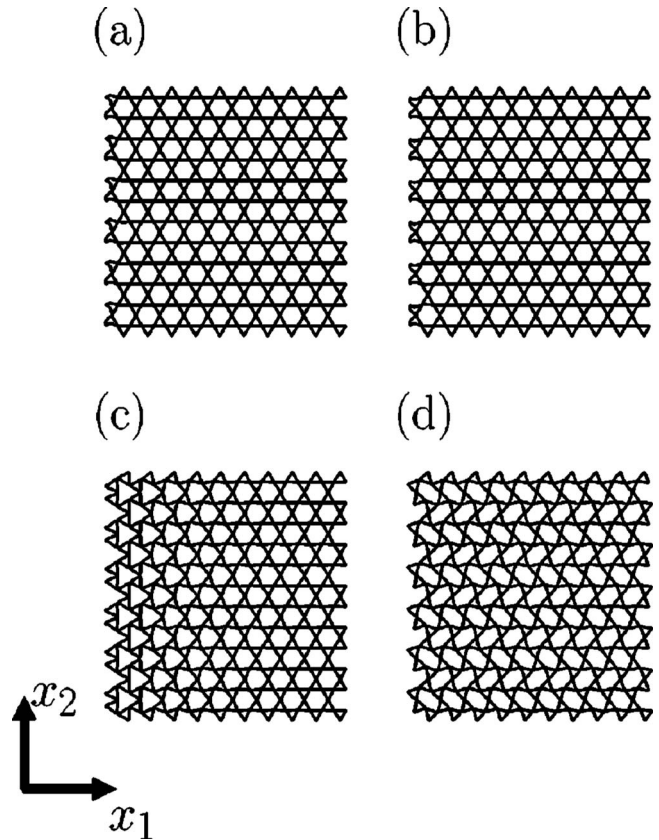


Fig. 5 Boundary layers in a Kagome lattice parallel to the x_2 direction for $\rho=0.05$: (a) eigenvector 1, (b) eigenvector 2, (c) eigenvector 3, and (d) eigenvector 4

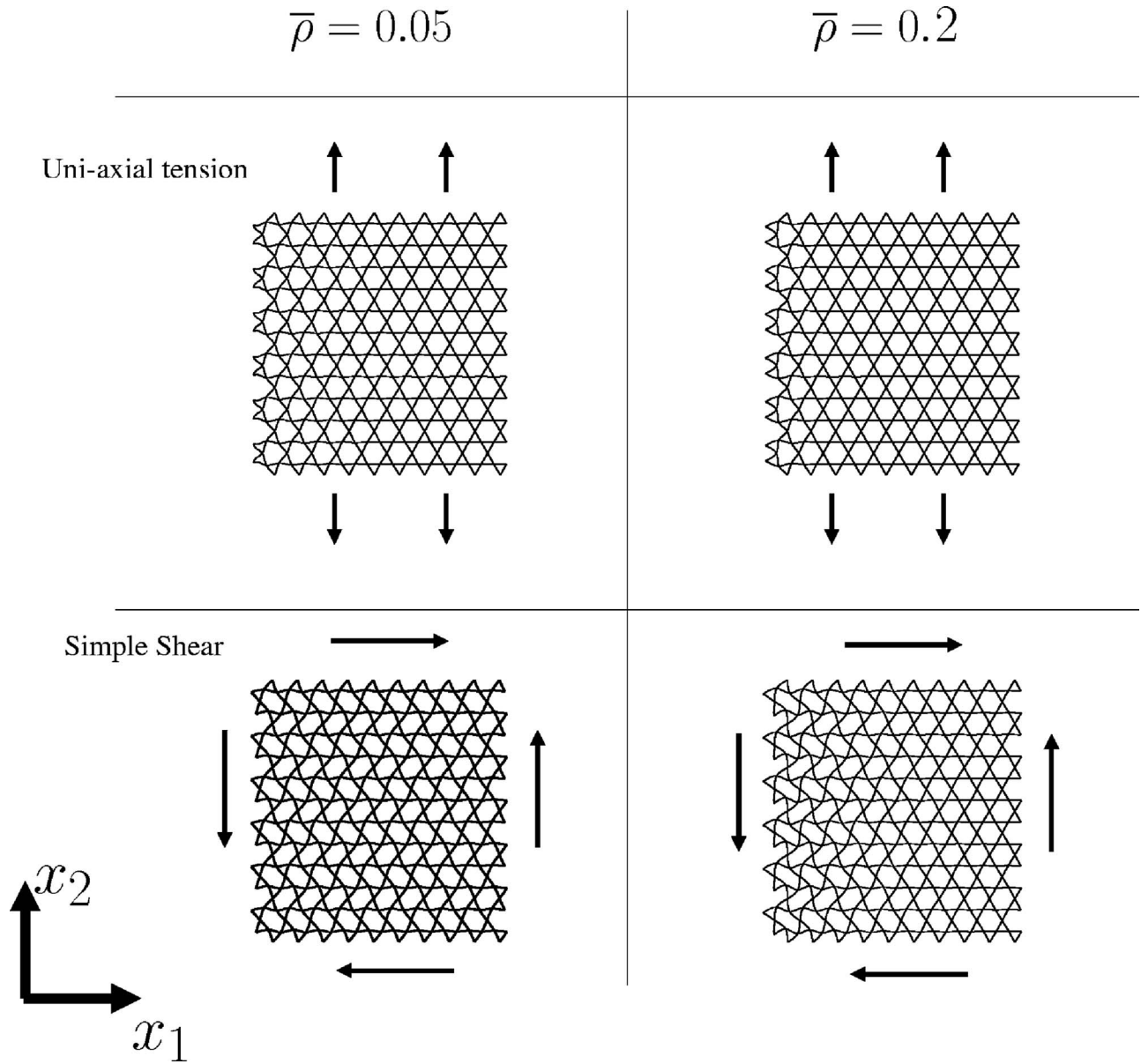


Fig. 6 Elastic boundary layers in a Kagome lattice subjected to macroscopic uniaxial tension along the x_2 direction and subjected to simple shear

2. Specify the pair ($\omega=0, k_2=0$) and form the matrices **A**, **B**, and **C** in the eigenvalue problem in Eq. (19).
3. Identify the subset of waves for which the eigenvalue has the property ($|\gamma|=|e^{(-ik_1 l_1)}| < 1$) as the characteristic waves that constitute the complementary function.
4. Calculate the contribution of each eigenvector to the complementary function by matching with the *known* particular solution, thus completing the solution of the boundary value problem.

We adopt the above procedure in order to construct boundary layer solutions for quasistatic loading of each lattice in turn.

3.1 Kagome Lattice

3.1.1 Boundary Layers Parallel to the x_2 Direction. Consider a Kagome lattice of relative density $\bar{\rho}$ subjected to macroscopic uniaxial stretching in the x_2 direction, as shown in Fig. 2(a). The unit cell has six boundary displacement degrees of freedom, $\mathbf{q}_l \in R^6$. The particular solution represents the uniform state of stress

within the lattice. This is matched by a suitably chosen complementary function in order for the complete solution to satisfy the free edge boundary conditions. There are two joints, labeled 1 and 2, lying on the free edge for the choice of the unit cell shown in Fig. 4(a). Two bars meet at each joint. Three traction-free boundary conditions are specified on each joint in order to satisfy the free-edge boundary condition. Thus, the following six boundary conditions must be satisfied on the free edge:

$$\sum_{b=1}^2 M_b^j = 0 \quad \sum_{b=1}^2 V_b^j = 0 \quad \sum_{b=1}^2 T_b^j = 0 \quad j = 1, 2 \quad (22)$$

where M_b^j , V_b^j , and T_b^j denote the bending moment, shear force, and axial force, respectively, exerted by the bar b on the joint j .

Numerical calculations for a Kagome lattice in the quasistatic limit ($\omega=0, k_2=0$) show that there are six eigenvector solutions of the eigenvalue problem in Eq. (19): Two of these eigenvectors are the rigid-body displacements ($|\gamma|=1$), consistent with the earlier study of the present authors in Ref. [4]. The remaining four waves

comprise exponentially attenuating waves in the x_1 direction so that the eigenvalue of Eq. (19) satisfies the property $|\gamma| < 1$. The four attenuating waves are shown in Fig. 5 for the relative density $\bar{\rho}$ equal to 0.05. The joint forces associated with each eigenvector $\tilde{\mathbf{F}}_i$, $i=1, \dots, 4$, are calculated via Eq. (3). The relative contribution of each of the four eigenvectors to the complementary function is evaluated as follows.

Let the vector $\mathbf{F} \in R^6$ denote the forces due to the particular solution acting on the two joints 1 and 2 as labeled in Fig. 4(a). This vector is obtained by considering the equilibrium of the infinite lattice under prescribed macroscopic loading (see Refs. [10,11] for details). The free-edge conditions in Eq. (22) are matched by finding a complementary force vector $\tilde{\mathbf{F}} \in R^6$ such that

$$\mathbf{F} + \tilde{\mathbf{F}} = \mathbf{0} \quad (23)$$

The complementary solution $\tilde{\mathbf{F}}$ is expressed as a linear superposition of the independent eigenforce vectors associated with each of the exponentially decaying eigenvectors as

$$\tilde{\mathbf{F}} = \sum_{i=1}^4 a_i \tilde{\mathbf{F}}_i \quad (24)$$

where $|\tilde{\mathbf{F}}_i|$ is normalized to unity for each eigenforce vector. The amplitudes a_i of each of these exponentially decaying eigenvector are obtained from the matching condition in Eq. (23). For definiteness, the amplitudes are scaled such that the maximum value of the set is equal to unity.

The amplitudes of the four attenuating eigenvectors in the case of macroscopic uniaxial tensile loading are $\mathbf{a}=[0, 0.7, 1, 0]$. In the case of macroscopic simple shear loading, the amplitudes are $\mathbf{a}=[0.4, 0, 0, 1]$.

The above results are consistent with the reflective symmetry of the Kagome lattice and loading states. Recall that the unit cell of the Kagome lattice in Fig. 4(a) has a horizontal axis of reflective symmetry, as indicated by the dashed line. About this axis of symmetry, the tensile loading is symmetric, while the simple shear loading is antisymmetric. It can be seen from Fig. 5 that the deformation of the unit cell in second and third eigenvectors exhibits reflective symmetry about the horizontal axis, whereas the first and fourth eigenvectors show antisymmetric deformation. Hence, only symmetric eigenvectors contribute to the complementary function in the case of uniaxial tensile loading, whereas only antisymmetric eigenvectors contribute in the case of simple shear loading. The resultant boundary layer in each case is shown in Fig. 6 for two values of relative density equal to 0.05 and 0.2, respectively. It can be seen that the boundary layer is much deeper for the case of lower relative density.

In each loading case, the smallest value of δ_i , $i=1, \dots, 4$, for which $|a_i| > 0$ governs the boundary layer thickness. The dependence of the attenuation constant $\lambda \equiv \delta_1 l_1$ on the relative density $\bar{\rho}$ is shown in Fig. 7 for each of the eigenvectors. For waves 1 and 2, λ is approximately equal to 4, independent of $\bar{\rho}$. These waves decay rapidly within one unit cell. In contrast, for waves 3 and 4, λ scales linearly with $\bar{\rho}$, and these waves decay only gradually from one unit cell to the next. Recall that the boundary layer for uniaxial tension involves waves 2 and 3, whereas the boundary layer for shear involves waves 1 and 4. Consequently, the depth of the boundary layers in uniaxial tension and in shear both scale as $1/\bar{\rho}$. We conclude that very deep boundary layers exist for lattices of low relative density. A similar dependence of boundary layer depth on $\bar{\rho}$ was noted previously by Fleck and Qiu in Ref. [1]. The specimen size effect observed in the experimental studies of Kueh et al. [2] can now be explained by the presence of an elastic boundary layer.

3.1.2 Boundary Layers Parallel to the x_1 Direction. Next, consider the case when the Kagome lattice is loaded along the x_1

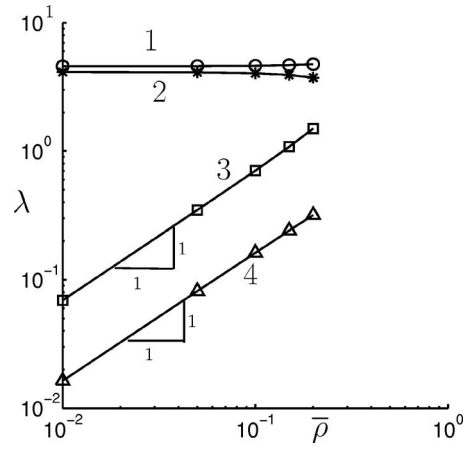


Fig. 7 The attenuation of each of the four eigenvectors versus $\bar{\rho}$ in a Kagome lattice

direction, following the same analysis as that described above. There is now only one joint of the unit cell, labeled 3 in Fig. 4(a), that lies on the free edge. Hence, the unit cell has three boundary displacement degrees of freedom, $\mathbf{q}_t \in R^3$. In the quasistatic limit ($\omega=0$, $k_1=0$), we find that there are three eigenvector solutions of the eigenvalue problem in Eq. (19), of which two are rigid-body displacements and one is an exponentially attenuating wave in the x_2 direction for which the eigenvalue of Eq. (19) satisfies the property $|\gamma| < 1$.

The eigenvector with the property $|\gamma| < 1$ is sketched in Fig. 8. It is characterized by a displacement field $u_2=0$, $u_1 > 0$ together with joint rotation which decay rapidly with depth x_2 . This eigenvector is activated when the top face of a Kagome lattice is subjected to a uniform displacement in the x_1 direction, with unconstrained rotation of the joints.

The above eigenvalue analysis was repeated for selected values of relative density in the range 10^{-2} – 10^{-1} . It was found that λ equals 3.8 in all cases, with a unique eigenvector.

3.2 Hexagonal and Triangular Lattices. An eigenvalue analysis to extract the spatially decaying waves has been performed for a hexagonal lattice and a fully triangulated lattice. The unit cells employed are sketched in Figs. 4(b) and 4(c). Eigenwaves parallel to the x_1 and x_2 directions are investigated for both lattices for selected values of relative density $\bar{\rho}$ in the range 10^{-2} – 10^{-1} .

Consider first the boundary layer solutions parallel to the x_2 direction in a hexagonal lattice. There is only one joint of the unit cell, labeled 1 in Fig. 4(b), that lies on the free edge. Hence $\mathbf{q}_t$

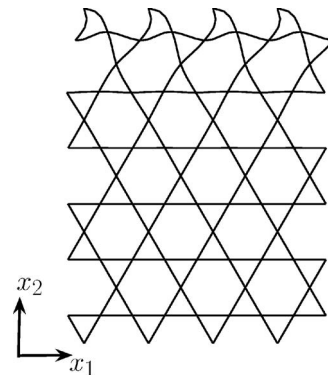


Fig. 8 Boundary layer in a Kagome lattice parallel to the x_1 direction for $\bar{\rho}=0.05$

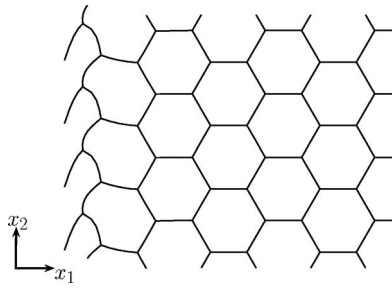


Fig. 9 Boundary layer in a hexagonal lattice parallel to the x_2 direction, for $\bar{\rho}=0.05$

$\in R^3$. In the quasistatic limit ($\omega=0$, $k_2=0$), we find that there are three eigenvector solutions of the eigenvalue problem in Eq. (19), of which two are rigid-body displacements and one is an exponentially attenuating wave in the x_1 direction so that the eigenvalue of Eq. (19) satisfies the property $|\gamma| < 1$. The deformation associated with this eigenvector is sketched in Fig. 9. It exhibits a rapid decay of the deformation within one unit cell of the lattice. The eigenvalue analysis was repeated for selected values of relative density in the range 10^{-2} – 10^{-1} . The eigenvector was unchanged and the value of $\lambda \equiv \delta_1 l_1$ equals 5.5 in all cases.

Next, consider the boundary layer solutions parallel to the x_1 direction. There are now two joints of the unit cell, labeled 2 and 3 in Fig. 4(b), that lie on the free edge. Hence, the unit cell has six boundary displacement degrees of freedom, $\mathbf{q}_l \in R^6$. In the quasistatic limit ($\omega=0$, $k_1=0$), we find that there are six eigenvector solutions, of which two are rigid-body displacements and four are exponentially attenuating waves in the x_2 direction such that $|\gamma| < 1$ in Eq. (19). The elastic deformation corresponding to the four eigenvectors is shown in Fig. 10. Note that the eigenvector deformations in Figs. 10(a) and 10(c) are symmetric about the vertical

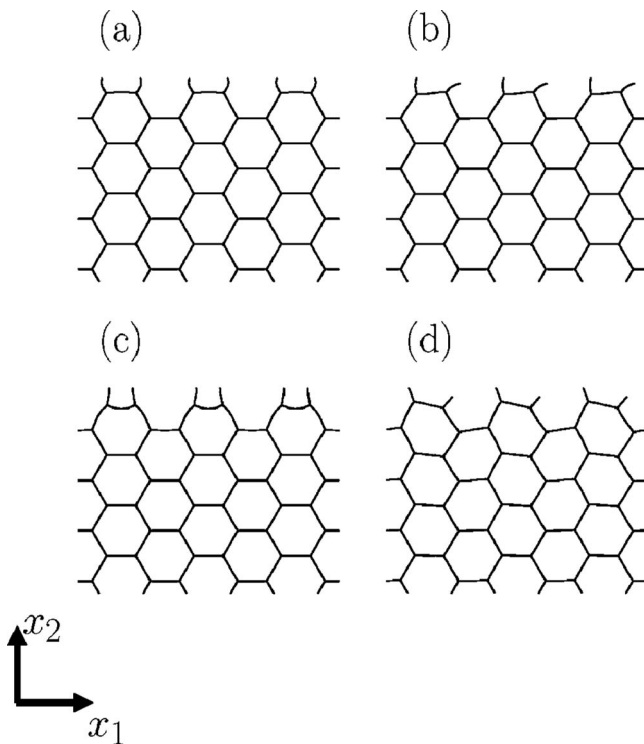


Fig. 10 Boundary layers in a hexagonal lattice parallel to the x_1 direction for $\bar{\rho}=0.05$: (a) eigenvector 1, (b) eigenvector 2, (c) eigenvector 3, and (d) eigenvector 4

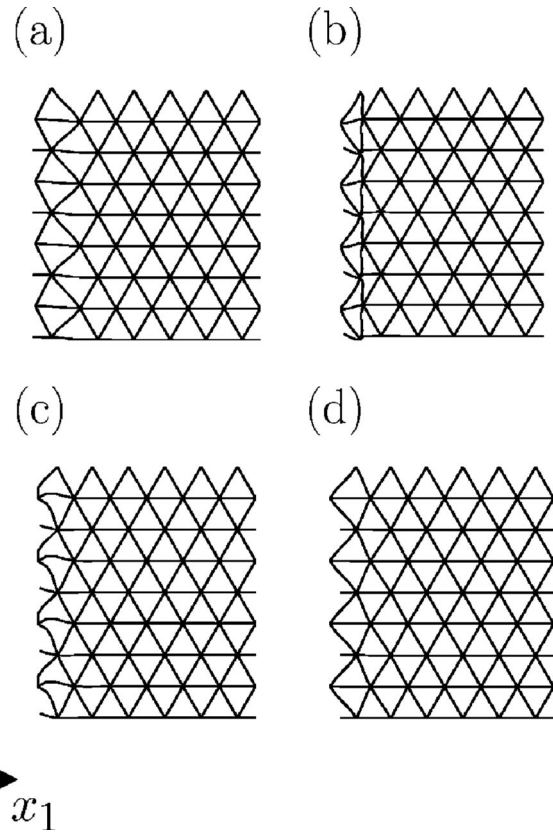


Fig. 11 Boundary layers in a fully triangulated lattice parallel to the x_2 direction for $\bar{\rho}=0.05$: (a) eigenvector 1, (b) eigenvector 2, (c) eigenvector 3, and (d) eigenvector 4

axis of reflective symmetry of the unit cell in Fig. 4(b), while the eigenvector deformations in Figs. 10(b) and 10(d) are antisymmetric. The eigenvalue analysis was repeated for selected values of relative density in the range 10^{-2} – 10^{-1} . It was found that $\lambda \equiv \delta_2 l_2$ equals 8.8, 5.3, 3.6, and 1.7 for the four waves shown in Figs. 10(a)–10(d), respectively. Also, each eigenvector is insensitive to the magnitude of relative density.

A similar analysis has been performed for the triangular lattice to obtain boundary layers parallel to the x_1 and x_2 directions. Consider first a boundary layer parallel to the x_2 direction. The unit cell has six boundary degrees of freedom, $\mathbf{q}_l \in R^6$, since there are two joints on the free edge, labeled 1 and 2 in Fig. 4(c). The elastic deformations of the lattice associated with the four exponentially decaying waves in the quasistatic limit ($\omega=0$, $k_2=0$) are sketched in Figs. 11(a)–11(d). It can be seen that the region of elastic deformation is confined to one unit cell for all eigenvectors. The eigenvalue analysis was repeated for selected values of relative density in the range 10^{-2} – 10^{-1} . The eigenvectors remain unchanged, with $\lambda \equiv \delta_1 l_1$ equal to 9.2, 2.5, 2.4, and 1.4 for the four waves shown in Figs. 11(a)–11(d), respectively.

Finally, consider possible boundary layer along the x_1 direction. A single eigenvector decays in the x_2 direction, as sketched in Fig. 12. Recall that the unit cell has three boundary degrees of freedom, $\mathbf{q}_l \in R^3$, since there is only one joint on the free edge, labeled 3 in Fig. 4(c). The shape of the eigenvector is fixed and $\lambda \equiv \delta_2 l_2$ equals 3.9 for a relative density in the range 10^{-2} – 10^{-1} .

The above study reveals that the boundary layer depth is on the order of the unit-cell size and hence has a negligible influence on the elastic stiffness of a finite-width panel made from a hexagonal lattice or a fully triangulated lattice. These observations are consistent with the earlier study of Fleck and Qiu [1].

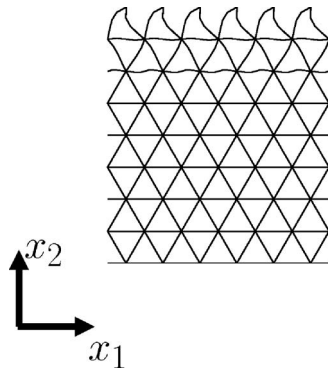


Fig. 12 Boundary layer in a fully triangulated lattice parallel to the x_1 direction for $\bar{\rho}=0.05$.

4 Concluding Remarks

A general formulation based on Floquet-Bloch theory has been developed to compute the elastic boundary layer solutions in two-dimensional periodic lattices. The special case of boundary layers in the quasistatic limit of zero frequency and infinite wavelength along the free edge of the lattice is investigated for three isotropic lattices: a semiregular Kagome lattice, a regular hexagonal lattice, and a fully triangulated lattice.

The Kagome lattice displays deep boundary layers when loaded along particular directions. The analysis presented here provides a theoretical justification for the observation of Fleck and Qiu [1] that the boundary layer depth scales as $1/\bar{\rho}$.

The eigenvector analyses for a hexagonal and a fully triangulated lattice reveal that the characteristic decay length is on the

order of the strut length, independent of the relative density. Consequently, elastic boundary layers have negligible influence on the stiffness of the finite specimen made from these lattices.

This study has been restricted to lattices under quasistatic loading. In the general case of finite frequency dynamic loading, it is of practical importance to know if free surface waves of Rayleigh type exist in these periodic media. The formalism developed here can be employed to search for free surface waves in lattices, and this is the subject of a future publication.

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Three-Dimensional Repeated Elasto-Plastic Point Contacts, Rolling, and Sliding

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Accumulative plastic deformation due to repeated loading is crucial to the lives of many mechanical components, such as gears, stamping dies, and rails in rail-wheel contacts. This paper presents a three-dimensional numerical model for simulating the repeated rolling or sliding contact of a rigid sphere over an elasto-plastic half-space. This model is a semi-analytical model based on the discrete convolution and fast Fourier transform algorithm. The half-space behaves either elastic-perfectly plastically or kinematic plastically. The analyses using this model result in histories of stress, strain, residual displacement, and plastic strain volume integral (PV) in the half-space. The model is examined through comparisons of the current results with those from the finite element method for a simple indentation test. The results of rolling contact obtained from four different hardening laws are presented when the load exceeds the theoretical shakedown limit. Shakedown and ratcheting behaviors are discussed in terms of the PV variation. The effect of friction coefficient on material responses to repeated sliding contacts is also investigated. [DOI: 10.1115/1.2755171]

1 Introduction

The wear experiment reported in [1] of a copper pin sliding repeatedly against a steel ring revealed a severe shear plastic deformation in the near surface material layer. The extensive review by Johnson [2] presented several regimes of behaviors of an elasto-plastic body subjected to repeated rolling contacts: (1) at a sufficiently small load, the material may respond purely elastically; (2) although the yield limit may be reached at the early cycles, the residual stress, the strain hardening, and the conformingly deformed geometry may “shakedown” the material; thereafter, the material will respond elastically; and (3) if the load exceeds a certain value, known as the “shakedown limit,” each cycle of loading can result in the repeated increment of plastic strain in the material (known as the “ratcheting”). Accumulative “ratcheting” in an elasto-plastic body may consequently lead to ductile fracture, which has been considered to be a mechanism of metallic wear [3,4].

Johnson [2] utilized a static shakedown theory to investigate the theoretical shakedown limit of repeated rolling contacts. The effect of friction coefficient on the shakedown limit was also studied. Ponter et al. [5] employed a kinematic shakedown theory, which considered the history of plastic deformation to study an elastic-perfectly plastic (EPP) solid over whose surface a prescribed rolling or sliding traction is repeatedly applied. Kapoor et al. [6] introduced a term, named “plasticity index in repeated sliding,” to include the influence of surface roughness on the shakedown behaviors of materials. Johnson and Shercliff [7] specifically investigated the shakedown in the two-dimensional sliding contacts through considering the asperity profile variations.

The finite element method (FEM) was used to model the two-dimensional repeated rolling contacts of rail steel with a kinematic hardening behavior by Bhargava et al. [8]. Kulkarni et al. [9,10] developed a three-dimensional FEM model for a half-space in frictionless repeated rolling contacts. Residual stresses, strains, and other related quantities were calculated when the relative peak pressure p_0/k_s was at the theoretical shakedown limit and above

the theoretical shakedown limit, respectively. Jiang et al. [11] used FEM to investigate a three-dimensional rolling contact problem, where the shear tractions in both rolling and perpendicular directions were considered, when p_0/k_s was above the theoretical shakedown limit. Furthermore, a partial slip condition was studied in [12]. Yu et al. [13] presented a novel and efficient direct FEM approach to obtain the steady-state solution of a linear kinematic plastic-hardening solid in a three-dimensional repeated point contact. In these FEM simulations, the geometric changes of contact surfaces were neglected, and a prescribed Hertz contact pressure was allowed to traverse repeatedly over a half-space surface.

Recently, a fast semi-analytical method (SAM) was developed by Jacq et al. [14] to study the elasto-plastic counterformal contacts. Compared to FEM, SAM is more efficient because only the contact region needs to be meshed and simulated. In addition, SAM yields a more accurate solution, because it fully considers the surface geometry variation due to plastic deformation. Thermoelastic deformation has been added in this model by Boucly et al. [15] to account for the frictional heating effect. Wang and Keer [16] investigated the influence of the type of strain-hardening laws on the elasto-plastic behaviors of typical steels. The discrete convolution and fast Fourier Transform (DC-FFT) algorithm, outlined by Liu et al. [17], was embedded in the model to accelerate the linear convolution calculations involved in the elasto-plastic contact problems.

The current investigation, based on Jacq’s model [14], aims to develop a three-dimensional elasto-plastic model for point contacts subjected to the repeated rolling or sliding traction. This model accounts for the conformity of contact geometry induced by surface profile variation under cyclic contacts. For the study of rolling contacts, four types of strain hardening laws are employed to examine repeated contact performances of materials with different hardening behaviors. Shakedown and ratcheting phenomena are investigated for various relative peak pressure values and different strain hardening laws in terms of the plastic strain volume integral (PV) in the entire space. In order to simulate sliding contact, the shear traction is assumed to be the product of normal pressure and a specified friction coefficient. The influence of friction coefficient on stress-strain states is then examined.

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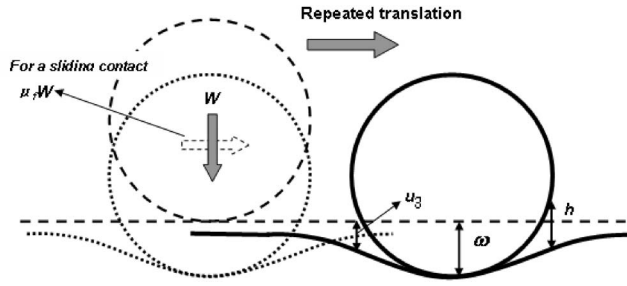


Fig. 1 Repeated rolling or sliding contacts of a rigid sphere on the surface of an elasto-plastic half-space

2 Theoretical Background

2.1 Description of an Elasto-Plastic Contact. The repeated rolling or sliding contacts of a rigid sphere with an elasto-plastic half-space are shown in Fig. 1. The general contact model based on elasticity used by many researchers [2,14–16] is summarized as follows:

$$W = \int_{\Gamma_c} p(x,y) d\Gamma \quad (1)$$

$$h(x,y) = h_i(x,y) + u_3(x,y) - \omega \geq 0 \quad (2)$$

$$p(x,y) \geq 0 \quad h(x,y)p(x,y) = 0$$

$$p(x,y) = 0 \subset \Gamma_c \quad h(x,y) = 0 \not\subset \Gamma_c \quad (3)$$

where W is the applied load, Γ_c the real contact surface, p the normal contact pressure, u_3 the normal displacement of the half-space, and $h_i(x,y)$ and ω are the initial gap and the contact interference, respectively. Equation (1) is the equilibrium condition, while Eqs. (2) and (3) give the surface clearance and boundary constraints, respectively. Based on the corresponding Green's functions (the Boussinesq and Cerruti formulas [2]), as indicated in Eq. (4), the elastic surface displacement caused by contact pressure p and shear traction s (along the x -axis) is given in Eq. (5):

$$G^p(x,y) = \frac{1}{\pi E^* r}, \quad G^s(x,y) = \frac{x}{\pi \mu_e r^2} \quad (4)$$

$$u_3^e(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [G^p(x-x', y-y') p(x', y') + G^s(x-x', y-y') s(x', y')] dx' dy' = G^p * p + G^s * s \quad (5)$$

where $r = \sqrt{x^2 + y^2}$, $E^* = E/(1-\nu^2)$, and $\mu_e = 2E/(1+\nu)(1-2\nu)$. Therefore, an elastic contact problem can be described by a linear equation system subjected to the constraints of nontensile contact pressure and impenetrable contact bodies. The iterative method based on the conjugate gradient method [18] is utilized to solve this system including equations and inequalities for rough-surface contact problems efficiently, with which contact pressure and contact area can be determined simultaneously.

In order to include the influence of plastic deformation, Jacq et al. [14] developed an exact solution for residual displacement u_3^r based on the reciprocal theorem, which is expressed as a volume integral in Eq. (6) as

$$u_3^r(x,y) = 2\mu \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varepsilon_{ij}^p(x', y', z') \varepsilon_{ij}^*(x-x', y-y', z') dx' dy' dz' \quad (6)$$

where $\varepsilon_{ij}^*(x-x', y-y', z')$ is the elastic-strain component in the

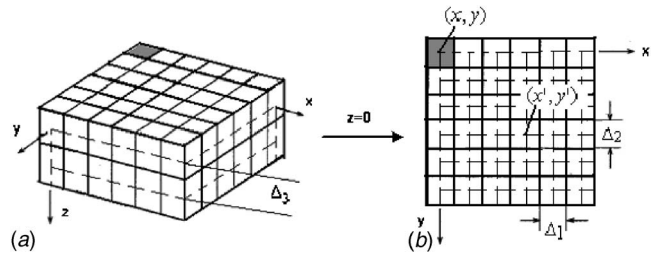


Fig. 2 Description of the mesh system: (a) the simulated domain with the mesh in a three-dimensional view and (b) the simulated contact surface with the mesh

half-space at point (x', y', z') , which is induced by a unit concentrated normal force applied on a surface point (x, y) . The residual displacement can be directly added into the total displacement u_3 in Eq. (2) to obtain the solution of an elasto-plastic contact.

As shown in Fig. 2, a rectangular mesh system is used to digitize the simulation domain. The numerical evaluation needs discrete influence coefficients (ICs) i.e., D_j instead of the continuous Green's function. The general form of one-dimensional ICs is the integral of the product of the shape function $Y(x)$ with Green's function $G(x)$ over $[-\Delta/2, \Delta/2]$. A rectangular pulse function is usually used as the shape function:

$$D_j = \int_{-\Delta/2}^{\Delta/2} G(\Delta j - \xi) Y(\xi) d\xi \quad (7)$$

where Δ is the mesh size. The displacement u_i can be expressed in a linear convolution as

$$u_i = \sum_{r=0}^{N-1} p_r D_{\text{mod}(i-r) \text{ of } N} \quad i = 0, \dots, N-1 \quad (8)$$

For instance, the influence coefficients of elastic displacement due to normal pressure were discussed in [19], as shown by

$$D_{ij}^p = \frac{1}{\pi E^*} [f(x_u, y_u) + f(x_l, y_l) - f(x_u, y_l) - f(x_l, y_u)] \quad (9)$$

where $x_{u,l} = \Delta(i \pm 1/2)$, $y_{u,l} = \Delta(j \pm 1/2)$, and $f(x,y) = x \ln(y+r) + y \ln(x+r)$. The closed-form ICs of displacements due to shear traction and plastic strains were discussed in [14,19].

The subsurface stress field should be calculated to determine the plastic strain zone. The total stress can be decomposed into an elastic part and a residual part:

$$\sigma_{ij} = \sigma_{ij}^e + \sigma_{ij}^r \quad (10)$$

The elastic stress in a solid due to surface tractions is expressed as follows in the form of discrete convolution:

$$\sigma_{ij}(x_m, y_n, z) = \sum_{\xi} \sum_{\eta} [p_{\xi\eta} D_{Nij}^{m-\xi, n-\eta, z} + s_{\xi\eta} D_{Sij}^{m-\xi, n-\eta, z}] \quad i, j = 1, 2, 3 \quad (11)$$

where the ICs for the elastic stress ($D_{Nij}^{m,n,z}$ and $D_{Sij}^{m,n,z}$) can be found in [20]. The evaluation of residual stress needs to superpose the contributions of all yield elements with nonzero plastic strains after unloading:

$$\sigma_{ij}^r(M) = \sum_{C=1}^{N_V} D_{ijkl}^r(M, C) \varepsilon_{kl}^p(C) \quad (12)$$

The ICs for the residual stress D_{ijkl}^r were discussed in detail in [14]. The DC-FFT algorithm [17,21] can be utilized to efficiently evaluate the linear convolutions existing in Eqs. (8), (11), and (12).

Using the subsurface stress values and the plasticity model,

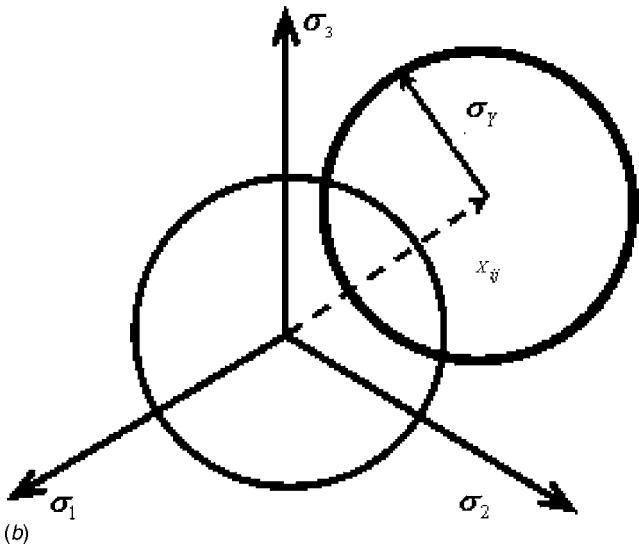
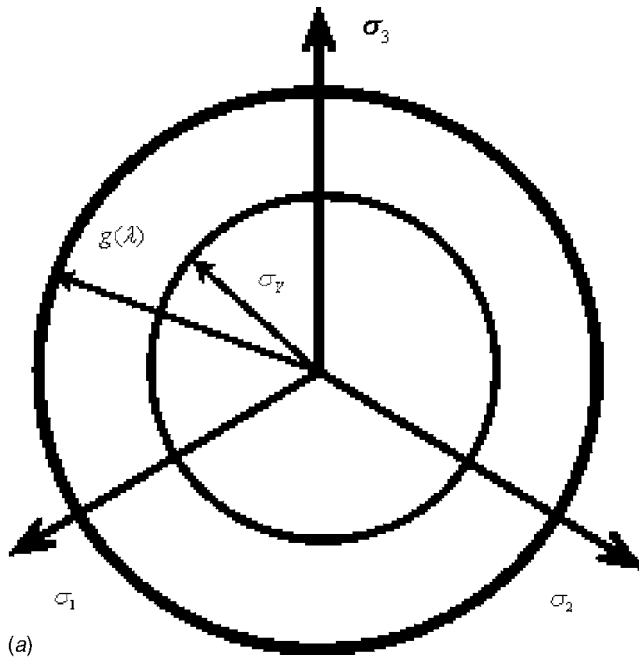


Fig. 3 Strain hardening laws: (a) isotropic and (b) kinematic

plastic deformation can be determined by the increment-based approach [14], where the variations of plastic strains in each loading step are expressed as a function of current stresses σ_{ij} , variations of stresses $\delta\sigma_{ij}$, prestrain ε_{ij}^p , and strain-hardening parameters:

$$\delta\varepsilon_{ij}^p = f(\sigma_{ij}, \delta\sigma_{ij}, \varepsilon_{ij}^p, \text{hardening parameters}) \quad (13)$$

2.2 Plasticity Consideration. Plasticity is the irreversible behavior of a material in response to load application. The von Mises yield criterion function, as indicated in Eq. (14), is utilized to identify the transition from elastic to plastic deformation as

$$f = \sigma_{VM} - g(\lambda) = \sqrt{\frac{3}{2}} S_{ij} : S_{ij} - g(\lambda) \quad (14)$$

Here, σ_{VM} is the von Mises equivalent stress, $\lambda = \sum d\lambda = \sum (\sqrt{2d\varepsilon_{ij}^p d\varepsilon_{ij}^p} / 3)$ the effective accumulative plastic strain, g the yield strength function ($g(0)$ equals the initial yield strength, σ_Y), and $S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$ the deviatoric stress. Equation (15) presents

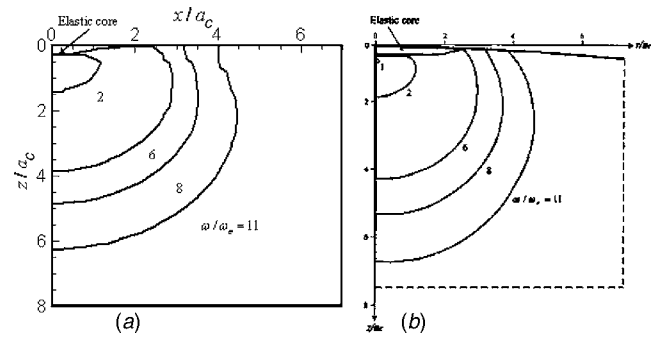


Fig. 4 Evolutions of the plastic deformation region for $1 \leq \omega/\omega_c \leq 11$: (a) from the current model and (b) from the FEM analysis in [26]

the loading/unloading constraints for the increment of effective plastic strain $d\lambda$ and the yield function f :

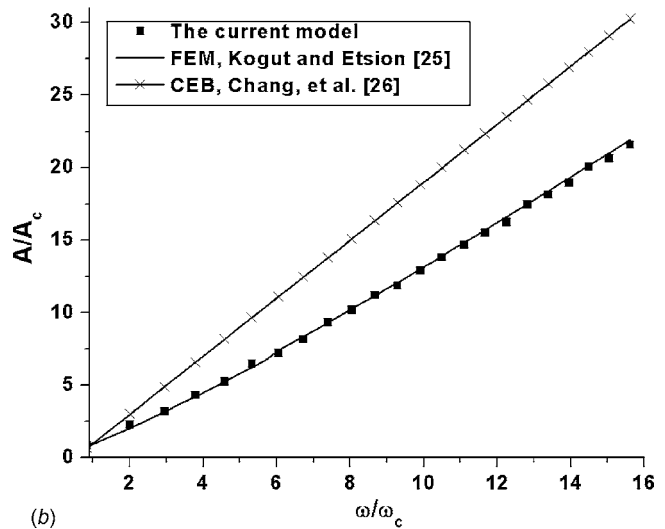
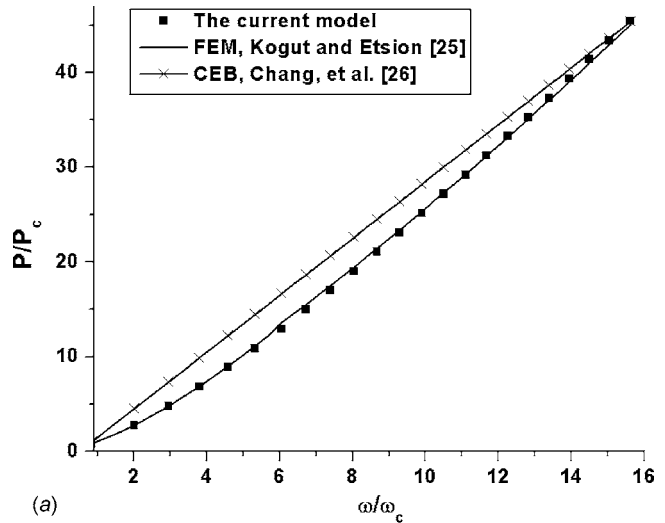


Fig. 5 Model verifications: (a) the dimensionless contact load versus the dimensionless interference and (b) the dimensionless contact area versus the dimensionless interference

Table 1 Parameters and material properties in the simulations

Terms	Value
E	210 GPa
ν	0.3
σ_Y	383.30 MPa
B	787.68 MPa
C	0.00082
n	0.132
E_T	$0.2E$
K	1782.24 MPa
γ	15.80
R	18 mm
Element size, Δ	$12 \times 12 \times 12 \mu\text{m}$
Grid number	$64 \times 64 \times 30$

$$f \leq 0 \quad d\lambda \geq 0 \quad fd\lambda = 0 \quad (15)$$

The plastic strain variation is governed by the plastic flow rule [22], and expressed in Eq. (16) when the von Mises yield criterion is used as:

$$d\epsilon_{ij}^p = d\lambda \frac{\partial f}{\partial \sigma_{ij}} = d\lambda \frac{3S_{ij}}{2\sigma_{VM}} \quad (16)$$

The simplest plastic model is to assume that materials possess the elastic-perfectly plastic (EPP) behavior, in which the yield strength $g(\lambda)$ always remains at the initial value σ_Y . In fact, work hardening usually happens after the first occurrence of plastic strain to resist further plastic deformation. There are two basic ways, i.e., isotropic and kinematic hardening, to model the strain hardening effect.

Isotropic Hardening Law. With an isotropic hardening law, the yield surface increases in size, but keeps the same shape as the plastic strain, as shown in Fig. 3(a). In a quasi-static loading process, materials deform at a very low strain rate. Therefore, a rate-independent law is adequate. In the following study, two isotropic hardening laws are applied: the Swift power hardening law and the linear hardening law. The Swift law is expressed as:

$$g(\lambda) = B(C + \lambda)^n \quad (17)$$

Here, B , C , and n are work hardening parameters, and $\sigma_Y = BC^n$. The linear hardening law is given as follows:

$$g(\lambda) = \sigma_Y + \frac{E_T}{1 - E_T/E} \lambda \quad (18)$$

where E_T is the elasto-plastic tangential modulus. However, an isotropic hardening law alone is generally not suitable for materials subjected to repeated loadings.

Kinematic Hardening Law. On the other hand, the kinematic hardening law translates the yield surface without changing its shape and size, as shown in Fig. 3(b), to account for the effect of cyclic plastic deformation. The yield surface is dragged along the direction of increasing stress. Thus, materials become harder for a further increased load and softer for a reversed load (i.e., the Bauschinger effect). The back stress X_{ij} is the center of a new yield surface in the stress space, and the deviatoric stress becomes $S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} - X_{ij}$.

The back stress depends on the history of plastic deformation, and the back stress variation can be modeled by the Armstrong and Frederick's law [23]:

$$dX_{ij} = d\lambda \left[K \frac{S_{ij}}{\sigma_{VM}} - \gamma X_{ij} \right] \quad X_{ij} = 0 \text{ when } \lambda = 0 \quad (19)$$

where K and γ are kinematic hardening parameters.

However, the plastic behaviors of some common engineering materials are too complicated to be described with neither isotro-

pic nor kinematic alone. Therefore, the kinematic law should be used together with the isotropic law, and then the von Mises yield function becomes:

$$f = \sqrt{\frac{3}{2}(\sigma'_{ij} - X_{ij})(\sigma'_{ij} - X_{ij})} - g(\lambda) \quad (20)$$

Here, the first term is the von Mises equivalent stress indicated by the kinematic law, and $g(\lambda)$ the yield strength modeled by the isotropic law.

2.3 Plastic Strain Increment. Fotiu and Nemat-Nasser [24] developed a universal integration algorithm for constitutive equations of elastoplasticity, including isotropic and kinematic hardening, as well as thermal softening. The method seems to be unconditionally stable and accurate. Nelias et al. [27] implemented this algorithm in their code to improve the convergence of plasticity loop. The current study follows the idea in [27] to use this algorithm to calculate the increment of plastic strain. A yield occurs when $f(\lambda) > 0$, i.e., the equivalent von Mises stress is larger than the current yield strength. The actual increment of the effective plastic strain $\Delta\lambda$ should satisfy the condition expressed by $f(\lambda + \Delta\lambda) = 0$, in the plastic zone. Thus, the Newton-Raphson iteration scheme is utilized to find the solution of this nonlinear equation. The yield function can be expanded approximately as:

$$f^{(n+1)} = f^{(n)} + \Delta\lambda^{(n)} f'_{,\lambda} = 0 \quad (21)$$

The correction of effective plastic strain $\Delta\lambda^{(n)}$ between two consecutive iterative steps is expressed as:

$$\Delta\lambda^{(n)} = -\frac{f^{(n)}}{f'_{,\lambda}} = \frac{f^{(n)}}{g'_{,\lambda} - \sigma'_{VM,\lambda}} \quad (22)$$

where

$$\frac{\partial \sigma_{VM}}{\partial \lambda} = -3\mu - K + \frac{3\gamma X_{ij} S_{ij}}{2\sigma_{VM}} \quad (23)$$

The detailed derivation of $\partial \sigma_{VM} / \partial \lambda$ is given in the Appendix. All of the related variables are updated as follows:

$$\sigma_{VM}^{(n+1)} = \sigma_{VM}^{(n)} + \sigma'_{VM,\lambda} \Delta\lambda^{(n)} \quad \lambda^{(n+1)} = \lambda^{(n)} + \Delta\lambda^{(n)}$$

$$g^{(n+1)} = g(\lambda^{(n+1)})$$

$$X_{ij}^{(n+1)} = X_{ij}^{(n)} + \Delta\lambda^{(n)} \left[K \frac{S_{ij}^{(n)}}{\sigma_{VM}^{(n)}} - \gamma X_{ij}^{(n)} \right] \quad S_{ij}^{(n+1)} = \frac{\sigma_{VM}^{(n+1)}}{\sigma_{VM}^{(1)}} \sigma_{ij}^{(1)} - X_{ij}^{(n+1)} \quad (24)$$

Here, $\lambda^{(1)}$, $X_{ij}^{(1)}$, $\sigma_{VM}^{(1)}$, and $\sigma_{ij}^{(1)}$ are the initial effective plastic strain, back stress, equivalent von Mises stress, and Cauchy stress components, respectively. The computation ends if the convergence condition is satisfied:

$$\left| \frac{f^{(n+1)}}{g^{(n+1)}} \right| = \left| \frac{\sigma_{VM}^{(n+1)} - g^{(n+1)}}{g^{(n+1)}} \right| < \text{tolerance} \quad (25)$$

The steps indicated in Eqs. (22)–(25) are repeated until the iteration converges. The estimation of the plastic strain increment is determined by the plastic-flow rule shown below

$$\Delta\epsilon_{ij}^p = [\lambda^{(n+1)} - \lambda^{(1)}] \frac{3S_{ij}^{(n+1)}}{2\sigma_{VM}^{(n+1)}} \quad (26)$$

3 Model Verification

The contact between an elasto-plastic sphere and a rigid half-space was investigated by Kogut and Etsion [25] using FEM and by Chang et al. [26] using a volume conservation model (the CEB model). The current model is verified by comparing results with these previous numerical solutions. In order to be consistent with [25], this part of the work uses the same elastic-perfectly plastic

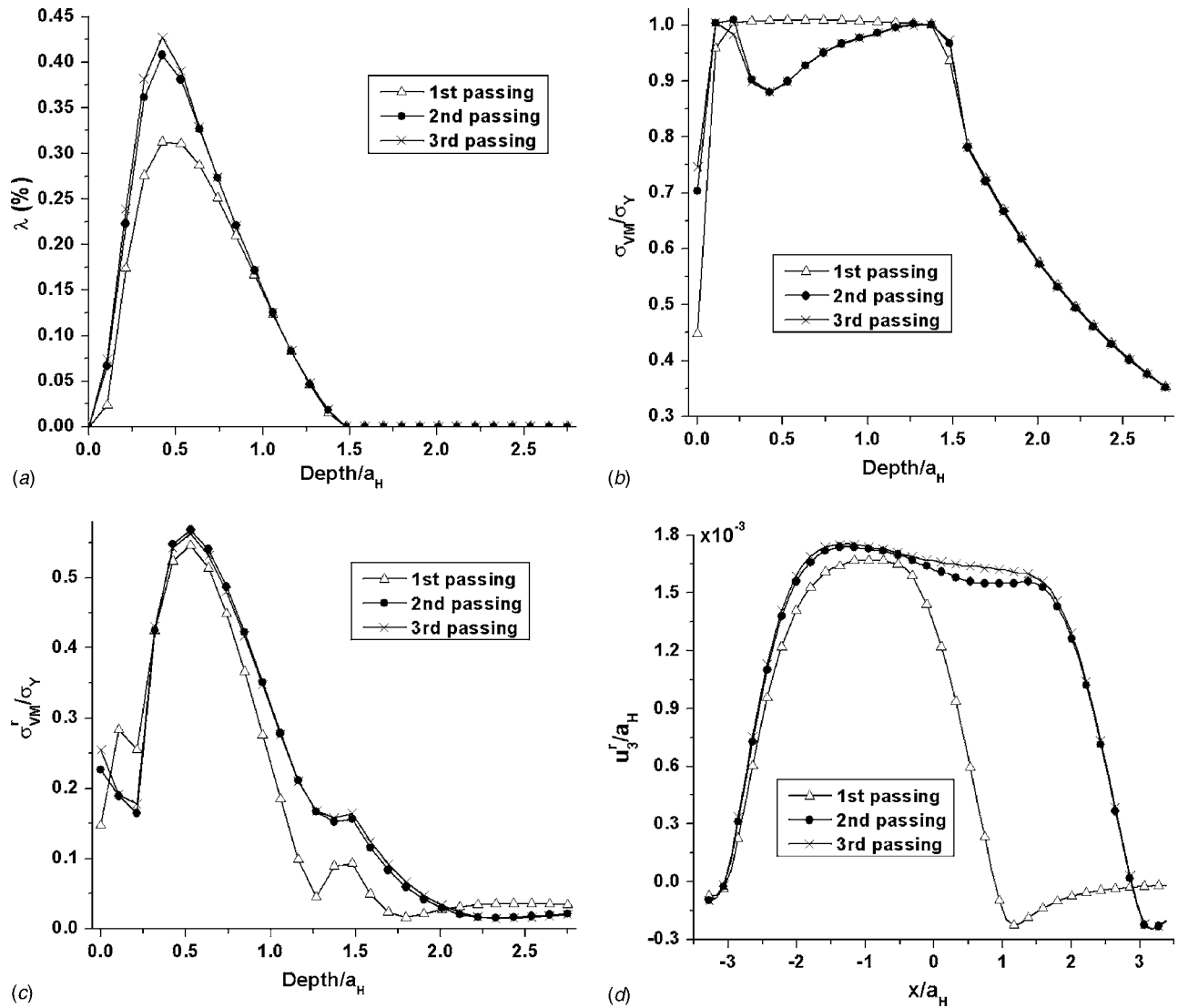


Fig. 6 Simulation results obtained using the KP hardening law when the indenter passes the origin for the first three rolling contacts: (a) the effective plastic strain along the z-axis, (b) the dimensionless total von Mises stress along the z-axis, (c) the dimensionless residual von Mises stress along the z-axis, and (d) the residual surface normal displacement along the x-axis (positive for the inward displacement and negative for the outward displacement)

material property. The ratio of Young's modulus E to the yield strength σ_Y is 500, $\nu=0.3$, and the spherical radius R is 8 mm. The simulation results are given as a function of the dimensionless interference ω/ω_c , where ω_c is the critical interference, indicating the transition from an elastic contact to an elasto-plastic contact:

$$\omega_c = \left(\frac{\pi KH}{2E} \right)^2 R \quad (27)$$

Here, H is the hardness of the sphere equal to $2.8\sigma_Y$. The hardness coefficient K is related to the Poisson ratio by $K=0.454+0.41\nu$.

Evolutions of the plastic region versus the dimensionless interference are plotted in Fig. 4, where a_c is the Hertz contact radius at the critical interference ω_c . A good agreement is found between the results obtained from the current model and the FEM model presented in [25]. The plastic region lies under the surface first, and then touches the surface as the interference increases up to about $\omega/\omega_c=6$.

Further verifications are made for the dimensionless contact load and the dimensionless contact area, when increasing the dimensionless interference value up to 15.6. In Fig. 5, the results

from the current semi-analytical model are compared with the solutions from the FEM [25] and the CEB [26] models. Here, A_c and P_c are the critical contact area and the critical load, respectively, when the contact interference equals ω_c . As indicated in Fig. 5(a), the contact load obtained from this model agrees with the FEM results very well in the entire loading range investigated, and the relative error is less than 2%. The CEB model predicts a higher contact load than the FEM and the current models do, because of the assumptions of volume conservation and constant mean contact pressure. Figure 5(b) also indicates a satisfied agreement between the contact area from the current model and the FEM results even at the large contact interference, and the maximum relative error is about 2.6%. Similarly, the CEB model predicts a higher contact area as compared to the results from the current and the FEM models. Considering the fact that contact conditions of commonly used engineering components are within the range of $\omega < 15.6\omega_c$, the current model can be utilized to simulate the elasto-plastic contacts in a wide range of applications accurately and efficiently.

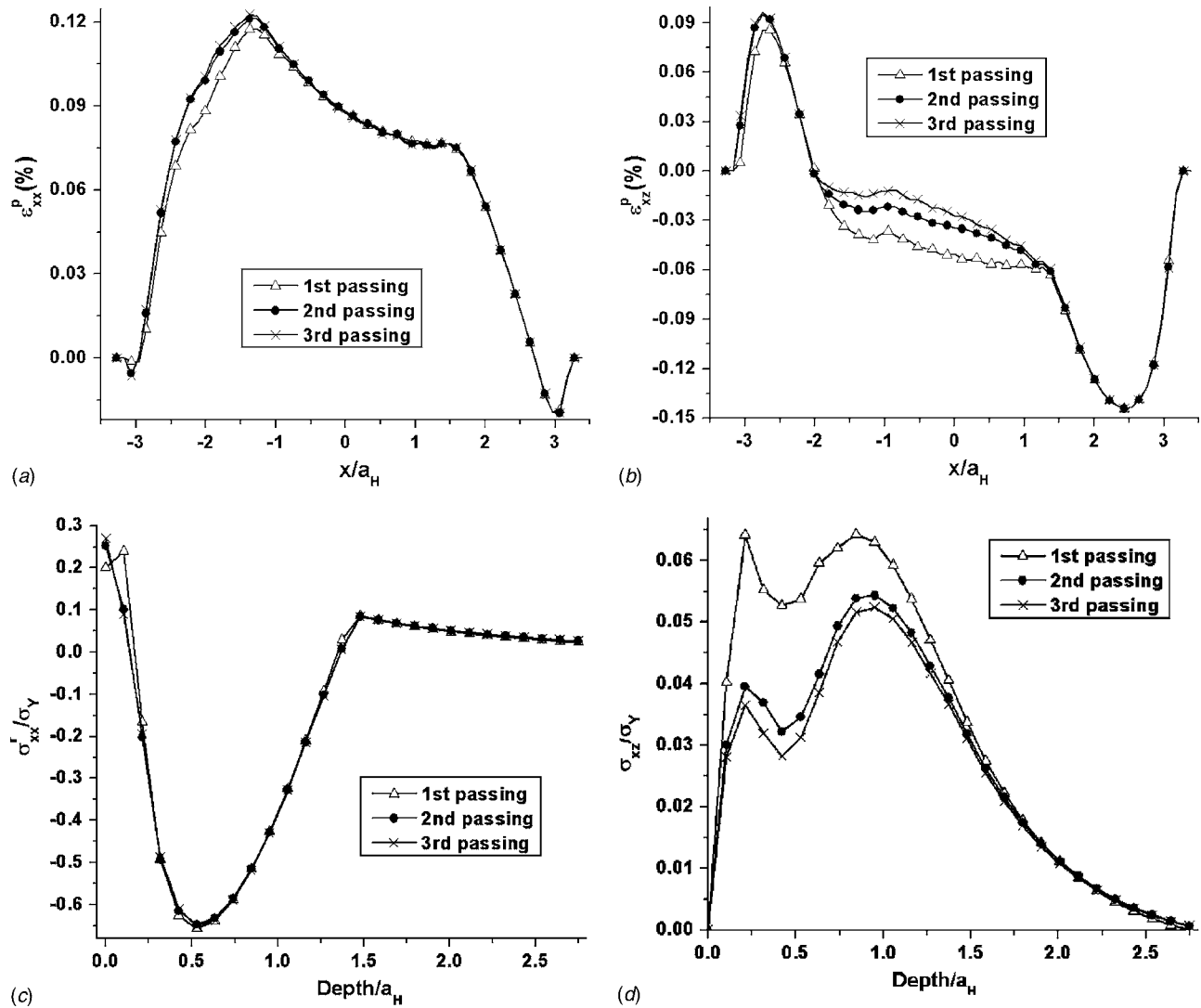


Fig. 7 Simulation results obtained using the KP hardening law when the indenter passes $x=2a_H$ for the first three rolling contacts: (a) the normal plastic strain ϵ_{xx}^p along the x-axis at $z=0.48a_H$, (b) the shear plastic strain ϵ_{xz}^p along the x-axis at $z=0.48a_H$, (c) the normal residual stress σ_{xx}^r along the z-axis, and (d) the shear residual stress σ_{xz}^r along the z-axis

4 Results and Discussion

Three-dimensional simulations are conducted using the current model for a repeated rolling or sliding contact involving a half-space and a punch, as shown in Fig. 1. Suppose that the punch is a rigid sphere with a radius of $R=18$ mm and the elasto-plastic half-space has the material properties of DP600 high strength steel. Both surfaces are assumed to be smooth. The punch is pushed into the half-space by a normal load at first; and then it is translated across the half-space surface for a certain distance when the normal load remains. After that, the indenter is disengaged from the surface and drawn back to the beginning point. Cyclic frictionless rolling contacts are simulated by repeating the whole process. Similarly, cyclic sliding contact can be simulated with shear traction applied on the interface. The repeated contact analyses result in histories of stress-strain states and plastic strain volume integral (PV) in the elasto-plastic half-space. Simulation parameters and material properties are listed in Table 1. In the following results, the stresses are normalized by the initial yield strength σ_Y , the space variables by the Hertz contact radius a_H , and the strains are represented in the form of percentage. As shown in Fig. 2, the x - and y -axes are laid on the surface, while the z -axis points into the half-space downwards.

4.1 Results of Repeated Rolling Contacts. For the simulations of rolling contacts, the maximum normal compressive load remains 25 N, corresponding to the relative Hertz peak pressure $p_0/k_s=5.2$, the Hertz contact radius $a_H=113.5$ μm , and the Hertz interference $\omega/\omega_c=3.46$. In each rolling contact cycle, the ball indenter is moved along the x -axis from $(-2a_H, 0)$ to $(2a_H, 0)$.

Results Obtained From the Kinematic Law. The simulation results of the first three rolling cycles are plotted in Figs. 6 and 7 for the kinematic plasticity hardening behavior (KP). As indicated in Fig. 6(a), the effective plastic strain along the z -axis increases with repeated rolling contacts; however, the plastic strain increment drops substantially between two consecutive cycles. The maximum increment of the effective plastic strain reduces from 0.31% at the first rolling pass to 0.02% at the third rolling pass. In addition, there is no obvious change of the plastic zone range with the cycle number. These are consistent with the observations reported by Kulkarni et al. [10].

Figure 6(b) shows that equivalent stress intensity is equal to the initial yield strength σ_Y , when the indenter passes the origin for the first time. As the rolling traction is translated repeatedly, the stress intensity increases in the layer near the surface, and de-

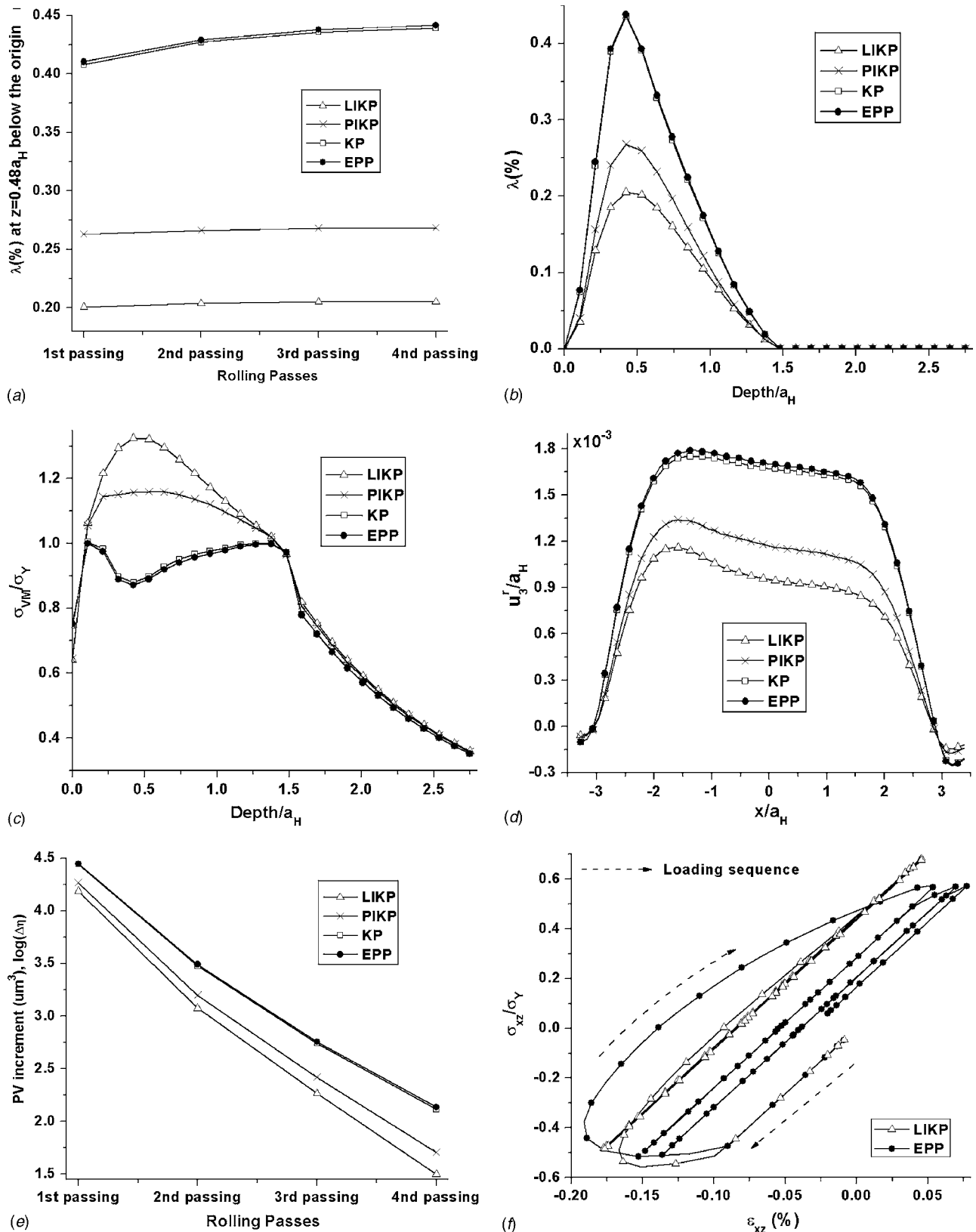


Fig. 8 Comparisons of the results from different strain hardening laws for repeated rolling contacts: (a) variations of the effective plastic strain at $z=0.48a_H$ below the origin as a function of the number of passes, (b) the effective plastic strain along the z-axis after the third rolling pass, (c) the dimensionless total von Mises stress along the z-axis when the indenter passes the origin for the third time, (d) the residual surface normal displacement along the x-axis after the third rolling pass, (e) increments of the plastic strain volume integral as a function of the number of passes, and (f) curves of the shear strain component ϵ_{xz} versus the shear stress component σ_{xz} at $z=0.48a_H$ below the origin

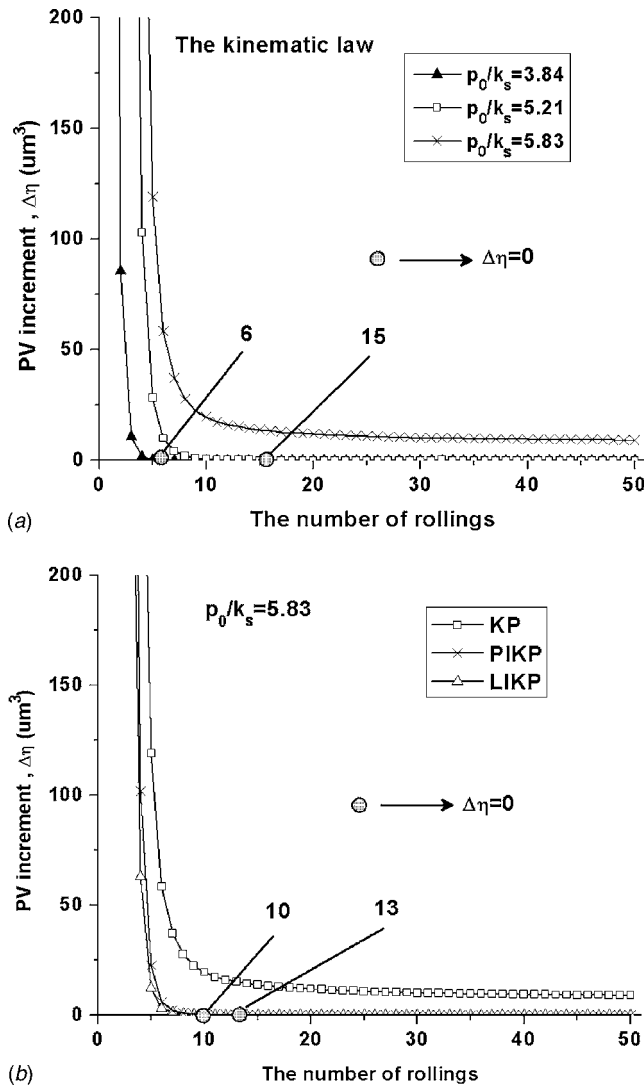


Fig. 9 Shakedown and ratchetting behaviors: (a) the increment of the plastic strain volume integral versus the rolling pass number for different relative peak pressure values p_0/k_s and (b) the PV increment versus the rolling pass number for different strain hardening laws (the numbers indicate the cycle number when shakedown occurs)

creases within the plastic zone. However, the profiles of von Mises stress do not have an obvious change, and keep the same shape of the elastic solution outside the plastic zone. The reason can be found in the residual stress history presented in Fig. 6(c). The residual stress induced by the accumulative plastic strain counteracts the elastic stress field; it is the main factor leading to the material shakedown. As compared to the results obtained after the first passing, the residual stress intensity between the surface and the Hertz depth of $0.48a_H$ in the following cycles is reduced, while the residual stress intensity is enhanced below the Hertz depth. In addition, the residual stress decays fast and has a negligibly small influence on the elastic stress field outside the plastic zone. Figure 6(d) presents the variation of the normal residual displacement along the x -axis; it indicates an increment of the depth of the residual dent induced by the cyclic rolling contact. A plowing material buildup wedge formed by plastic deformation can be found ahead of the ball indenter.

Figures 7(a) and 7(b) show the histories of plastic strain components ϵ_{xx}^p and ϵ_{xz}^p along the rolling direction at $z=0.48a_H$, respectively. The normal plastic strain ϵ_{xx}^p does not have much

variation with repeated rolling contacts. The lateral deformation along the rolling direction after the first passing can be detected from the negative value of plastic shear strain ϵ_{xz}^p . The following rolling contacts lift ϵ_{xz}^p upwards to zero and reduce the lateral deformation on the surface. Figures 7(c) and 7(d) present the evolutions of the profiles of residual stress components σ_{xx}^r and σ_{xz}^r along the z -axis, respectively. The normal residual stress σ_{xx}^r is found to be tensile in the layer near the surface. Below this layer and within the plastic zone, σ_{xx}^r is found to be compressive. The maximum compressive σ_{xx}^r is at $z=0.5a_H$; however, the maximum σ_{xx}^r does not change with repeated rolling contacts. On the other hand, the shear residual stress σ_{xz}^r decreases obviously with repeated rolling contacts.

Comparisons of Different Hardening Laws. Four different plasticity hardening behaviors: elastic-perfectly plastic (EPP), kinematic plastic (KP), linear-isotropic-kinematic plastic (LIKP), and power-isotropic-kinematic plastic (PIKP) are included in this model for comparison. Table 1 lists the work hardening parameters used in the calculations. Figure 8 shows the comparisons of the results obtained from these plasticity hardening laws.

The variation of the effective plastic strain versus the number of rolling passes at the Hertz depth of $0.48a_H$ below the origin is presented in Fig. 8(a), and the profiles of effective plastic strain along the depth at the origin after the third passing are plotted in Fig. 8(b) for different hardening laws. After the third passing, the maximum effective plastic strains obtained from the EPP, KP, PIKP, and LIKP laws are 0.437%, 0.435%, 0.268%, and 0.205%, respectively. In addition, the increments of the effective plastic strain corresponding to the PIKP and LIKP laws drop faster than those to the EPP and KP laws do. The range of the plastic zone is not affected by the strain-hardening laws. Figure 8(c) shows the dimensionless von Mises stress along the z -axis when the indenter passes the origin for the third time. Due to the strong counteracting effect of the residual stress induced by the plastic strain in the neighboring space, the von Mises stresses from the EPP and KP laws are less than the initial yield strength σ_Y within the plastic zone. However, the von Mises stresses from the PIKP and LIKP laws are larger than σ_Y , because of the weak effect of residual stress and the increased yield limit caused by work hardening. As indicated in Fig. 8(d), the surface residual dent caused by the repeated rolling contact on the EPP material is deeper than those on the materials with other hardening behaviors. The shallow dent on the surface of LIKP materials implies a strong work hardening effect, indicated by the linear isotropic hardening law.

The plastic strain volume integral (PV) η is defined in Eq. (28) as an index used to measure the volume summation of the plastic deformation in the entire space:

$$\eta = \int \int \int_V \lambda dV = \Delta\Omega \sum_{i=1}^{N_v} \lambda(i) \quad (28)$$

where $\Delta\Omega$ is the elementary volume, N_v the number of yield elements where the plastic strain has a nonzero value, and $\lambda(i)$ the effective plastic strain in the i th element. The increments of PV with the number of rolling passes for different hardening laws are presented in Fig. 8(e). The PV increments obtained using all hardening laws drop significantly. In each rolling cycle, the PV increment in the EPP material is the largest, while that in the LIKP material is the smallest.

Figure 8(f) shows the curves of shear strain ϵ_{xz} versus shear stress σ_{xz} at $z=0.48a_H$ below the origin under the repeated rolling contacts. At the beginning, the purely elastic loading curves obtained using the EPP and LIKP laws overlap. When no plastic deformation occurs at the point of $(0, 0, 0.48a_H)$, the slopes of the stress-strain curves are the same as those of the purely elastic loading curves. For materials with the EPP behavior, ϵ_{xz} changes with σ_{xz} in each cycle, although the ϵ_{xz} increment decreases with

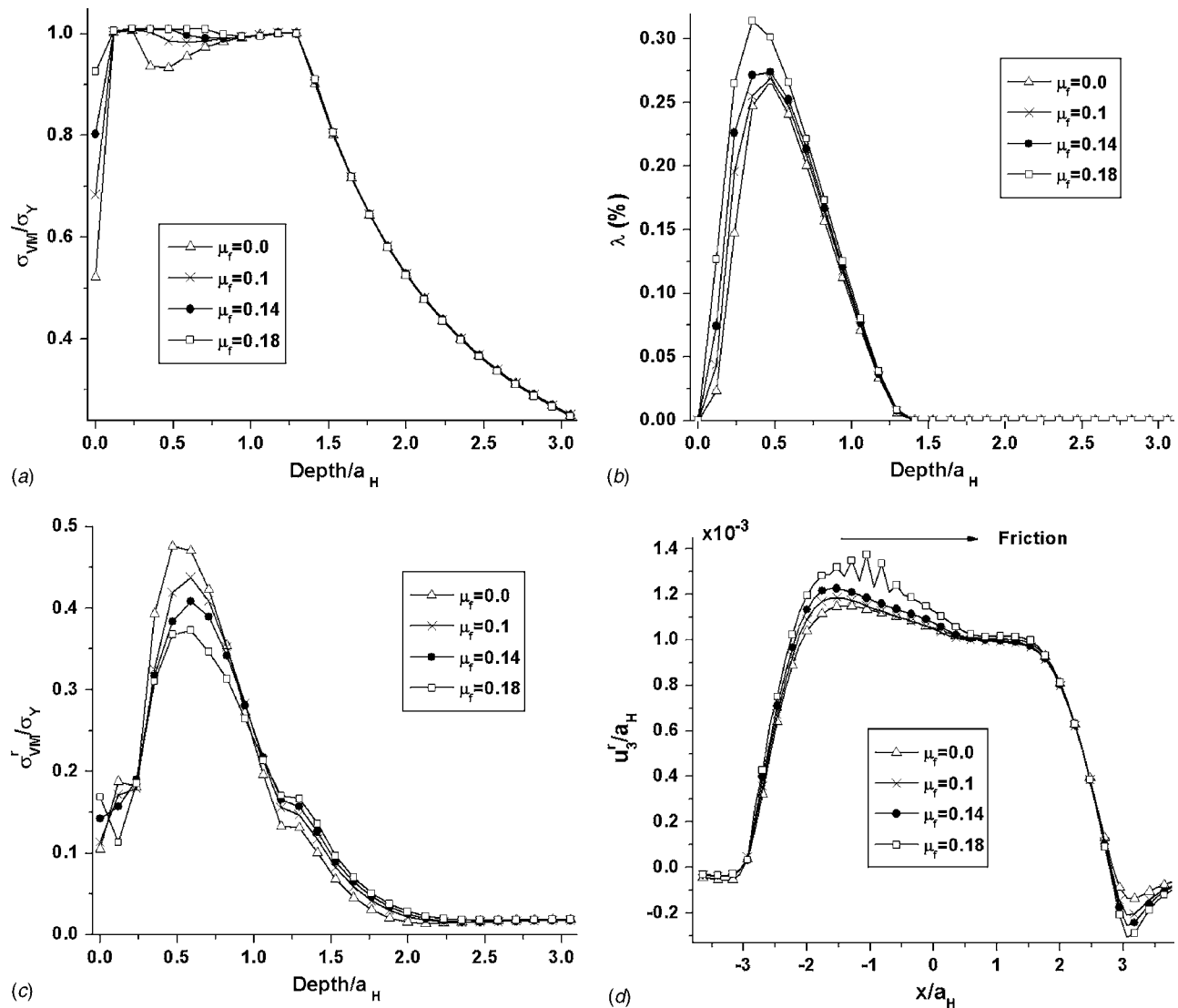


Fig. 10 Results of the repeated sliding contacts for different friction coefficients when the indenter passes the origin for the second time: (a) the dimensionless total von Mises stress along the z-axis, (b) the effective plastic strain along the z-axis, (c) the dimensionless residual von Mises stress along the z-axis, and (d) the residual surface displacement u'_3 along the x-axis

cyclic σ_{xz} . On the other hand, for the materials with the LIKP behavior, the curve of ε_{xz} versus σ_{xz} becomes almost reversed after the first rolling pass.

Shakedown and ratchetting. The steady-state (shakedown) and accumulative plastic deformation (ratchetting) in repeated rolling contacts are investigated in terms of the plastic strain volume integral. “Shakedown,” indicates the state where the PV increment vanishes beyond a certain number of rolling passes. On the other hand, “ratchetting” means the ceaseless accumulation of PV in the half-space under cyclic rolling contacts.

Johnson [2] discussed the shakedown phenomenon in terms of the relative peak pressure p_0/k_s and the theoretical shakedown limit for the three-dimensional spherical rolling contact of an elastic-perfectly plastic solid is $p_0/k_s=4.68$. Three different relative peak pressure values: $p_0/k_s=3.84$, 5.21, and 5.83, were employed in this part of the simulation work. The PV increments $\Delta\eta$ as a function of the number of rolling passes are presented in Fig. 9(a) for the material with the KP hardening behavior. The half-space can reach the shakedown state when the relative peak pressure value is 3.84 or 5.21. Actually, the increase in the peak pressure can elongate the period leading to the state of shakedown.

For $p_0/k_s=5.83$, the PV increment drops fast and converges to one stable value (about $9 \mu\text{m}^3$), by the amount of which the half-space involves a “ratchetting” of PV in each cycle. The half-space experiences shakedown (at $p_0/k_s=5.21$) above the theoretical shakedown limit for the rolling contact because the current model considers the influences of conformingly deformed contact geometry and strain hardening. In addition, the type of strain hardening laws can change the shakedown and ratchetting behaviors. Figure 9(b) shows the PV increment versus the number of passes for different hardening laws when $p_0/k_s=5.83$. For the KP material, repeated ratchetting of plastic deformation occurs under this condition, while the PIKP material shakedowns at the rolling pass number of 13 and the LIKP material shakedowns at an even lower rolling pass number.

4.2 Results of Repeated Sliding Contacts. In the simulations of repeated sliding contacts, the indenter is brought into contact with the half-space by a normal load of 18.2 N, corresponding to $p_0/k_s=4.68$, the Hertz radius $a_H=102 \mu\text{m}$, and the Hertz interference $\omega/\omega_c=2.8$. At the same time, a surface shear traction, equal to the production of friction coefficient, μ_f , and normal pressure,

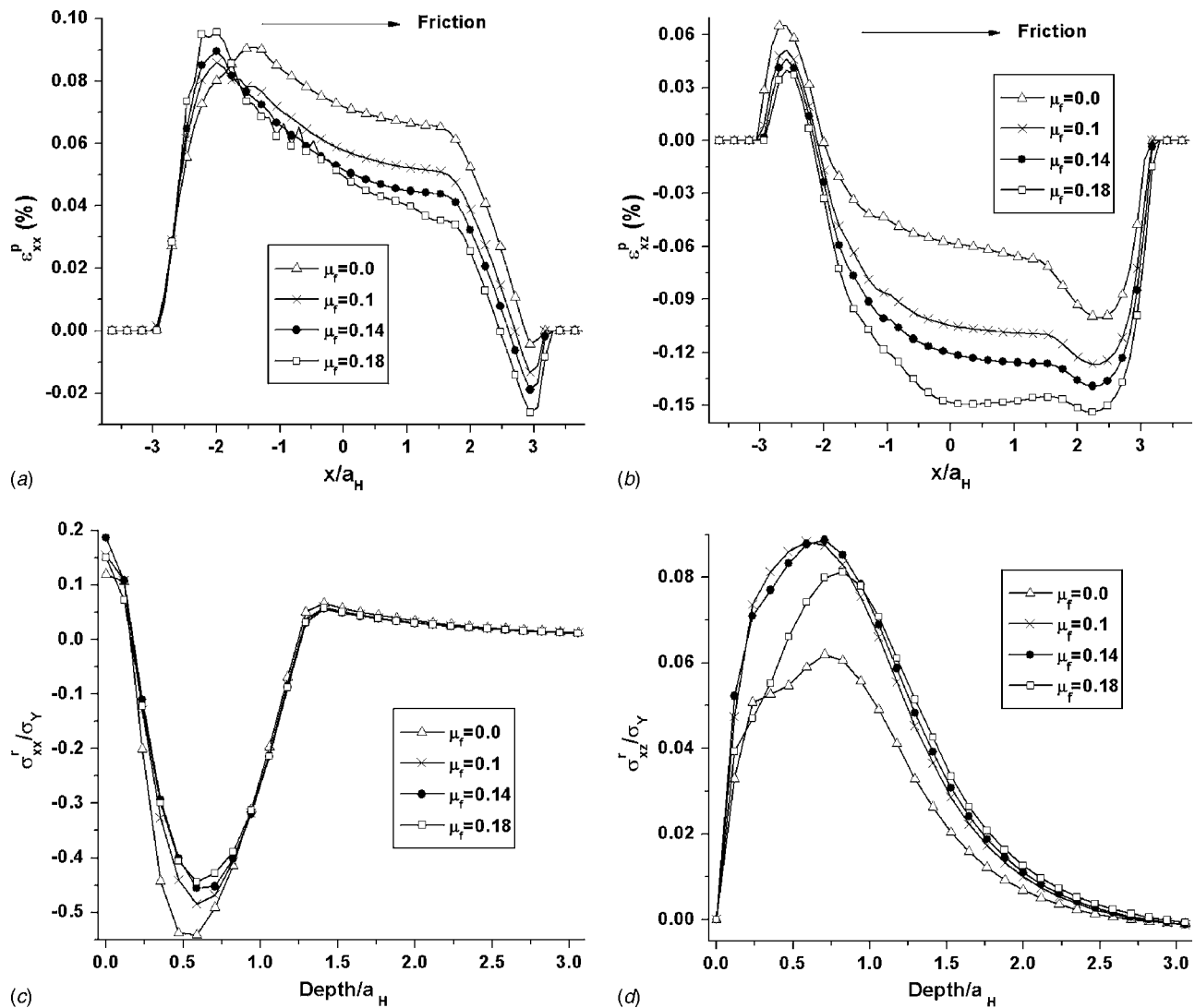


Fig. 11 Results of the repeated sliding contacts for different friction coefficients after the second passing: (a) the normal plastic strain ϵ_{xx}^p along the x-axis at $z=0.48a_H$, (b) the shear plastic strain ϵ_{xz}^p along the x-axis at $z=0.48a_H$, (c) the normal residual stress σ_{xx}^r along the z-axis, and (d) the shear residual stress σ_{xz}^r along the z-axis

is applied along the positive x -axis on the contact interface. Similarly, the rigid ball indenter slides from $(-2a_H, 0)$ to $(2a_H, 0)$ in each sliding pass, and the half-space possesses a KP hardening behavior. In order to investigate the effect of shear traction on the repeated sliding contact, various friction coefficients $\mu_f = 0.0$ (rolling), 0.1, 0.14, and 0.18 are used.

Figure 10 presents the comparisons of simulation results obtained for different friction coefficients when the indenter passes the origin for the second time. As shown in Fig. 10(a), the von Mises stress intensity increases with friction coefficient in the near surface layer and the plastic zone, while it remains unchanged below the plastic zone. The increment in stress intensity induced by shear traction can lead to more plastic deformation; it may make the materials experience ratchetting under a lighter load. Therefore, the shakedown limit can be reduced by increasing shear traction. This is consistent with the well known conclusion drawn by Johnson in [2]. Figure 10(b) indicates that the friction coefficient increment enhances the effective plastic strain, and also lifts the position of the maximum effective plastic strain towards the surface. However, the depth of plastic zone is not influenced by friction. Figure 10(c) presents the residual stress profiles along the z -axis for various friction coefficients. The increase in friction coefficient reduces the residual stress intensity in the

plastic zone. Contrary to the residual stress intensity in repeated rolling contacts, the residual stress intensity decreases with the increased effective plastic strain in repeated sliding contacts. Figure 10(d) shows that the residual dent becomes deeper and the buildup wedge ahead of the indenter is higher when the contact interface has a larger friction coefficient.

The detailed information of stress-strain states for different friction coefficients is presented in Fig. 11 after the second sliding pass. The profiles of plastic strain components ϵ_{xx}^p and ϵ_{xz}^p along the x -axis at $z=0.48a_H$ are plotted in Figures 11(a) and 11(b), respectively. The increase in friction coefficient reduces the normal plastic strain ϵ_{xx}^p within the sliding zone. This is because the reversed tangential load due to the sliding contact can relax the normal plastic deformation along the sliding direction. On the other hand, the shear plastic strain component ϵ_{xz}^p along the sliding direction increases significantly with the increasing friction coefficient. The reason is that the intenser shear stress field due to the increasing friction coefficient can generate larger irreversible shear plastic strain. In addition, the relatively large plastic strains ϵ_{xx}^p and ϵ_{xz}^p indicate the presence of surface lateral deformation drifting along the sliding direction. The increase of friction coefficient actually augments the degree of tangential plowing. Fig-

ures 11(c) and 11(d) show the profiles of residual stress components σ'_{xx} and σ'_{xz} along the z -axis after unloading. Figure 11(c) indicates that the compressive normal residual stress σ'_{xx} in the plastic zone decreases with friction coefficient. This behavior is consistent with the trend of the normal plastic strain component ϵ^p_{xx} . The increase in friction coefficient first enhances the tensile residual stress component σ'_{xx} near the surface, but further friction increment reduces the tensile part of σ'_{xx} . Figure 11(d) shows that the shear residual stress component σ'_{xz} first increases and then drops with the increasing friction coefficient.

5 Conclusions

A three-dimensional elasto-plastic contact model has been developed for repeated rolling and sliding contacts of a spherical indenter over a half-space. This model employed a universal integration algorithm for elasto-plasticity involving isotropic and kinematic hardening. Verification of this model was made through comparing the results obtained from the current model with published numerical solutions.

- (1) The current model was utilized to simulate the evolutions of plastic strain, elastic and residual stress, and residual normal surface displacement under the repeated rolling contacts for kinematic plastic hardening (KP) materials. As the rolling traction is translated repeatedly, the effective plastic strain increases and the total von Mises stress intensity decreases in the plastic zone, while the range of the plastic zone remains fixed.
- (2) The elastic-perfectly plastic (EPP), kinematic plastic (KP), and linear/power-isotropic-kinematic plastic (LIKP/PIKP) hardening behaviors of materials have been simulated. In terms of the capability of resisting further plastic deformation, the LIKP material is the strongest and the EPP material the weakest.
- (2) The plastic strain volume integral in the half-space was used to study the shakedown and ratchetting behaviors. The shakedown state may be readily achieved at a light load in a solid with the isotropic-kinematic plastic hardening property.
- (4) The current model was applied to simulate the stress and strain histories in repeated sliding contacts with Coulomb shear traction applied on the surface. The friction coefficient increment enhances the effective plastic strain and the total von Mises stress intensity; it reduces the residual stress intensity in the plastic zone. The presence of shear traction increases the depth of residual dent, the degree of tangential plowing, and the height of buildup ahead of the indenter.

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Nomenclature

- a_H = contact radius of the Hertz solution, mm
- B, C, n = Swift isotropic hardening law parameters, B (MPa)
- D = influence coefficients (ICs)
- E = Young's modulus, GPa
- E^* = equivalent Young's modulus, GPa, $E^* = E/(1 - \nu^2)$
- E_T = elasto-plastic tangential modulus
- $g(\lambda)$ = yield strength function, MPa
- G = Green's functions

- h, h_i = surface gap, initial gap, mm
- k_s = von Mises shear yield strength, $k_s = \sigma_Y/\sqrt{3}$, MPa
- K = Armstrong and Frederick kinematic law coefficients, MPa
- p_0 = peak pressure of the Hertz solution
- p, s = pressure and shear traction, MPa
- R = radius of the spherical punch, mm
- S_{ij} = deviatoric stress, MPa
- u_3, u_3^e, u_3^r = total normal displacement, elastic, and residual normal displacement
- W = applied contact load, N
- x, y, z = space coordinates
- X_{ij} = back stress components, MPa
- Y = shape function

Greek Letters

- γ = Armstrong and Frederick kinematic law coefficients
- Γ_c = real contact area
- Δ = mesh size, μm
- $\epsilon^p_{ij}, \epsilon_{ij}$ = plastic and total strain component
- η = plastic strain volume integral (PV), μm^3
- $d\lambda, \lambda$ = effective plastic incremental and accumulative strain
- μ = shear modulus, $\mu = 2E/(1 + \nu)$, GPa
- μ_e = equivalent shear modulus, $1/\mu_e = (1 + \nu)(1 - 2\nu)/2E$, GPa
- μ_f = friction coefficient
- ν = Poisson ratio
- $\sigma_{ij}, \sigma^e_{ij}, \sigma^r_{ij}$ = Cauchy stress components, elastic, and residual stress components, MPa
- σ_{VM} = von Mises equivalent stress, MPa
- σ_Y = initial yield strength with strain hardening, MPa
- ω = contact interference, mm

Special Marks

- $*$ = continuous convolution
- $()'$ = deviatoric operator
- $()_{,}$ = partial differential operator
- EPP = elastic-perfectly plastic hardening behavior
- KP = kinematic-plastic hardening behavior
- LIKP/PIKP = linear/power-isotropic-kinematic-plastic hardening behavior

Appendix: Derivation of Partial Differential $\partial\sigma_{VM}/\partial\lambda$

Based on Hooke's law,

$$\sigma'_{ij} = 2\mu(\epsilon'_{ij} - \epsilon^p_{ij}) \quad (A1)$$

Considering the volume conservation of the plastic deformation $\epsilon^p_{kk} = 0$, Eq. (A1) becomes

$$\sigma'_{ij} = 2\mu(\epsilon'_{ij} - \epsilon^p_{ij}) \quad (A2)$$

When the Armstrong and Frederick kinematic hardening law is used, one has the following:

$$S_{ij} = \sigma'_{ij} - X_{ij} = 2\mu(\epsilon'_{ij} - \epsilon^p_{ij}) - X_{ij}$$

$$\frac{\partial X_{ij}}{\partial \lambda} = K \frac{S_{ij}}{\sigma_{VM}} - \gamma X_{ij} \quad (A3)$$

and the total strain ϵ'_{ij} is assumed to be rate independent if the plastic strain increment $\Delta\lambda$ is sufficiently small in one loading step. Thus,

$$\frac{\partial \sigma_{VM}}{\partial \lambda} = \frac{\partial(\sqrt{3S_{ij}S_{ij}}/2)}{\partial \lambda} = \frac{3S_{ij}}{2\sigma_{VM}} \frac{\partial S_{ij}}{\partial \lambda} = \frac{3S_{ij}}{2\sigma_{VM}} \left[-2\mu \frac{\partial \epsilon_{ij}^p}{\partial \lambda} - \frac{\partial X_{ij}}{\partial \lambda} \right] \quad (A4)$$

In light of the flow rule, i.e., $d\epsilon_{ij}^p = d\lambda 3S_{ij}/2\sigma_{VM}$, one has:

$$\frac{\partial \sigma_{VM}}{\partial \lambda} = \frac{3S_{ij}}{2\sigma_{VM}} \left[-3\mu \frac{S_{ij}}{\sigma_{VM}} - K \frac{S_{ij}}{\sigma_{VM}} + \gamma X_{ij} \right] = -3\mu - K + \frac{3\gamma X_{ij} S_{ij}}{2\sigma_{VM}} \quad (A5)$$

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Stresses in a Multilayer Thin Film/Substrate System Subjected to Nonuniform Temperature

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Current methodologies used for the inference of thin film stress through curvature measurements are strictly restricted to uniform film stress and system curvature states over the entire system of a single thin film on a substrate. By considering a circular multilayer thin film/substrate system subjected to nonuniform temperature distributions, we derive relations between the stresses in each film and temperature, and between the system curvatures and temperature. These relations featured a "local" part that involves a direct dependence of the stress or curvature components on the temperature at the same point, and a "nonlocal" part, which reflects the effect of temperature of other points on the location of scrutiny. We also derive relations between the film stresses in each film and the system curvatures, which allow for the experimental inference of such stresses from full-field curvature measurements in the presence of arbitrary nonuniformities. These relations also feature a "nonlocal" dependence on curvatures making full-field measurements of curvature a necessity for the correct inference of stress. The interfacial shear tractions between the films and between the film and substrate are proportional to the gradient of the first curvature invariant, and can also be inferred experimentally.
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Keywords: multilayer thin films, nonuniform film temperatures and stresses, nonuniform system curvatures, nonlocal stress-curvature relations, interfacial shears

1 Introduction

Substrates formed of suitable solid-state materials may be used as platforms to support various thin film structures. Integrated electronic circuits, integrated optical devices and optoelectronic circuits, microelectromechanical systems deposited on wafers, three-dimensional electronic circuits, systems-on-a-chip structures, lithographic reticles, and flat panel display systems are examples of such thin film structures integrated on various types of plate substrates. The stress buildup in the thin film is important to the reliability and performance of these devices and systems.

Stoney [1] studied a system composed of a thin film of thickness h_f , deposited on a relatively thick substrate, of thickness h_s , and derived a simple relation between the curvature κ of the system and the stress $\sigma^{(f)}$ of the film as follows:

$$\sigma^{(f)} = \frac{E_s h_s^2 \kappa}{6 h_f (1 - \nu_s)} \quad (1.1)$$

In the above, the subscripts "f" and "s" denote the thin film and substrate, respectively, and E and ν are the Young's modulus and Poisson's ratio. Equation (1.1) is called the Stoney formula, and it has been extensively used in the literature to infer film stress changes from experimental measurement of system curvature changes [2].

Stoney's formula was based on a number of assumptions:

- (i) Both the film thickness h_f and the substrate thickness h_s are uniform and $h_f \ll h_s \ll R$, where R represents the characteristic length in the lateral direction (e.g., system radius R shown in Fig. 1);

- (ii) The strains and rotations of the plate system are infinitesimal;
- (iii) Both the film and substrate are homogeneous, isotropic, and linearly elastic;
- (iv) The film stress states are in-plane isotropic or equibiaxial (two equal stress components in any two, mutually orthogonal in-plane directions) while the out-of-plane direct stress and all shear stresses vanish;
- (v) The system's curvature components are equibiaxial (two equal direct curvatures) while the twist curvature vanishes in all directions; and
- (vi) All surviving stress and curvature components are spatially constant over the plate system's surface, a situation that is often violated in practice.

Despite the explicitly stated assumptions of spatial stress and curvature uniformity, the Stoney formula is often, arbitrarily, applied to cases of practical interest where these assumptions are violated. This is typically done by applying Stoney's formula pointwise, and thus extracting a local value of stress from a local measurement of the curvature of the system. This approach of inferring film stress clearly violates the uniformity assumptions of the analysis and, as such, its accuracy as an approximation is expected to deteriorate as the levels of curvature nonuniformity become more severe.

Following the initial formulation by Stoney, a number of extensions have been derived to relax some assumptions. Such extensions of the initial formulation include relaxation of the assumption of equibiaxiality as well as the assumption of small deformations/deflections. A biaxial form of Stoney formula (with different direct stress values and nonzero in-plane shear stress) was derived by relaxing the assumption (v) of curvature equibiaxiality [2]. Related analyses treating discontinuous films in the form of bare periodic lines [3] or composite films with periodic line structures (e.g., bare or encapsulated periodic lines) have also been derived [4–6]. These latter analyses have removed the assumptions (iv) and (v) of equibiaxiality and have allowed the

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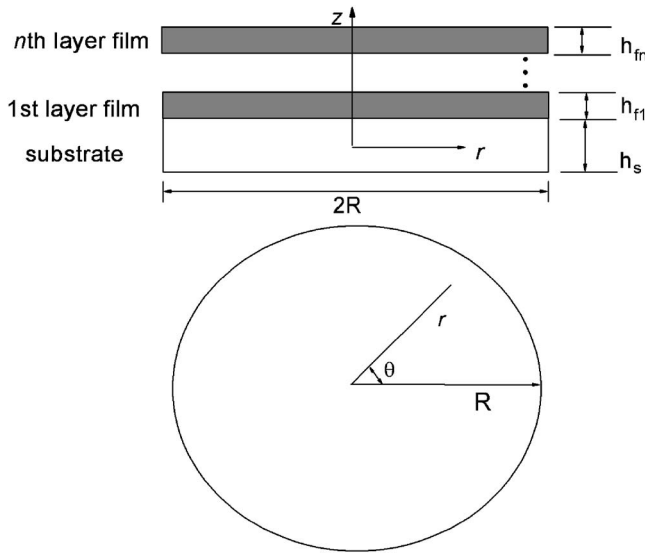


Fig. 1 A schematic diagram of a multilayer thin film/substrate system, showing the cylindrical coordinates (r, θ, z)

existence of three independent curvature and stress components in the form of two, nonequal, direct components and one shear or twist component. However, the uniformity assumption (vi) of all of these quantities over the entire plate system was retained. In addition to the above, single, multiple, and graded films and substrates have been treated in various “large” deformation analyses [7–10]. These analyses have removed both the restrictions of an equibiaxial curvature state as well as the assumption (ii) of infinitesimal deformations. They have allowed for the prediction of kinematically nonlinear behavior and bifurcations in curvature states that have also been observed experimentally [11,12]. These bifurcations are transformations from an initially equibiaxial to a subsequently biaxial curvature state that may be induced by an increase in film stress beyond a critical level. This critical level is intimately related to the system aspect ratio, i.e., the ratio of in-plane to thickness dimension and the elastic stiffness. These analyses also retain the assumption (vi) of spatial curvature and stress uniformity across the system. However, they allow for deformations to evolve from an initially spherical shape to an energetically favored shape (e.g., ellipsoidal, cylindrical, or saddle shapes) that features three different, still spatially constant, curvature components [6,11].

The above-discussed extensions of Stoney’s methodology have not relaxed the most restrictive of Stoney’s original assumption (vi) of spatial uniformity that does not allow film stress and system curvature components to vary in the thin film/substrate system. This crucial assumption is often violated in practice, since film stresses and the associated system curvatures are nonuniformly distributed. Recently, Huang et al. [13] and Huang and Rosakis [14] relaxed the assumption (vi) (and also (iv) and (v)) to study the thin film/substrate system subjected to nonuniform, axisymmetric misfit strain (in thin film) and temperature change (in both thin film and substrate), respectively, while Huang and Rosakis [15] and Ngo et al. [16] studied the thin film/substrate system subject to arbitrarily nonuniform (e.g., nonaxisymmetric) misfit strain and temperature. The most important result is that *the film stresses depend nonlocally on the system curvatures*; i.e., they depend on curvatures of the entire system. The relations between film stresses and system curvatures are established for arbitrarily nonuniform misfit strain and temperature change, and such relations degenerate to Stoney’s formula for uniform, equibiaxial stresses and curvatures.

Feng et al. [17] relaxed part of the assumption (i) to study the thin film and substrate of different radii. Ngo et al. [18] com-

pletely relax the assumption (i) to study arbitrarily nonuniform thickness of the thin film. They derived an analytical relation between the film stresses and system curvatures that allows for the accurate experimental inference of film stress from full-field curvature measurements once the film thickness distribution is known. Brown et al. [19] used two independent types of X-ray microdiffraction to measure both substrate slope and film stress across the diameter of an axisymmetric thin film/substrate specimen composed of a Si substrate on which a smaller circular W film island was deposited. The substrate slopes, measured by polychromatic (white beam) X-ray microdiffraction, were used to calculate curvature fields and to, thus, infer the film stress distribution using both the “local” Stoney formula and the new, nonlocal relation. The variable film thickness, which was independently measured, was also an input to the new relation. These were then compared with the film stress measured independently through monochromatic X-ray diffraction in the sample to validate the new analytical relation [18].

Many thin film/substrate systems involve multiple layers of thin films. The main purpose of this paper is to extend the above analyses by Huang, Rosakis, and co-workers to a system composed of multilayer thin films on a substrate subjected to nonuniform temperature distribution. We will relate stresses in each film and system curvatures to the temperature distribution, and ultimately derive a relation between the stresses in each film and system curvatures that would allow for the accurate experimental inference of film stresses from full-field and real-time curvature measurements.

2 Axisymmetric Temperature Distribution

We first consider a system of multilayer thin films deposited on a substrate subjected to axisymmetric temperature distribution $T(r)$, where r is the radial coordinate (Fig. 1). The thin films and substrate are circular in the lateral direction and have a radius R . The deformation is axisymmetric and is therefore independent of the polar angle θ , where (r, θ, z) are cylindrical coordinates with the origin at the center of the substrate (Fig. 1).

2.1 Governing Equations. Let $h_f (i=1, \dots, n)$ denote the thickness of the i th thin film (Fig. 1). The total film thickness $h_f = \sum_{i=1}^n h_{f_i}$ of all n thin films is much less than the substrate thickness h_s , and both are much less than R ; i.e., $h_f \ll h_s \ll R$. The Young’s modulus, Poisson’s ratio, and coefficient of thermal expansion of the i th film and substrate are denoted by E_{f_i} , ν_{f_i} , α_{f_i} , E_s , ν_s , and α_s , respectively.

The substrate is modeled as a plate since it can be subjected to bending and $h_s \ll R$. The thin films are modeled as membranes that have no bending rigidities due to their small thickness $h_{f_i} \ll h_s$. Therefore, they all have the same in-plane displacement $u_f(r)$ in the radial direction. The strains are $\epsilon_{rr} = du_f/dr$ and $\epsilon_{\theta\theta} = u_f/r$. The stresses in the i th thin film can be obtained from the linear thermo-elastic constitutive model as

$$\begin{aligned} \sigma_{rr}^{(i)} &= \frac{E_{f_i}}{1 - \nu_{f_i}^2} \left[\frac{du_f}{dr} + \nu_{f_i} \frac{u_f}{r} - (1 + \nu_{f_i}) \alpha_{f_i} T \right] \\ \sigma_{\theta\theta}^{(i)} &= \frac{E_{f_i}}{1 - \nu_{f_i}^2} \left[\nu_{f_i} \frac{du_f}{dr} + \frac{u_f}{r} - (1 + \nu_{f_i}) \alpha_{f_i} T \right] \end{aligned} \quad (2.1)$$

The membrane forces in the i th thin film are

$$N_r^{(f_i)} = h_{f_i} \sigma_{rr}^{(i)} \quad N_\theta^{(f_i)} = h_{f_i} \sigma_{\theta\theta}^{(i)} \quad (2.2)$$

For a nonuniform temperature distribution $T=T(r)$, the shear stress tractions at the film/substrate and film/film interfaces do not vanish, and are denoted by $\tau^{(i)}(r) (i=1, \dots, n)$ as shown in Fig. 2. The normal stress tractions σ_{zz} still vanish because thin films have

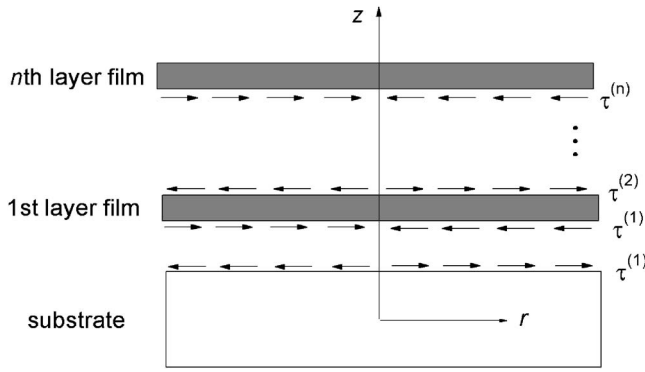


Fig. 2 A schematic diagram of the nonuniform shear traction distribution at the film/film and film/substrate interfaces

no bending rigidities. The equilibrium equations for thin films, accounting for the effect of interface shear stress tractions, become

$$\begin{cases} \frac{dN_r^{(f_1)}}{dr} + \frac{N_r^{(f_1)} - N_\theta^{(f_1)}}{r} - (\tau^{(1)} - \tau^{(2)}) = 0 \\ \frac{dN_r^{(f_2)}}{dr} + \frac{N_r^{(f_2)} - N_\theta^{(f_2)}}{r} - (\tau^{(2)} - \tau^{(3)}) = 0 \\ \vdots \\ \frac{dN_r^{(f_n)}}{dr} + \frac{N_r^{(f_n)} - N_\theta^{(f_n)}}{r} - \tau^{(n)} = 0 \end{cases} \quad (2.3)$$

Substitution of Eqs. (2.1)–(2.3) and the summation of its left-hand side yield

$$\sum_{i=1}^n \frac{E_{f_i} h_{f_i}}{1 - \nu_{f_i}^2} \left(\frac{d^2 u_f}{dr^2} + \frac{1}{r} \frac{du_f}{dr} - \frac{u_f}{r^2} \right) = \tau^{(1)} + \sum_{i=1}^n \frac{E_{f_i} h_{f_i} \alpha_{f_i}}{1 - \nu_{f_i}^2} \frac{dT}{dr} \quad (2.4)$$

Let u_s denote the displacement in the radial (r) direction at the neutral axis ($z=0$) of the substrate, and w the displacement in the normal (z) direction. The forces and bending moments in the substrate are obtained from the linear thermo-elastic constitutive model as

$$\begin{aligned} N_r^{(s)} &= \frac{E_s h_s}{1 - \nu_s^2} \left[\frac{du_s}{dr} + \nu_s \frac{u_s}{r} - (1 + \nu_s) \alpha_s T \right] \\ N_\theta^{(s)} &= \frac{E_s h_s}{1 - \nu_s^2} \left[\nu_s \frac{du_s}{dr} + \frac{u_s}{r} - (1 + \nu_s) \alpha_s T \right] \\ M_r &= \frac{E_s h_s^3}{12(1 - \nu_s^2)} \left(\frac{d^2 w}{dr^2} + \frac{\nu_s}{r} \frac{dw}{dr} \right) \\ M_\theta &= \frac{E_s h_s^3}{12(1 - \nu_s^2)} \left(\nu_s \frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} \right) \end{aligned} \quad (2.5)$$

The shear stress $\tau^{(1)}$ at the film/substrate interface is equivalent to the distributed axial force $\tau^{(1)}(r)$ and bending moment $(h_s/2)\tau^{(1)}(r)$ applied at the neutral axis ($z=0$) of the substrate. The in-plane force equilibrium equation of the substrate then becomes

$$\frac{dN_r^{(s)}}{dr} + \frac{N_r^{(s)} - N_\theta^{(s)}}{r} + \tau^{(1)} = 0 \quad (2.7)$$

The out-of-plane force and moment equilibrium equations are given by

$$\frac{dM_r}{dr} + \frac{M_r - M_\theta}{r} + Q - \frac{h_s}{2} \tau^{(1)} = 0 \quad (2.8)$$

$$\frac{dQ}{dr} + \frac{Q}{r} = 0 \quad (2.9)$$

where Q is the shear force normal to the neutral axis. Substitution of Eq. (2.5) into Eq. (2.7) yields

$$\frac{d^2 u_s}{dr^2} + \frac{1}{r} \frac{du_s}{dr} - \frac{u_s}{r^2} = (1 + \nu_s) \alpha_s \frac{dT}{dr} - \frac{1 - \nu_s^2}{E_s h_s} \tau^{(1)} \quad (2.10)$$

Elimination of Q from Eqs. (2.8) and (2.9), in conjunction with Eq. (2.6), gives

$$\frac{d^3 w}{dr^3} + \frac{1}{r} \frac{d^2 w}{dr^2} - \frac{1}{r^2} \frac{dw}{dr} = \frac{6(1 - \nu_s^2)}{E_s h_s^2} \tau^{(1)} \quad (2.11)$$

The continuity of displacement across the film/substrate interface requires

$$u_f = u_s - \frac{h_s}{2} \frac{dw}{dr} \quad (2.12)$$

Equations (2.4) and (2.10)–(2.12) constitute four ordinary differential equations for u_f , u_s , w , and $\tau^{(1)}$.

We can eliminate u_f , u_s , and w from these four equations to obtain the shear stress at the film/substrate interface in terms of temperature as

$$\tau^{(1)} = \frac{\sum_{i=1}^n \frac{E_{f_i} h_{f_i}}{1 - \nu_{f_i}^2} [(1 + \nu_s) \alpha_s - (1 + \nu_{f_i}) \alpha_{f_i}]}{1 + \sum_{i=1}^n 4 \frac{E_{f_i} h_{f_i}}{1 - \nu_{f_i}^2} \frac{1 - \nu_s^2}{E_s h_s}} \frac{dT}{dr} \quad (2.13)$$

which is a remarkable result that holds regardless of boundary conditions at the edge $r=R$. Therefore, the interface shear stress is proportional to the gradient of temperature. For uniform temperature $T=\text{constant}$, the interface shear stress vanishes; i.e., $\tau^{(1)}=0$.

Substitution of the above solution for shear stress $\tau^{(1)}$ into Eqs. (2.11) and (2.10) yields ordinary differential equations for displacements w and u_s in the substrate. Their solutions, at the limit $h_f \ll h_s$ are

$$\begin{aligned} \frac{dw}{dr} &= 6 \frac{1 - \nu_s^2}{E_s h_s^2} \sum_{i=1}^n \frac{E_{f_i} h_{f_i}}{1 - \nu_{f_i}^2} [(1 + \nu_s) \alpha_s - (1 + \nu_{f_i}) \alpha_{f_i}] \frac{1}{r} \int_0^r \eta T(\eta) d\eta \\ &\quad + \frac{B_1}{2} r \end{aligned} \quad (2.14)$$

$$u_s = (1 + \nu_s) \alpha_s \frac{1}{r} \int_0^r \eta T(\eta) d\eta + \frac{B_2}{2} r \quad (2.15)$$

where B_1 and B_2 are constants to be determined by boundary conditions. The displacement in the thin films is then obtained from Eq. (2.12) as

$$u_f = (1 + \nu_s) \alpha_s \frac{1}{r} \int_0^r \eta T(\eta) d\eta + \left(\frac{B_2}{2} - \frac{h_s B_1}{4} \right) r \quad (2.16)$$

The first boundary condition at the free edge $r=R$ requires that the net force vanish

$$\sum_{i=1}^n N_r^{(f_i)} + N_r^{(s)} = 0 \quad \text{at } r=R \quad (2.17)$$

which gives

$$B_2 = (1 - \nu_s)\alpha_s \bar{T} \quad (2.18)$$

for $h_f \ll h_s$, where $\bar{T} = (2/R^2) \int_0^R \eta T(\eta) d\eta = \int \int T dA / \pi R^2$ is the average temperature over the entire system. The second boundary condition at the free edge $r=R$ is vanishing of net moment, i.e.,

$$M_r - \frac{h_s}{2} \sum_{i=1}^n N_r^{(f_i)} = 0 \quad \text{at } r=R \quad (2.19)$$

which gives

$$B_1 = 6 \frac{1 - \nu_s^2}{E_s h_s^2} \sum_{i=1}^n \frac{E_{f_i} h_{f_i}}{1 - \nu_{f_i}^2} \left[\frac{(1 + \nu_{f_i})(1 - \nu_s)}{1 + \nu_s} (\alpha_s - \alpha_{f_i}) - (\nu_s - \nu_{f_i}) \alpha_s \right] \bar{T} \quad (2.20)$$

2.2 Stresses in Multilayer Thin Films and System Curvatures. The system curvatures are related to the out-of-plane displacement w by $\kappa_{rr} = d^2 w / dr^2$ and $\kappa_{\theta\theta} = dw / r dr$. Their sum is given by

$$\kappa_{rr} + \kappa_{\theta\theta} = 12 \frac{1 - \nu_s}{E_s h_s^2} \left[A_\alpha \bar{T} + \frac{1 + \nu_s}{2} A_{\nu\alpha} (T - \bar{T}) \right] \quad (2.21)$$

where $\bar{T}$ is the average temperature in the thin film/substrate system, and

$$A_\alpha \equiv \sum_{i=1}^n \frac{E_{f_i} h_{f_i}}{1 - \nu_{f_i}} (\alpha_s - \alpha_{f_i})$$

$$A_{\nu\alpha} \equiv \sum_{i=1}^n \frac{E_{f_i} h_{f_i}}{1 - \nu_{f_i}^2} [(1 + \nu_s) \alpha_s - (1 + \nu_{f_i}) \alpha_{f_i}] \quad (2.22)$$

The first term on the right-hand side corresponds to the (constant) average temperature $\bar{T}$, while the second term gives the deviation $T - \bar{T}$ from the constant temperature.

The difference between two system curvatures is

$$\kappa_{rr} - \kappa_{\theta\theta} = 6 \frac{1 - \nu_s^2}{E_s h_s^2} A_{\nu\alpha} \left[T - \frac{2}{r^2} \int_0^r \eta T(\eta) d\eta \right] \quad (2.23)$$

As compared to the system curvatures for a single thin film [14], Eqs. (2.21) and (2.23) can be obtained by replacing the single film properties by the sum of multilayer film properties in Eq. (2.22).

The stresses in the i th thin film can be obtained from the in-plane displacement u_f as

$$\sigma_{rr}^{(f_i)} + \sigma_{\theta\theta}^{(f_i)} = \frac{E_{f_i}}{1 - \nu_{f_i}} \{ 2(\alpha_s - \alpha_{f_i}) \bar{T} + [(1 + \nu_s) \alpha_s - 2\alpha_{f_i}] (T - \bar{T}) \} \quad (2.24)$$

$$\sigma_{rr}^{(f_i)} - \sigma_{\theta\theta}^{(f_i)} = \frac{E_{f_i}}{1 + \nu_{f_i}} (1 + \nu_s) \alpha_s \left[T - \frac{2}{r^2} \int_0^r \eta T(\eta) d\eta \right] \quad (2.25)$$

They are identical to Huang and Rosakis [14] for a single thin film if the Young's modulus, Poisson's ratio, and coefficient of thermal expansion are substituted by E_i , ν_i , and α_i of the i th thin film, respectively. The shear stresses along the film/film or film/substrate interface can be obtained from the equilibrium equation (2.3). Specifically, the shear stress of i th thin film is given by

$$\tau^{(i)} = \sum_{j=i}^n \frac{E_{f_j} h_{f_j}}{1 - \nu_{f_j}^2} [(1 + \nu_s) \alpha_s - (1 + \nu_{f_j}) \alpha_{f_j}] \frac{dT}{dr} \quad (2.26)$$

where the summation is from the i th thin film to the last (n th).

3 Extension of Stoney Formula for a Multilayer Thin Film/Substrate System Subjected to Axisymmetric Temperature Distribution

We extend the Stoney formula for a multilayer thin film/substrate system by eliminating the nonuniform axisymmetric temperature in order to establish a direct relation between the stresses in the i th thin film and system curvatures. Both $\kappa_{rr} - \kappa_{\theta\theta}$ in Eq. (2.23) and $\sigma_{rr}^{(f_i)} - \sigma_{\theta\theta}^{(f_i)}$ in Eq. (2.25) are proportional to $T - (2/r^2) \int_0^r \eta T(\eta) d\eta$, and therefore can be directly related by

$$\sigma_{rr}^{(f_i)} - \sigma_{\theta\theta}^{(f_i)} = \frac{E_s h_s^2 \alpha_s}{6(1 - \nu_s)} \frac{E_{f_i}}{1 + \nu_{f_i}} \frac{\kappa_{rr} - \kappa_{\theta\theta}}{A_{\nu\alpha}} \quad (3.1)$$

where $A_{\nu\alpha}$ is given in Eq. (2.22). We define the average system curvature $\kappa_{rr} + \kappa_{\theta\theta}$ as

$$\overline{\kappa_{rr} + \kappa_{\theta\theta}} = \frac{1}{\pi R^2} \int \int_A (\kappa_{rr} + \kappa_{\theta\theta}) \eta d\eta d\theta = \frac{2}{R^2} \int_0^R \eta (\kappa_{rr} + \kappa_{\theta\theta}) d\eta \quad (3.2)$$

which can be related to the average temperature $\bar{T}$ by averaging both sides of Eq. (2.21), i.e.,

$$\overline{\kappa_{rr} + \kappa_{\theta\theta}} = 12 \frac{1 - \nu_s}{E_s h_s^2} A_\alpha \bar{T} \quad (3.3)$$

where A_α is given in Eq. (2.22). The deviation from the average curvature $\kappa_{rr} + \kappa_{\theta\theta} - \overline{\kappa_{rr} + \kappa_{\theta\theta}}$ can be related to the deviation from the average temperature $T - \bar{T}$ from Eq. (2.21) as

$$\kappa_{rr} + \kappa_{\theta\theta} - \overline{\kappa_{rr} + \kappa_{\theta\theta}} = 6 \frac{1 - \nu_s^2}{E_s h_s^2} A_{\nu\alpha} (T - \bar{T}) \quad (3.4)$$

Elimination of temperature deviation $T - \bar{T}$ and average temperature $\bar{T}$ from Eqs. (3.3), (3.4), and (2.24) gives the sum of stresses in the i th thin film in terms of curvature as

$$\sigma_{rr}^{(f_i)} + \sigma_{\theta\theta}^{(f_i)} = \frac{E_s h_s^2}{6(1 - \nu_s)} \frac{E_{f_i}}{1 - \nu_{f_i}} \left[\frac{\alpha_s - \alpha_{f_i}}{A_\alpha} \overline{\kappa_{rr} + \kappa_{\theta\theta}} + \frac{(1 + \nu_s) \alpha_s - 2\alpha_{f_i}}{(1 + \nu_s) A_{\nu\alpha}} (\kappa_{rr} + \kappa_{\theta\theta} - \overline{\kappa_{rr} + \kappa_{\theta\theta}}) \right] \quad (3.5)$$

Equations (3.1) and (3.5) provide direct relations between stresses in each thin film and system curvatures. Stresses at a point in each thin film depend not only on curvatures at the same point (local dependence), but also on the average curvature in the entire substrate (nonlocal dependence).

The interface stress $\tau^{(i)}$ can also be directly related to system curvatures via

$$\tau^{(i)} = \frac{E_s h_s^2}{6(1 - \nu_s^2)} \frac{\sum_{k=i}^n \frac{E_{f_k} h_{f_k}}{1 - \nu_{f_k}^2} [(1 + \nu_s) \alpha_s - (1 + \nu_{f_k}) \alpha_{f_k}]}{A_{\nu\alpha}} \frac{d(\kappa_{rr} + \kappa_{\theta\theta})}{dr} \quad (3.6)$$

This provides a remarkably simple way to estimate the interface shear stress from radial gradients of the two nonzero system curvatures.

4 Arbitrary Temperature Distribution

Similar to Huang and Rosakis [15] for a single thin film on a substrate, we expand the arbitrary nonuniform temperature distribution $T(r, \theta)$ to the Fourier series:

$$T(r, \theta) = \sum_{m=0}^{\infty} T_c^{(m)}(r) \cos m\theta + \sum_{m=0}^{\infty} T_s^{(m)}(r) \sin m\theta \quad (4.1)$$

where

$$T_c^{(0)}(r) = \frac{1}{2\pi} \int_0^{2\pi} T(r, \theta) d\theta$$

$$T_c^{(m)}(r) = \frac{1}{\pi} \int_0^{2\pi} T(r, \theta) \cos m\theta d\theta$$

and

$$T_s^{(m)}(r) = \frac{1}{\pi} \int_0^{2\pi} T(r, \theta) \sin m\theta d\theta \quad (m \geq 1)$$

The analysis is similar to Huang and Rosakis [15], except it is now for multilayer thin films on a substrate.

The system curvatures are

$$\kappa_{rr} = \frac{\partial^2 w}{\partial r^2} \quad \kappa_{\theta\theta} = \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \quad \kappa_{r\theta} = \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial w}{\partial \theta} \right)$$

The sum of system curvatures is related to the temperature by

$$\begin{aligned} \kappa_{rr} + \kappa_{\theta\theta} = & 12 \frac{1 - \nu_s}{E_s h_s^2} \left\{ A_\alpha \bar{T} + \frac{1 + \nu_s}{2} A_{\nu\alpha} (T - \bar{T}) + (1 + \nu_s) \left(\frac{4}{3 + \nu_s} A_\alpha \right. \right. \\ & \left. \left. - A_{\nu\alpha} \right) \sum_{m=1}^{\infty} (m+1) \frac{r^m}{R^{2m+2}} \left[\cos m\theta \int_0^R \eta^{m+1} T_c^{(m)}(\eta) d\eta \right. \right. \\ & \left. \left. + \sin m\theta \int_0^R \eta^{m+1} T_s^{(m)}(\eta) d\eta \right] \right\} \end{aligned} \quad (4.2)$$

where $\bar{T} = (2/R^2) \int_0^R \eta T_c^{(0)}(\eta) d\eta = (1/\pi R^2) \int_A T(\eta, \varphi) dA$ is the average temperature over the entire area A of the thin film, $dA = \eta d\eta d\varphi$, and A_α and $A_{\nu\alpha}$ are given in Eq. (2.22).

The difference between two curvatures, i.e., $\kappa_{rr} - \kappa_{\theta\theta}$, and the twist $\kappa_{r\theta}$ are given by

$$\begin{aligned} \kappa_{rr} - \kappa_{\theta\theta} = & 6 \frac{1 - \nu_s^2}{E_s h_s^2} A_{\nu\alpha} \left[T - \frac{2}{r^2} \int_0^r \eta T_c^{(0)} d\eta - \sum_{m=1}^{\infty} \frac{m+1}{r^{m+2}} \left(\cos m\theta \int_0^r \eta^{m+1} T_c^{(m)} d\eta + \sin m\theta \int_0^r \eta^{m+1} T_s^{(m)} d\eta \right) \right. \\ & \left. - \sum_{m=1}^{\infty} (m-1) r^{m-2} \left(\cos m\theta \int_r^R \eta^{1-m} T_c^{(m)} d\eta + \sin m\theta \int_r^R \eta^{1-m} T_s^{(m)} d\eta \right) \right] + 6 \frac{1 - \nu_s^2}{E_s h_s^2} \left(\frac{4}{3 + \nu_s} A_\alpha - A_{\nu\alpha} \right) \sum_{m=1}^{\infty} \frac{m+1}{R^{m+2}} \left[m \left(\frac{r}{R} \right)^m \right. \\ & \left. - (m-1) \left(\frac{r}{R} \right)^{m-2} \right] \left(\cos m\theta \int_0^R \eta^{m+1} T_c^{(m)} d\eta + \sin m\theta \int_0^R \eta^{m+1} T_s^{(m)} d\eta \right) \end{aligned} \quad (4.3)$$

$$\begin{aligned} \kappa_{r\theta} = & 3 \frac{1 - \nu_s^2}{E_s h_s^2} A_{\nu\alpha} \left[- \sum_{m=1}^{\infty} \frac{m+1}{r^{m+2}} \left(\sin m\theta \int_0^r \eta^{m+1} T_c^{(m)} d\eta - \cos m\theta \int_0^r \eta^{m+1} T_s^{(m)} d\eta \right) + \sum_{m=1}^{\infty} (m-1) r^{m-2} \left(\sin m\theta \int_r^R \eta^{1-m} T_c^{(m)} d\eta \right. \right. \\ & \left. \left. - \cos m\theta \int_r^R \eta^{1-m} T_s^{(m)} d\eta \right) \right] - 3 \frac{1 - \nu_s^2}{E_s h_s^2} \left(\frac{4}{3 + \nu_s} A_\alpha - A_{\nu\alpha} \right) \sum_{m=1}^{\infty} \frac{m+1}{R^{m+2}} \left[m \left(\frac{r}{R} \right)^m - (m-1) \left(\frac{r}{R} \right)^{m-2} \right] \left(\sin m\theta \int_0^R \eta^{m+1} T_c^{(m)} d\eta \right. \\ & \left. - \cos m\theta \int_0^R \eta^{m+1} T_s^{(m)} d\eta \right) \end{aligned} \quad (4.4)$$

As compared to the system curvatures for a single thin film [15], Eqs. (4.2)–(4.4) can be obtained by replacing the film properties by the sum of multilayer film properties in Eqs. (2.22).

The sum of stresses $\sigma_{rr}^{(fi)} + \sigma_{\theta\theta}^{(fi)}$ in the i th thin film is related to the temperature by

$$\begin{aligned} \sigma_{rr}^{(fi)} + \sigma_{\theta\theta}^{(fi)} = & \frac{E_{fi}}{1 - \nu_{fi}} \left\{ 2(\alpha_s - \alpha_{fi}) \bar{T} + [(1 + \nu_s) \alpha_s - 2\alpha_{fi}] (T - \bar{T}) + 2(1 - \nu_s) \alpha_s \sum_{m=1}^{\infty} \frac{m+1}{R^{2m+2}} r^m \left(\cos m\theta \int_0^R \eta^{m+1} T_c^{(m)} d\eta \right. \right. \\ & \left. \left. + \sin m\theta \int_0^R \eta^{m+1} T_s^{(m)} d\eta \right) \right\} \end{aligned} \quad (4.5)$$

The difference between stresses, i.e., $\sigma_{rr}^{(fi)} - \sigma_{\theta\theta}^{(fi)}$, and shear stress $\sigma_{r\theta}^{(fi)}$ are given by

$$\begin{aligned} \sigma_{rr}^{(fi)} - \sigma_{\theta\theta}^{(fi)} = & \frac{E_{fi}}{1 + \nu_{fi}} (1 + \nu_s) \alpha_s \left\{ T - \frac{2}{r^2} \int_0^r \eta T_c^{(0)} d\eta - \sum_{m=1}^{\infty} \frac{m+1}{r^{m+2}} \left(\cos m\theta \int_0^r \eta^{m+1} T_c^{(m)} d\eta + \sin m\theta \int_0^r \eta^{m+1} T_s^{(m)} d\eta \right) - \sum_{m=1}^{\infty} (m-1) r^{m-2} \left(\cos m\theta \int_r^R \eta^{1-m} T_c^{(m)} d\eta + \sin m\theta \int_r^R \eta^{1-m} T_s^{(m)} d\eta \right) \right. \\ & \left. - \sum_{m=1}^{\infty} \frac{m+1}{R^{m+2}} \left[m \left(\frac{r}{R} \right)^m - (m-1) \left(\frac{r}{R} \right)^{m-2} \right] \left(\cos m\theta \int_0^R \eta^{m+1} T_c^{(m)} d\eta + \sin m\theta \int_0^R \eta^{m+1} T_s^{(m)} d\eta \right) \right\} \end{aligned} \quad (4.6)$$

$$\sigma_{r\theta}^{(f_i)} = \frac{E_{f_i}}{2(1+\nu_{f_i})} (1+\nu_s)\alpha_s \left\{ - \sum_{m=1}^{\infty} \frac{m+1}{r^{m+2}} \left(\sin m\theta \int_0^r \eta^{m+1} T_c^{(m)} d\eta - \cos m\theta \int_0^r \eta^{m+1} T_s^{(m)} d\eta \right) + \sum_{m=1}^{\infty} (m-1)r^{m-2} \left(\sin m\theta \int_r^R \eta^{1-m} T_c^{(m)} d\eta \right. \right. \\ \left. \left. - \cos m\theta \int_r^R \eta^{1-m} T_s^{(m)} d\eta \right) + \sum_{m=1}^{\infty} \frac{m+1}{R^{m+2}} \left[m \left(\frac{r}{R} \right)^m - (m-1) \left(\frac{r}{R} \right)^{m-2} \right] \left(\sin m\theta \int_0^R \eta^{m+1} T_c^{(m)} d\eta - \cos m\theta \int_0^R \eta^{m+1} T_s^{(m)} d\eta \right) \right\} \quad (4.7)$$

Equations (4.5)–(4.7) are identical to Huang and Rosakis [15] for a single thin film if the Young's modulus, Poisson's ratio, and coefficient of thermal expansion are substituted by E_i , ν_i , and α_i of the i th thin film, respectively.

The shear stresses $\tau_r^{(i)}$ and $\tau_\theta^{(i)}$ at the film/film and film/substrate interfaces are related to the temperature by

$$\tau_r^{(i)} = \sum_{j=i}^n \frac{E_{f_j} h_{f_j}}{1-\nu_{f_j}^2} [(1+\nu_s)\alpha_s - (1+\nu_{f_j})\alpha_{f_j}] \frac{\partial T}{\partial r} + 2 \left\{ \sum_{j=i}^n \frac{E_{f_j} h_{f_j}}{1-\nu_{f_j}^2} (\alpha_s - \alpha_{f_j}) - \sum_{j=i}^n \frac{E_{f_j} h_{f_j}}{1-\nu_{f_j}^2} [(1+\nu_s)\alpha_s - (1+\nu_{f_j})\alpha_{f_j}] \right\} \sum_{m=1}^{\infty} m(m+1) \frac{r^{m-1}}{R^{2m+2}} \left(\cos m\theta \int_0^R \eta^{m+1} T_c^{(m)} d\eta + \sin m\theta \int_0^R \eta^{m+1} T_s^{(m)} d\eta \right) \quad (4.8)$$

$$\tau_\theta^{(i)} = \sum_{j=i}^n \frac{E_{f_j} h_{f_j}}{1-\nu_{f_j}^2} [(1+\nu_s)\alpha_s - (1+\nu_{f_j})\alpha_{f_j}] \frac{1}{r} \frac{\partial T}{\partial \theta} - 2 \left\{ \sum_{j=i}^n \frac{E_{f_j} h_{f_j}}{1-\nu_{f_j}^2} (\alpha_s - \alpha_{f_j}) - \sum_{j=i}^n \frac{E_{f_j} h_{f_j}}{1-\nu_{f_j}^2} [(1+\nu_s)\alpha_s - (1+\nu_{f_j})\alpha_{f_j}] \right\} \sum_{m=1}^{\infty} m(m+1) \frac{r^{m-1}}{R^{2m+2}} \left(\sin m\theta \int_0^R \eta^{m+1} T_c^{(m)} d\eta - \cos m\theta \int_0^R \eta^{m+1} T_s^{(m)} d\eta \right) \quad (4.9)$$

where the summation is from the i th thin film to the last (n th).

5 Extension of Stoney Formula for a Multilayer Thin Film/Substrate System Subjected to Arbitrary Temperature Distribution

We extend the Stoney formula for a multilayer thin film/substrate system by establishing the direct relation between the stresses in each thin film and system curvatures. Similar to Huang and Rosakis [15] for a single thin film, we first define the coefficients C_m and S_m , related to the system curvatures by

$$C_m = \frac{1}{\pi R^2} \int \int_A (\kappa_{rr} + \kappa_{\theta\theta}) \left(\frac{\eta}{R} \right)^m \cos m\varphi dA \\ S_m = \frac{1}{\pi R^2} \int \int_A (\kappa_{rr} + \kappa_{\theta\theta}) \left(\frac{\eta}{R} \right)^m \sin m\varphi dA \quad (5.1)$$

where the integration is over the entire area A of the thin film, and $dA = \eta d\eta d\varphi$. Elimination of temperature gives the stresses in each thin film in terms of system curvatures by

$$\sigma_{rr}^{(f_i)} - \sigma_{\theta\theta}^{(f_i)} = \frac{E_s h_s^2}{6(1-\nu_s)} \frac{E_{f_i}}{1+\nu_{f_i} A_{\nu\alpha}} \alpha_s \left\{ \kappa_{rr} - \kappa_{\theta\theta} - \sum_{m=1}^{\infty} (m+1) \right. \\ \left. \times \left[m \left(\frac{r}{R} \right)^m - (m-1) \left(\frac{r}{R} \right)^{m-2} \right] \right. \\ \left. \times (C_m \cos m\theta + S_m \sin m\theta) \right\} \quad (5.2)$$

$$\sigma_{r\theta}^{(f_i)} = \frac{E_s h_s^2}{6(1-\nu_s)} \frac{E_{f_i}}{1+\nu_{f_i} A_{\nu\alpha}} \alpha_s \left\{ \kappa_{r\theta} + \frac{1}{2} \sum_{m=1}^{\infty} (m+1) \left[m \left(\frac{r}{R} \right)^m \right. \right. \\ \left. \left. - (m-1) \left(\frac{r}{R} \right)^{m-2} \right] (C_m \sin m\theta - S_m \cos m\theta) \right\} \quad (5.3)$$

$$\sigma_{rr}^{(f_i)} + \sigma_{\theta\theta}^{(f_i)} = \frac{E_s h_s^2}{6(1-\nu_s)} \frac{E_{f_i}}{1-\nu_{f_i}} \left\{ \frac{\alpha_s - \alpha_{f_i}}{A_\alpha} \overline{\kappa_{rr} + \kappa_{\theta\theta}} \right. \\ + \frac{(1+\nu_s)\alpha_s - 2\alpha_{f_i}}{(1+\nu_s)A_{\nu\alpha}} (\kappa_{rr} + \kappa_{\theta\theta} - \overline{\kappa_{rr} + \kappa_{\theta\theta}}) \\ + \left[\frac{3+\nu_s}{1+\nu_s} \frac{\alpha_s - \alpha_{f_i}}{A_\alpha} - 2 \frac{(1+\nu_s)\alpha_s - 2\alpha_{f_i}}{(1+\nu_s)A_{\nu\alpha}} \right] \\ \left. \times \sum_{m=1}^{\infty} (m+1) \left(\frac{r}{R} \right)^m (C_m \cos m\theta + S_m \sin m\theta) \right\} \quad (5.4)$$

where $\overline{\kappa_{rr} + \kappa_{\theta\theta}} = C_0 = (1/\pi R^2) \int \int_A (\kappa_{rr} + \kappa_{\theta\theta}) dA$ is the average curvature over entire area A of the thin film, and A_α and $A_{\nu\alpha}$ are given in Eq. (2.22). Equations (5.2)–(5.4) provide direct relations between individual film stresses and system curvatures. It is important to note that stresses at a point in each thin film depend not only on curvatures at the same point (local dependence), but also on the curvatures in the entire substrate (nonlocal dependence) via the coefficients C_m and S_m .

The shear stresses $\tau_r^{(i)}$ and $\tau_\theta^{(i)}$ at the film/film and film/substrate interfaces can also be directly related to system curvatures via

$$\tau_r^{(i)} = \frac{E_s h_s^2}{6(1-\nu_s^2)} \left(\frac{\sum_{k=i}^n \frac{E_{f_k} h_{f_k}}{1-\nu_{f_k}^2} [(1+\nu_s)\alpha_s - (1+\nu_{f_k})\alpha_{f_k}]}{A_{\nu\alpha}} \frac{\partial}{\partial r} (\kappa_{rr} + \kappa_{\theta\theta}) \right. \\ - \frac{1}{2R} \left[\frac{\sum_{k=i}^n \frac{E_{f_k} h_{f_k}}{1-\nu_{f_k}^2} [(1+\nu_s)\alpha_s - (1+\nu_{f_k})\alpha_{f_k}]}{A_{\nu\alpha}} \right. \\ \left. - \frac{(3+\nu_s) \sum_{k=i}^n \frac{E_{f_k} h_{f_k}}{1-\nu_{f_k}^2} (\alpha_s - \alpha_{f_k})}{A_\alpha} \right] \left. \right)$$

$$\times \sum_{m=1}^{\infty} m(m+1)(C_m \cos m\theta + S_m \sin m\theta) \left(\frac{r}{R}\right)^{m-1} \Bigg), \quad (5.5)$$

$$\tau_{\theta}^{(i)} = \frac{E_s h_s^2}{6(1-\nu_s^2)} \left(\frac{\sum_{k=i}^n \frac{E_{f_k} h_{f_k}}{1-\nu_{f_k}^2} [(1+\nu_s)\alpha_s - (1+\nu_{f_k})\alpha_{f_k}]}{A_{v\alpha}} \frac{1}{r} \frac{\partial}{\partial \theta} (\kappa_{rr}) \right.$$

$$+ \kappa_{\theta\theta} \Bigg) + \frac{1}{2R} \left\{ \frac{4 \sum_{k=i}^n \frac{E_{f_k} h_{f_k}}{1-\nu_{f_k}^2} [(1+\nu_s)\alpha_s - (1+\nu_{f_k})\alpha_{f_k}]}{A_{v\alpha}} \right.$$

$$\left. - \frac{(3+\nu_s) \sum_{k=i}^n \frac{E_{f_k} h_{f_k}}{1-\nu_{f_k}^2} (\alpha_s - \alpha_{f_k})}{A_{\alpha}} \right\}$$

$$\times \sum_{m=1}^{\infty} m(m+1)(C_m \sin m\theta - S_m \cos m\theta) \left(\frac{r}{R}\right)^{m-1} \Bigg) \quad (5.6)$$

This provides a way to determine the interface shear stresses from the system curvatures. It also displays a nonlocal dependence via the coefficients C_m and S_m .

6 Concluding Remarks

The analytical solution is obtained for a multilayer thin film/substrate system subjected to arbitrary temperature distribution. The stresses in each thin film and system curvatures are obtained in terms of the temperature. The direct relation between the stresses in each thin film and system curvatures is also obtained. The dependence of the film stresses on curvatures is not generally "local," i.e., the stress components at a point on the film will depend on both the local value of the curvature components (at the same point) and on the value of curvatures of all other points (nonlocal dependence).

The presence of nonlocal contributions in such relations also has implications regarding the nature of diagnostic methods needed to perform wafer-level film stress measurements. Notably, the existence of nonlocal terms necessitates the use of full-field methods capable of measuring curvature components over the entire surface of the plate system (or wafer). Furthermore, measurement of all independent components of the curvature field is necessary. This is because the stress state at a point depends on curvature contributions from the entire plate surface.

The nonuniform temperature distribution also results in shear stresses along the film/film and film/substrate interfaces. The relation between the shear stresses and system curvatures provides an effective method to estimate the shear stresses. Since film

delamination is a commonly encountered form of failure during wafer manufacturing, the ability to estimate the level and distribution of such stresses from wafer-level metrology might prove to be invaluable in enhancing the reliability of such systems.

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Anomalous Frictional Behavior in Collisions of Thin Disks Revisited

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In prior work, Calsamiglia et al. (1999, "Anomalous Frictional Behavior in Collisions of Thin Disks," ASME J. Appl. Mech., 66, pp. 146–152) reported experimental results of collisions between thin plastic disks and a relatively rigid steel barrier. In those experiments, it was observed that, contrary to a commonly held assumption in rigid body collision modeling, the ratio of tangential to normal components of the contact impulse could be substantially less than the friction coefficient even for collisions where the disk contact point did not reverse its velocity direction (i.e., for sliding collisions). In those experiments, the disk's edges were rounded to make the contact less sensitive to machining imperfections. While such impact/contact is nominally at a single point, the rounded edges make the interaction three dimensional (from the view point of analyzing deformations). Here, we revisit that problem computationally, but model the edges as flat, making the problem two dimensional. Our finite element calculations (ABAQUS) do not reproduce the anomalous frictional interactions observed in those experiments, suggesting that rounding of the edges, among other possibilities, may have played a significant role in the experimental results. [DOI: 10.1115/1.2793131]

Keywords: rigid body collision, sliding collision, friction, disks

1 Introduction

In prior work, Calsamiglia et al. [1] reported experimental result of collision experiments involving thin disks with a rigid barrier. They studied plain disks, disks with holes in them, as well as disks with smaller disks glued on to them. The disks were made of Delrin (a plastic), and the "rigid barrier" was a steel plate. The disks slid on a near-frictionless air table.

In those experiments, it was observed that, contrary to a commonly held assumption in rigid body collision modeling, the ratio of tangential to normal components of the contact impulse could be substantially less than the friction coefficient even for collisions where the disk contact point did *not* reverse its velocity direction (i.e., for sliding collisions). Just into the sliding regime, wherein the contact point velocity has the same direction before and after impact, the ratio of impulses was about half (≈ 0.08) of the separately measured friction coefficient μ (≈ 0.16 or 0.18). The ratio of impulses, using a fitted line extrapolated to grazing incidence, predicted an impulse ratio of about 0.16 in that limit. In those experiments, also, the disk's edges were rounded to make the contact point location less sensitive to machining imperfections. From the viewpoint of examining the details of the de-

formation in the disks, however, this made the interaction three dimensional.

Here, we revisit the frictional sliding collision of a disk with a rigid barrier using finite element computations (ABAQUS), but take the disk's edges as flat, and model the disk's deformation as two dimensional (plane stress). Our aim here is to see if such a two-dimensional model reproduces the anomalous frictional interactions observed in the experiments [1].

With a view to motivating the analysis presented below, we consider a hypothetical collision with a rigid barrier of a deformable body wherein all deformations are strongly localized in a small region, of negligible inertia, surrounding the nominal contact point. The microscopic near-contact region, in essence, mediates the interaction between the rigid barrier and a perfectly rigid body (for extended related discussion see Chatterjee and Ruina [2,3]). Under these assumptions, the dynamics during contact is governed by global inertia properties of the "rigid" body, local deformation behavior of the contact region, and friction. In this context, note that in the experiments [1], changing the inertia properties (changing the radius of gyration without changing the outer radius) did not have a significant effect. Thus, the anomalous frictional interaction observed in these experiments may, it seems, be largely a consequence of contact friction and local deformation behavior.

For purposes of approximate modeling of impacts in, say, granular flows, the relatively small friction coefficient may well play a correspondingly small role in important aspects of the overall dynamics. For this reason, perhaps, some readers of Ref. [1] have taken active note only of a single data point where the coefficient of normal restitution appears to slightly exceed unity, and conclusively demonstrated such extra bounce both in experiment and simulation [4,5]. However, comparable investigations, corroborative or otherwise, of the anomalous frictional interaction have not yet been reported in the literature. We mention that Stronge has noted the mismatch in frictional interaction, but looked at the data in the context of his own simplified rigid body collision model (see Ref. [6], especially Fig. 5.15, p. 109); he has obtained a horizontal line, drawn through the rising data points, that serves his modeling purposes. In this context, the experimental data raise a question in the mechanics of deformable solids (as opposed to rigid bodies, however modeled); the answer to this question lies outside the rigid body collision models. It is with this motivation that we undertake a detailed elastodynamic simulation of some contact interactions that seem relevant to the experiments.

In this work, we model the contact using Coulomb friction, which we expect to give reasonable results. There is consistency in the impulse ratios observed in the experiments for several disks with different inertia properties, indicating that the experimental observation is in some sense robust: it might survive under small errors in the friction modeling. The local deformation behavior in the disks poses more difficult questions. For example, since the disks are flat and thin, can elastodynamics in two dimensions capture this behavior? Or, since the edges are rounded, is elastodynamics in three dimensions needed to capture this behavior? Is dissipation (e.g., viscoelastic behavior) or material nonlinearity (e.g., plastic deformation) crucial? We do not know.

In this brief note, we report the results of the most straightforward investigation possible, namely, that of a purely elastic disk in plane stress hitting a rigid barrier. A more detailed and careful study of the three-dimensional interaction, we hope, may be undertaken in the future. The present two-dimensional calculations, which do *not* reproduce the experimentally observed anomalous behavior, serve to motivate that computationally more demanding three-dimensional study.

2 Model Details

We have used ABAQUS Version 6.5 (explicit) for all finite element calculations. We modeled the circular disk in two dimensions (plane stress) with high mesh refinement near the contact

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region, with 1920 elements of the four noded quadrilateral type. The rigid barrier's surface is represented as such in the model, as allowed by the software [7].

The geometrical and material properties of the disk are purely notional, because the outcome of the calculation is determined by disk radius, density, Young's modulus, and velocity magnitude (all four have physical units), as well as the material's Poisson's ratio, the incident velocity direction and the Coulomb friction coefficient (all three quantities are dimensionless). Of the first four quantities, we will demonstrate negligible dependence on velocity magnitude, leaving only three important dimensional quantities. By suitably choosing units of mass, length, and time, these three may be assigned any arbitrary positive values we like. The diameter and thickness of the disk were taken as 40 mm and 1 mm, respectively (the software asks for a thickness even for plane stress, but the thickness is irrelevant for our purposes). The disk material's Young's modulus was arbitrarily chosen as $E=210 \times 10^9 \text{ N/m}^2$ (which matches steel, not plastic, but is inconsequential anyway), Poisson's ratio was $\nu=0.3$, and density was $\rho=7800 \text{ kg/m}^3$ (again, steel instead of plastic). Note that in the experiment the plastic disks collided against a steel barrier, which being much stiffer may be treated as rigid. In our model, the barrier was treated as rigid as well. The friction coefficient was taken as 0.2 (in Ref. [1], it was between about 0.16 and 0.18).

In our simulations, we considered three velocity magnitudes (0.4 m/s, 1.0 m/s, and 1.5 m/s). For each velocity magnitude held constant, the incidence angle θ was varied. Therefore, the normal component of collision of precollision contact velocity was proportional to $\cos \theta$. The initial angular velocity of the disk was zero in every case.

3 Calculation Details

We used ABAQUS's explicit solver. The contact time was first estimated for each impact configuration using one calculation with a reasonably small time step (the contact time was found to be on the order of 75–85 microseconds). Then the calculation was redone so that there were 5000 time steps within the estimated contact time to ensure acceptable accuracy.

At the end of each impact, there were residual vibrations in the disk. As a result, the postcollision velocity of the central node of the disk was not an accurate representation of the corresponding rigid body velocity. To eliminate the effect of these postcollisional residual vibrations (which would damp out rapidly in a real material), we numerically computed the integral of density times velocity over the disk volume (element by element, using a separate routine) to find the net postcollision momentum of the disk. The postcollision velocity of the disk center of mass was then obtained upon dividing by the disk mass. The difference between this calculated velocity (which we used) and the instantaneous velocity of the central node was typically found to be on the order of 1% or 2%.

The quantities calculated for comparison with Ref. [1] were normal and tangential restitution coefficients as well as the tangential to normal contact impulse ratio.

Normal restitution is computed directly using center of mass velocities. Tangential restitution calculations are more indirect.

First, using the change in the center of mass velocity, the impulse components are found. This gives the impulse ratio.

The tangential impulse component also gives the impulsive moment about the disk center, which in turn gives the change in the angular velocity of the disk. The postcollision contact point velocity is then calculated, and the tangential restitution coefficient obtained.

4 Results

In the range of velocities studied, it was found that results did not significantly depend on the velocity magnitude. This is con-

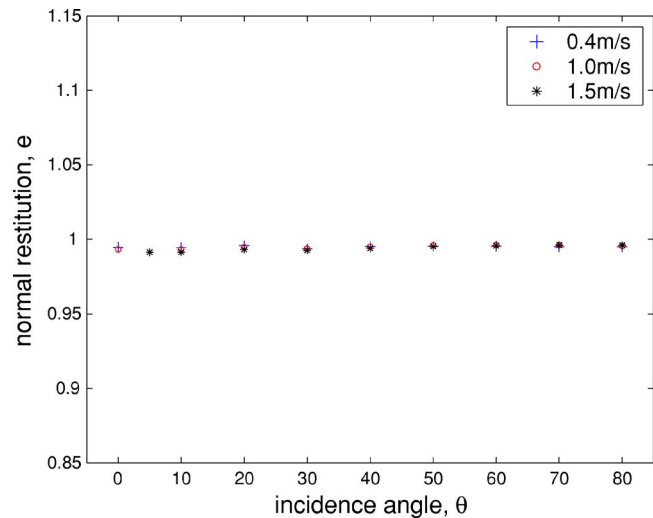


Fig. 1 Normal restitution coefficient

sistent with the almost-linear nature of two-dimensional contact mechanics, where the nonlinearity involves a logarithmic term (weak function) [8].

The coefficient of normal restitution e , defined as ratio of normal separation velocity to normal approach velocity, ranged between 0.991 and 0.996 for all three velocities shown in Fig. 1. It showed no significant dependence on incidence angle, consistent with Ref. [1].

The coefficient of tangential restitution, commonly used in modeling disk collisions and often assumed a constant [4,9], was found to vary significantly with incidence angle within the non-sliding regime (i.e., where the tangential velocity of the contact point reverses direction). These results (see Fig. 2) are qualitatively and loosely consistent though not identical with the experimental results of Ref. [1].

The impulse ratio was found to increase roughly linearly with incidence angle (measured from normal) up to about 50 deg and flatten out at about 0.2 thereafter (Fig. 3). In particular, for sliding collisions (in the sense used in rigid body collision modeling, i.e., where the tangential velocity of the contact point does not reverse direction), the impulse ratio is essentially equal to the friction coefficient. The value slightly larger than 0.2 at an incidence angle

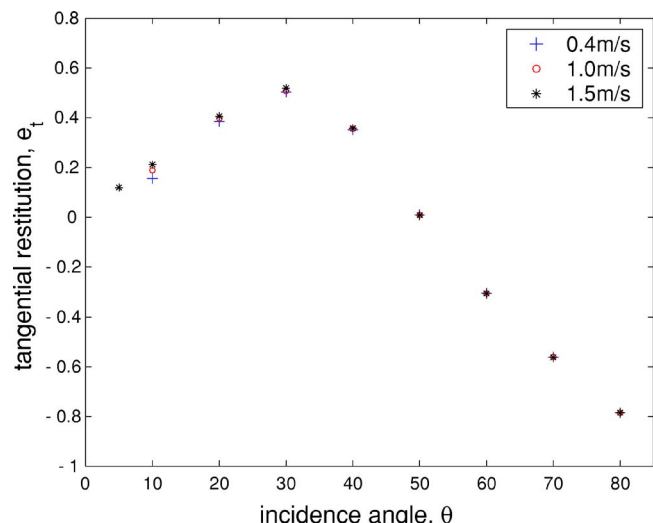


Fig. 2 Tangential restitution coefficient

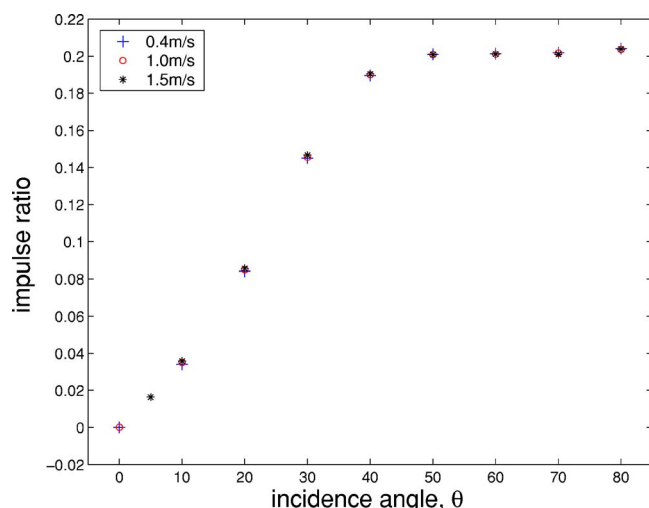


Fig. 3 Impulse ratio

of 80 deg is about 0.204, and its difference from 0.2 may be taken as representative of the numerical errors incurred in the finite element solution. The impulse ratios obtained do *not* match the experimental results of Ref. [1].

5 Discussion

The primary mismatch with the experimental results is in the impulse ratio. In Ref. [1], the disk's edges were rounded. Thus, we speculate that the contact behavior of those disks might be thought of as composed of an initially very soft three-dimensional Hertz contact in series with an initially much stiffer two-dimensional Hertz contact. For higher loads, the three-dimensional contact stiffens up much faster than the two-dimensional one. For sliding collisions that are yet far from grazing incidence, it seems likely that the tangential contact forces called into play are weak (recall that the friction coefficient is 0.2, so the tangential forces are in any case never more than a fifth of the net normal forces). Then

the tangential compliance encountered is much higher than the normal compliance; tangential sticking is possible for a relatively short duration during which the normal stiffness is quickly able to reverse the normal velocity component and terminate the collision. Finally, the localized tangential deformation near the contact region is suddenly released with no real consequence, and the disk in the experiment emerges from what looks like a sliding collision but where the impulse ratio is less than μ . The lack of such a significant difference between effective tangential and normal compliances in the purely two-dimensional case studied numerically in this note may be the reason why the anomalous frictional interaction observed in the experiments is not observed here. A full three-dimensional finite element model of impacting disks with rounded edges will be much more computationally demanding than the calculation presented here, but may shed further light on this problem in future work.

It must be acknowledged, however, that (barring some unlikely combination of experimental errors that might have conspired to suggest an anomaly where none exists) there are other possible explanations for the experimental results, chiefly material dissipation and nonlinearity.

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Stiff-String Basis Functions for Vibration Analysis of High Speed Rotating Beams

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A new rotating beam finite element is developed in which the basis functions are obtained by the exact solution of the governing static homogenous differential equation of a stiff string, which results from an approximation in the rotating beam equation. These shape functions depend on rotation speed and element position along the beam and account for the centrifugal stiffening effect. Using this new element and the Hermite cubic finite element, a convergence study of natural frequencies is performed, and it is found that the new element converges much more rapidly than the conventional Hermite cubic element for the first two modes at higher rotation speeds. The new element is also applied for uniform and tapered rotating beams to determine the natural frequencies, and the results compare very well with the published results given in the literature. [DOI: 10.1115/1.2775497]

Keywords: rotating beams, finite element method, free vibration, basis function, string

1 Introduction

The prediction of natural frequencies of a rotating beam is an important practical problem and is often done using the finite element method (FEM) [1,2]. An accurate approach to develop a finite element which has been recently proposed in the literature is to select shape functions which satisfy the static part of the homogenous governing differential equation for the problem [3]. In this paper, we seek to develop new shape functions using the exact solution of the governing static differential equation of a stiff string [4]. In order to simplify the analysis required to derive the shape functions, the centrifugal force is assumed as a constant for an element, which leads to the rotating beam equation becoming the stiff-string equation within the element due to the constant applied tension. Fortunately, the stiff-string equation captures the effect of the centrifugal force and is also analytically solvable.

2 Rotating Beam and Stiff-String Equation

The schematic of a tapered rotating beam is shown in Fig. 1. The partial differential equation for free vibration of a rotating beam is given by [1]

$$[EI(x)w''']'' + m(x)\ddot{w} - [T(x)w']' = 0 \quad (1)$$

where $T(x) = \int_x^L m(x) \Omega^2 (R+x) dx + F$ is the centrifugal tensile load at a distance x from the axis of rotation, $EI(x)$ is the flexural

stiffness, $m(x)$ is the mass per unit length, w is the bending displacement, Ω is the rotation speed, R is the hub radius, L is the beam length, and F is the axial force at the end of the beam. Ignoring the inertia term in Eq. (1) yields the static homogeneous equation, which for a uniform beam reduces to

$$(EIw'')'' - [T(x)w']' = 0 \quad (2)$$

where $T(x) = m\Omega^2[R(L-x) + (L^2 - x^2)/2] + F$. The complicated expression for the $T(x)$ term makes it possible to only obtain series solutions of Eq. (2). Most works on rotating beams use cubic shape functions, which result from the solution $EIw'''' = 0$, which means that the second term in Eq. (2) is completely ignored. Instead, let us consider an approximation which computes the centrifugal stiffening terms in an approximate sense: $T(x) = T = \text{constant}$. This approximation effectively reduces the rotating beam Eq. (1) to the stiff-string equation given by [4]

$$(EIw'')'' + m\ddot{w} - Tw'' = 0 \quad (3)$$

The static homogenous form of Eq. (3) is

$$(EIw'')'' - Tw'' = 0 \quad (4)$$

Until this point, no finite element discretization has been introduced and the process of assuming a constant tension as an approximation to the centrifugal stiffening effect may appear to be rather crude. However, if we consider the beam to be divided into N finite elements, T could be assumed to be constant within the element. The constant tension approximation would then become increasingly realistic as the number of elements increase. For an i th element along the beam, the relation between local coordinate ($\bar{x}$) and global coordinate (x) from Fig. 1 is given by $x = x_i + \bar{x}$, where $x_i = \sum_{j=1}^{i-1} l_j$. For a uniform mesh used in this paper, $x_i = (i-1)l$. Using $x = x_i + \bar{x}$ and assuming $EI = EI_i = \text{constant}$ for an element, and with the tension within the element as a constant (T_i), Eq. (4) can be expressed as

$$\frac{d^4 w}{d\bar{x}^4} - C_i^2 \frac{d^2 w}{d\bar{x}^2} = 0 \quad (5)$$

where $C_i = \sqrt{T_i/EI_i}$. Equation (5) is the governing static homogenous differential equation of a stiff string in terms of the element coordinate for element i . The constant T_i in the expression of C_i for an element is approximated by taking the maximum centrifugal tension that an element undergoes. The maximum centrifugal tension T_i for an i th element can be expressed as

$$T_i = \int_{x_i}^L m_j(x) \Omega^2 (R+x) dx + F = \sum_{j=i}^{j=N} \int_{x_j}^{x_{j+1}} m_j(x) \Omega^2 (R+x) dx + F \quad (6)$$

Here, x_i is the location of the left edge of the element i and $x_{N+1} = L$. The solution of Eq. (5) is used as the displacement field,

$$w(x) = a_0 + a_1 \bar{x} + a_2 e^{-C_i \bar{x}} + a_3 e^{C_i \bar{x}} \quad (7)$$

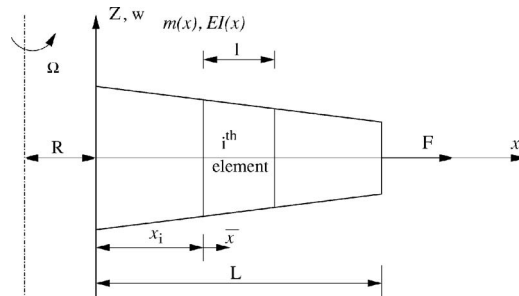


Fig. 1 Rotating tapered beam element geometry

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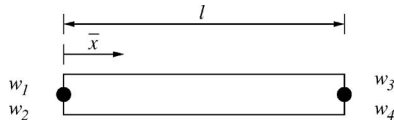


Fig. 2 Beam element

3 Stiff-String Basis Functions

Consider the two noded, four degree of freedom beam finite element shown in Fig. 2. The boundary conditions for the element of length l are given by $w(0)=w_1$, $dw(0)/dx=\theta_1=w_2$, $w(l)=w_3$, and $dw(l)/dx=\theta_2=w_4$. Putting Eq. (7) into the element boundary conditions yields $w_1=a_0+a_2+a_3$, $w_2=a_1-Ca_2+Ca_3$, $w_3=a_0+a_1l+a_2e^{-Cl}+a_3e^{Cl}$, and $w_4=a_1-a_2Ce^{-Cl}+a_3Ce^{Cl}$. Here, we have dropped the subscript i in C as the entire discussion here is relevant within the element. Solving for a_0 , a_1 , a_2 , and a_3 in terms of the nodal displacements and slopes using the above expressions, w can be approximated by

$$w = w_1N_1 + w_2N_2 + w_3N_3 + w_4N_4 \quad (8)$$

where N_1 , N_2 , N_3 , and N_4 are the shape functions and are given as

$$N_1 = \frac{R_1(\bar{x})}{D} \quad N_2 = \frac{R_2(\bar{x})}{CD} \quad N_3 = \frac{R_3(\bar{x})}{D} \quad N_4 = \frac{R_4(\bar{x})}{CD} \quad (9)$$

where

$$D = -4 + 2e^{Cl} + 2e^{-Cl} + Ce^{-Cl}l - Cle^{Cl} \quad (10)$$

$$R_1(\bar{x}) = -(-e^{Cl} - e^{-Cl} - Ce^{-Cl}l + 2 + Cle^{Cl} + C\bar{x}e^{-Cl} - C\bar{x}e^{Cl} - e^{-C\bar{x}+Cl} + e^{-C\bar{x}} + e^{C\bar{x}} - e^{C\bar{x}-Cl}) \quad (11)$$

$$R_2(\bar{x}) = e^{Cl} - Cle^{Cl} - e^{-Cl} - Ce^{-Cl}l + C\bar{x}e^{Cl} + C\bar{x}e^{-Cl} - 2C\bar{x} - e^{-C\bar{x}+Cl} + e^{-C\bar{x}+Cl}Cl + e^{-C\bar{x}} + e^{C\bar{x}-Cl} + e^{C\bar{x}-Cl}Cl - e^{C\bar{x}} \quad (12)$$

$$R_3(\bar{x}) = (e^{Cl} + e^{-Cl} - 2 + C\bar{x}e^{-Cl} - C\bar{x}e^{Cl} - e^{-C\bar{x}+Cl} + e^{-C\bar{x}} + e^{C\bar{x}} - e^{C\bar{x}-Cl}) \quad (13)$$

$$R_4(\bar{x}) = (2Cl - e^{Cl} + e^{-Cl} - 2C\bar{x} + C\bar{x}e^{Cl} + C\bar{x}e^{-Cl} - e^{-C\bar{x}+Cl} - e^{-C\bar{x}} + e^{-C\bar{x}+Cl} + e^{C\bar{x}} - e^{C\bar{x}-Cl} - e^{C\bar{x}-Cl}) \quad (14)$$

We call these the stiff-string basis functions. For a vibration analysis of a stiff string, these basis functions satisfy the static homogeneous part of the governing differential equation and therefore yield all the favorable properties discussed in Ref. [3]. For a rotating beam, their use is an approximation, which will be justified by numerical studies later in this paper. Note that the stiff-string basis functions are now also a function of the nondimensional rotational speed, element mass and stiffness, mass of outboard elements, beam length, and location of the element due to their dependence on $C = \sqrt{T_i/EI_i}$. These shape functions therefore capture the effect of the rotation speed on the element displacements as well as the fact that different locations contribute differently to the centrifugal stiffening effect.

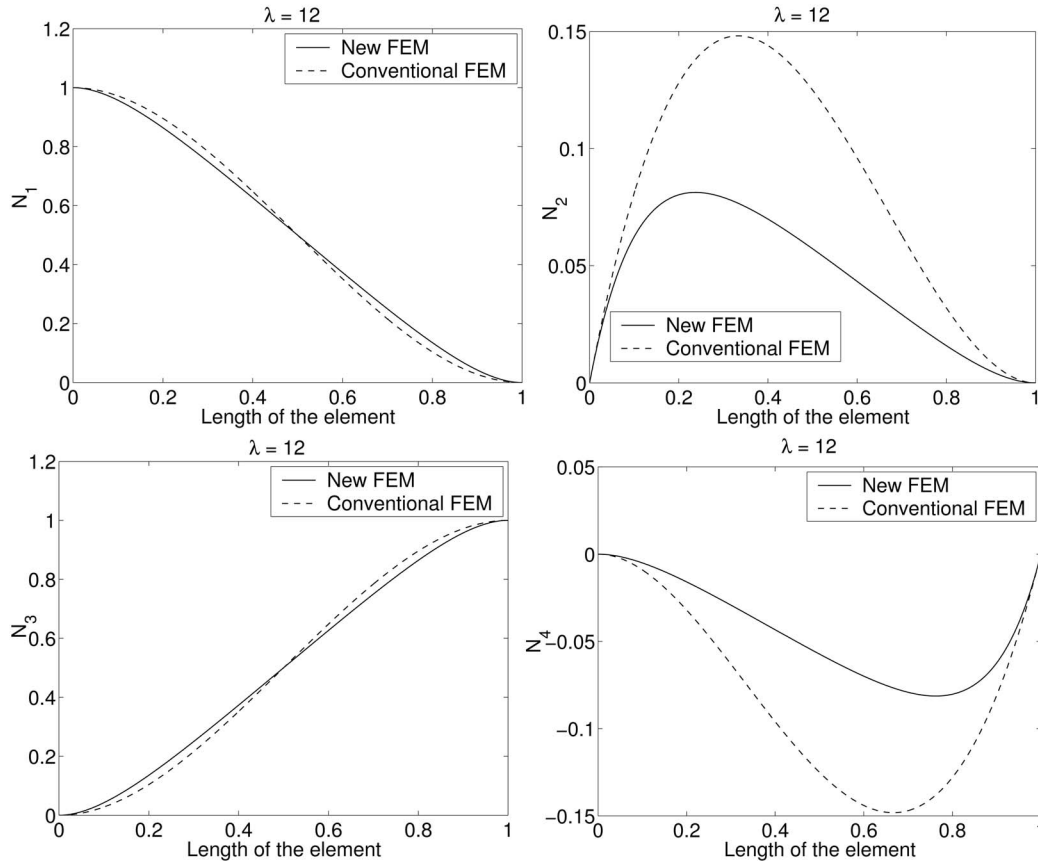


Fig. 3 Variation of shape functions along the elements ($N=1$, $\lambda=12$) with the new and conventional finite elements at high rotation speed

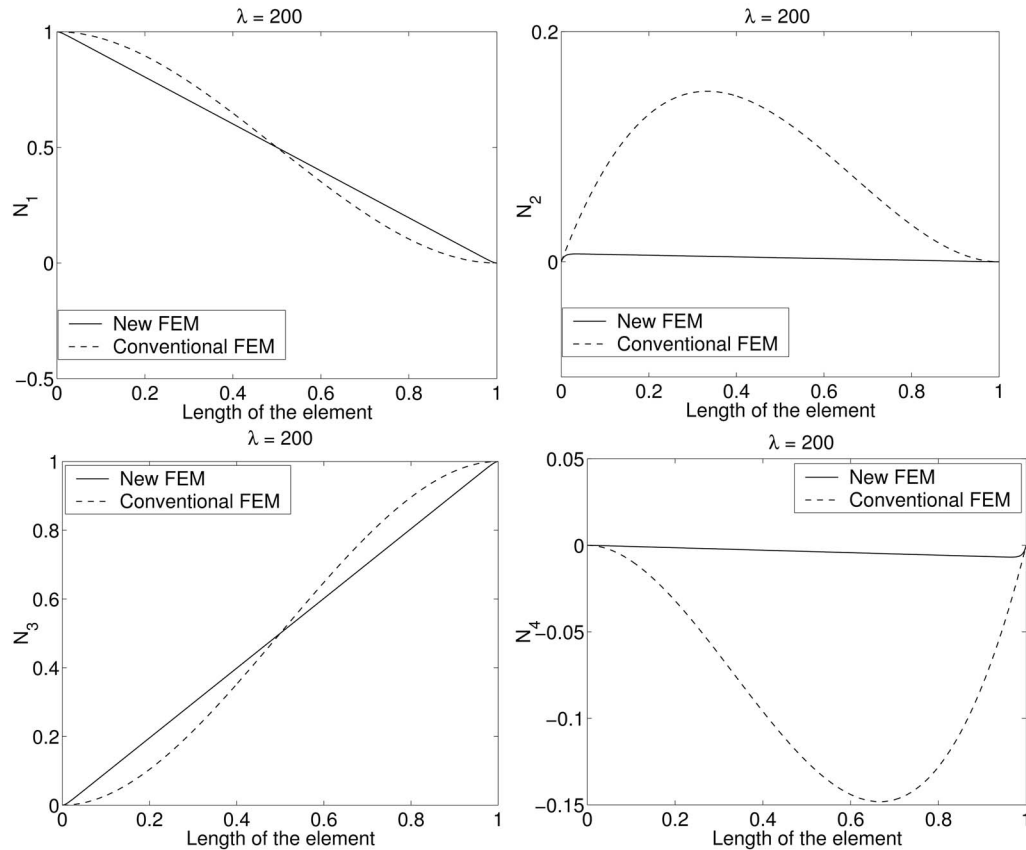


Fig. 4 Variation of shape functions along the elements ($N=1$, $\lambda=200$) with the new and conventional finite elements at very high rotation speed

The analytical limits of the stiff-string basis functions as the rotation speed tends to zero are shown in Eqs. (15) and (16) as the Hermite cubics,

$$\lim_{C \rightarrow 0} N_1 = \frac{2x^3 - 3x^2l + l^3}{l^3} \quad \lim_{C \rightarrow 0} N_2 = \frac{x^3 - 2x^2l + xl^2}{l^2} \quad (15)$$

$$\lim_{C \rightarrow 0} N_3 = \frac{-2x^3 + 3x^2l}{l^3} \quad \lim_{C \rightarrow 0} N_4 = \frac{-x^2l + x^3}{l^2} \quad (16)$$

As the rotation speed tends to infinity, the basis functions N_1 and N_3 become linear and N_2 and N_4 approach zero, as given in

$$\lim_{C \rightarrow \infty} N_1 = 1 - \frac{\bar{x}}{l} \quad \lim_{C \rightarrow \infty} N_2 = 0 \quad \lim_{C \rightarrow \infty} N_3 = \frac{\bar{x}}{l} \quad \lim_{C \rightarrow \infty} N_4 = 0 \quad (17)$$

The variation of the shape functions along the elements ($N=1$) is shown in Figs. 3 and 4 with the conventional Hermite cubic and the stiff-string basis functions at high ($\lambda=12$) and very high ($\lambda=200$) rotation speeds, respectively ($\lambda^2 = m\omega^2 L^4 / EI$). The difference between the two shape functions is clear. These shape functions therefore capture the effect of the rotation speed on the element displacements. As the rotation speed rises, the stiff-string basis functions become almost linear and the rotating beam approaches a rotating string, as is clear from Eq. (17). One can also observe from Eq. (5) that when $T_i=0$, w can be represented as a cubic polynomial and when $T_i \rightarrow \infty$, the second term dominates and w is a linear function.

4 Numerical Results

The stiff-string basis functions are used to develop the finite element equations for a free vibration of the rotating beam, and numerical results are obtained for a uniform beam and a tapered beam.

4.1 Uniform Beam. Tables 1 and 2 show a comparison of nondimensional natural frequencies of a rotating uniform cantilever and hinged beam, respectively, with results from Refs. [1,5,6]. Convergence for the first five modes was achieved using 75 uniform finite elements, and the results compare well.

The new element is now compared with the conventional element with cubic basis functions. A convergence study is done at two different rotation speeds ($\lambda=12$ and $\lambda=200$) on the first three modes since they are critical for dynamic modeling, and modes higher than 3 show little effect of rotation [2]. For the $\lambda=12$ results in Fig. 5, the convergence of the first mode is extremely

Table 1 Comparison of nondimensional natural frequencies of cantilever uniform beam

Mode	Present FEM	Wang and Wereley [6]	Wright et al. [5]	Hodges and Rutkowsky [1]
$\lambda=12$				
1	13.1702	13.1702	13.1702	13.1702
2	37.6031	37.6031	37.6031	37.6031
3	79.6145	79.6145	79.6145	79.6145
4	140.534	140.534	140.534	N/A
5	220.537	220.536	220.536	N/A

Table 2 Comparison of nondimensional natural frequencies of hinged uniform beam

Mode	Present FEM	Wang and Wereley [6]	Wright and Rutkowski [5]
$\lambda = 12$			
1	12.0000	12.0000	12.0000
2	33.7603	33.7603	33.7603
3	70.8373	70.8373	70.8373
4	126.431	126.431	126.431
5	201.123	201.122	201.122

good, though the second and third modes show slower convergence. For the $\lambda=200$ results in Fig. 6, convergence of the second mode also shows an improvement.

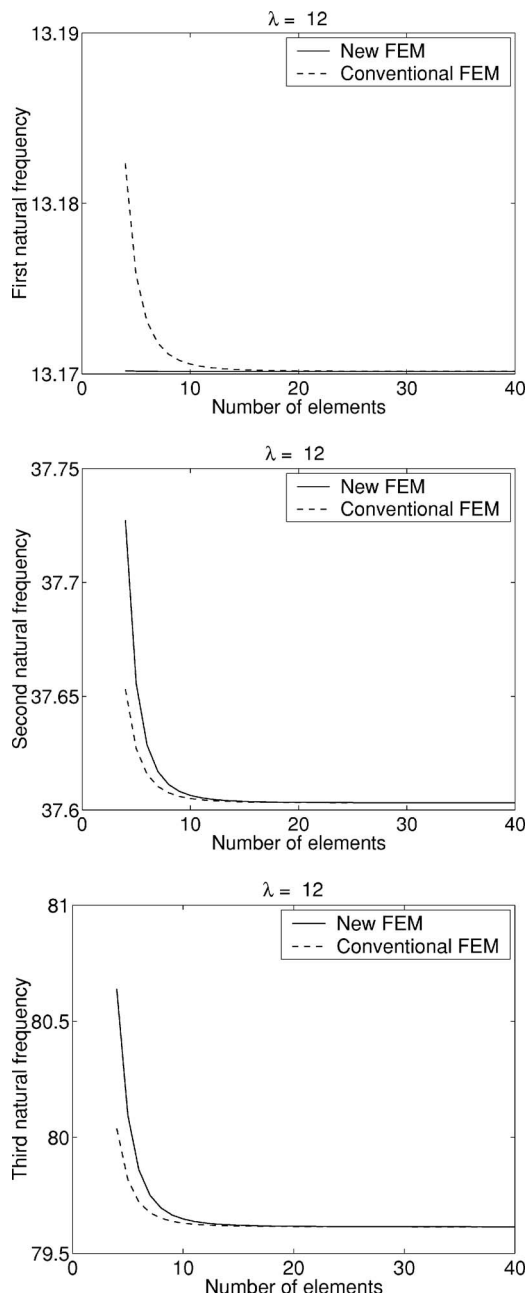


Fig. 5 Convergence of the natural frequencies with $\lambda = 12$

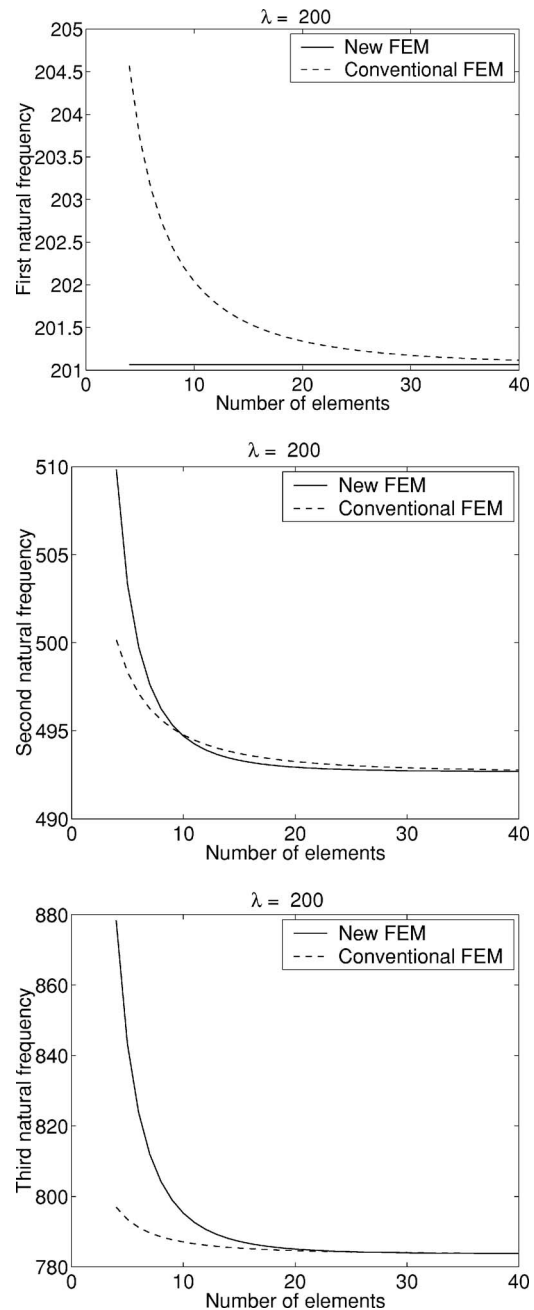


Fig. 6 Convergence of the natural frequencies with $\lambda = 200$

4.2 Tapered Rotating Beam. Consider the tapered beam used in Ref. [1] with $m(\xi) = m_0(1 - 0.5\xi)$ and $EI(\xi) = EI_0(1 - 0.5\xi)^3$. Here, m_0 and EI_0 correspond to the value of mass per unit length and flexural rigidity at the thick end of the beam ($\xi = 0$), respectively. Table 3 shows the present results and those in Refs. [1,6]. The comparison is very good.

5 Conclusions

In the present paper, new shape functions are derived for rotating beams by using the exact solution of the homogenous part of the governing static differential equation of a stiff string. In this case, the shape functions are not only functions of the element length but are also functions of the rotation speed, element location across the beam, element mass and stiffness, mass of out-board elements, and length of the beam. The element shows superior convergence of the first two modes at high rotation speed

Table 3 Comparison of nondimensional natural frequencies of tapered cantilever beam under different rotation speeds

λ	First mode			Second mode			Third mode		
	Present	Ref. [6]	Ref. [1]	Present	Ref. [6]	Ref. [1]	Present	Ref. [6]	Ref. [1]
1	3.9866	3.9866	3.9866	18.4740	18.4740	18.4740	47.4173	47.4173	47.4173
2	4.4368	4.4368	4.4368	18.9366	18.9366	18.9366	47.8717	47.8716	47.8716
3	5.0927	5.0927	5.0927	19.6839	19.6839	19.6839	48.6191	48.6190	48.6190
4	5.8788	5.8788	5.8788	20.6852	20.6852	20.6852	49.6457	49.6456	49.6456
5	6.7434	6.7434	6.7345	21.9053	21.9053	21.9053	50.9339	50.9338	50.9338
6	7.6551	7.6551	7.6551	23.3093	23.3093	23.3093	52.4633	52.4633	52.4633
7	8.5956	8.5956	8.5956	24.8647	24.8647	24.8647	54.2125	54.2124	54.2124
8	9.5540	9.5540	9.5540	26.5437	26.5437	26.5437	56.1596	56.1595	56.1595
9	10.5239	10.5239	10.5239	28.3227	28.3227	28.3227	58.2834	58.2833	58.2833
10	11.5015	11.5015	11.5015	30.1827	30.1827	30.1827	60.5640	60.5639	60.5639
11	12.4845	12.4845	12.4845	32.1085	32.1085	32.1085	62.9830	62.9829	62.9829
12	13.4711	13.4711	13.4711	34.0877	34.0877	34.0877	65.5238	65.5237	65.5237

over the conventional Hermite cubics and also presents a new shape function for rotating beams, which capture the effect of centrifugal force and element location. The poor convergence of the fundamental mode at high rotation speeds using the cubic polynomials is solved by using the stiff-string basis functions. This new element is also applied to determine the natural frequencies of uniform and tapered rotating beams, and the results compare very well with the published results.

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Dynamic Response of a Timoshenko Beam to a Moving Force

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We consider the dynamical response of a finite, simply supported Timoshenko beam loaded by a force moving with a constant velocity. The classical solution for the transverse displacement and the rotation of the cross section of a Timoshenko beam has a form of a sum of two infinite series, one of which represents the force vibrations (aperiodic vibrations) and the other one free vibrations of the beam. We show that one of the series, which represents aperiodic (force) vibrations of the beam, can be presented in a closed form. The closed form solutions take different forms depending if the velocity of the moving force is smaller or larger than the velocities of certain shear and bar velocities. [DOI: 10.1115/1.2775500]

Keywords: Timoshenko beam, moving force, vibrations

1 Introduction

The problem of a dynamic response of a structure subjected to moving loads is interesting and important. This problem occurs in dynamics of bridges, roadways, railways, and runways as well as missiles and aircrafts. Different types of structures and girders such as beams, plates, shells, and frames have been considered. Also, different models of moving loads have been assumed [1]. Deterministic and stochastic approaches have been presented [2,3]. It would be interesting to study the problem of the dynamic response of Timoshenko beam to moving loads. This problem has been considered, among others, in Refs. [4–15]. In the paper, we study dynamic response of a finite, simply supported Timoshenko beam subject to a moving force. The classical solution of the response of a finite, simply supported beam subjected to a force moving with a constant velocity, has a form of an infinite series. The main goal of this paper is to show that in the case when the finite, simply supported Timoshenko beam is loaded by a single force moving with a constant velocity, the aperiodic part of the solution can be presented in a closed form instead of an infinite series. Using the method of superposed deflections, Kązkowski [16] has shown for a simply supported Euler-Bernoulli beam that the aperiodic part of the solution can be presented in a closed form. Next, Reipert obtained a closed form solution for a beam with arbitrary boundary conditions [17] and for a frame [18]. In this paper, we use a different method to obtain the solutions in a closed form. The presented method of finding a solution in a closed form is based on the observation that the solution of the system of partial differential equations in the form of an infinite series is also a solution of an appropriate system of ordinary differential equations. This original method can also be applied in the analysis of vibrations of other types of girders loaded by moving forces. For a finite, simply supported Timoshenko beam, closed forms of the solutions take different forms whether the velocity of

the moving force is smaller or larger than the velocities of certain shear wave and bar velocity. This follows from the fact that for Timoshenko beam (contrary to Euler-Bernoulli beam), wave phenomena can occur.

2 Vibration of a Timoshenko Beam Under a Moving Force

2.1 Classical Solution. We consider vibrations of a simply supported Timoshenko beam of finite length l subjected to a force P moving with a constant velocity v . Vibrations of a beam are described by the equations

$$-\frac{GA}{\kappa} \frac{\partial^2 w(x,t)}{\partial x^2} + \frac{GA}{\kappa} \frac{\partial \varphi(x,t)}{\partial x} + A\rho \frac{\partial^2 w(x,t)}{\partial t^2} = P\delta(x-vt) \quad (1)$$

$$EI \frac{\partial^2 \varphi(x,t)}{\partial x^2} + \frac{GA}{\kappa} \frac{\partial w(x,t)}{\partial x} - \frac{GA}{\kappa} \varphi(x,t) - I\rho \frac{\partial^2 \varphi(x,t)}{\partial t^2} = 0 \quad (2)$$

where A and I denote the cross-section area and inertia momentum, respectively, E and G are the Young modulus and shear modulus, respectively, κ is the shear coefficient, ρ is the density, and $\delta(\cdot)$ is the Dirac delta.

The functions $w(x,t)$ and $\varphi(x,t)$ describe the transverse displacement and the rotation of the cross section of the beam, respectively. The bending moment $M(x,t)$ and the shear force $Q(x,t)$ are described by the relations

$$M(x,t) = -EI \frac{\partial \varphi(x,t)}{\partial x} \quad Q(x,t) = \frac{GA}{\kappa} \left[\frac{\partial w(x,t)}{\partial x} - \varphi(x,t) \right] \quad (3)$$

For a finite, simply supported beam, the boundary conditions have the forms

$$w(0,t) = w(l,t) = 0 \quad \left. \frac{\partial \varphi(x,t)}{\partial x} \right|_{x=0} = \left. \frac{\partial \varphi(x,t)}{\partial x} \right|_{x=l} = 0 \quad (4)$$

where l is the span length.

After introducing the dimensionless variables

$$\xi = \frac{x}{l} \quad T = \frac{vt}{l} \quad \xi \in [0,1] \quad T \in [0,1] \quad (5)$$

Eqs. (1) and (2) take the form

$$-w^{\text{II}}(\xi,T) + l\varphi^{\text{I}}(\xi,T) + \eta^2 \ddot{w}(\xi,T) = P_0 \delta(\xi - T) \quad (6)$$

$$lw^{\text{I}}(\xi,T) - l^2 \varphi(\xi,T) + r^2 \gamma^2 \varphi^{\text{II}}(\xi,T) - r^2 \eta^2 \ddot{\varphi}(\xi,T) = 0 \quad (7)$$

where $\eta = v/v_s$, $\gamma = v_g/v_s$, $r = \sqrt{I/A}$, $P_0 = Pl\kappa/AG$, $v_s = \sqrt{G/\kappa\rho}$, and $v_g = \sqrt{E/\rho}$. Roman numerals denote differentiation with respect to the spatial coordinate ξ and dots denote differentiation with respect to time T . For elastic materials, the inequality $v_g \geq v$ ($\gamma \geq 1$) holds true. The quantities v_s and v_g represent the shear wave velocity and bar velocity, respectively.

The boundary conditions have the forms

$$w(0,T) = w(1,T) = 0 \quad \varphi^{\text{I}}(0,T) = \varphi^{\text{I}}(1,T) = 0 \quad (8)$$

Let the initial conditions have the forms

$$w(\xi,0) = 0 \quad \dot{w}(\xi,0) = 0 \quad \varphi(\xi,0) = 0 \quad \dot{\varphi}(\xi,0) = 0 \quad (9)$$

The response of the beam $w(\xi,T)$ and $\varphi(\xi,T)$ for boundary conditions (8) is assumed to be in the form of sine and cosine series

$$w(\xi,T) = \sum_{n=1}^{\infty} y_n(T) \sin n\pi\xi \quad (10)$$

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$$\varphi(\xi, T) = \sum_{n=1}^{\infty} \varphi_n(T) \cos n\pi\xi \quad (11)$$

After substituting expressions (10) and (11) into Eqs. (6) and (7) and using orthogonality method [1], we obtain the set of ordinary differential equations

$$\ddot{y}_n(T) + \frac{(n\pi)^2}{\eta^2} y_n(T) - \frac{l(n\pi)}{\eta^2} \varphi_n(T) = \frac{2P_0}{\eta^2} \sin(n\pi T) \quad (12)$$

$$\ddot{\varphi}_n(T) + \left[\frac{l^2}{r^2\eta^2} + \frac{\gamma^2}{\eta^2} (n\pi)^2 \right] \varphi_n(T) + \frac{l(n\pi)}{r^2\eta^2} y_n(T) = 0 \quad (13)$$

These functions fulfill the initial conditions

$$y_n(0) = 0 \quad \dot{y}_n(0) = 0 \quad \varphi_n(0) = 0 \quad \dot{\varphi}_n(0) = 0 \quad (14)$$

Finally, the solutions of the system of Eqs. (6) and (7) are the sums of the particular integrals $w_A(\xi, T)$, $\varphi_A(\xi, T)$ and general integrals $w_S(\xi, T)$ and $\phi_S(\xi, T)$ and have the forms

$$\begin{aligned} w(\xi, T) &= w_A(\xi, T) + w_S(\xi, T) \\ &= 2P_0 \sum_{n=1}^{\infty} \frac{[(n\pi)^2 r^2 (\gamma^2 - \eta^2) + l^2] \sin n\pi T \sin n\pi\xi}{(n\pi)^2 [(n\pi)^2 (1 - \eta^2) r^2 (\gamma^2 - \eta^2) - l^2 \eta^2]} \\ &\quad + \sum_{n=1}^{\infty} (A_n \sin r_n T + B_n \sin s_n T) \sin n\pi\xi \end{aligned} \quad (15)$$

and

$$\begin{aligned} \varphi(\xi, T) &= \varphi_A(\xi, T) + \varphi_S(\xi, T) \\ &= 2P_0 l \sum_{n=1}^{\infty} \frac{\sin n\pi T \cos n\pi\xi}{(n\pi) [(n\pi)^2 (1 - \eta^2) r^2 (\gamma^2 - \eta^2) - l^2 \eta^2]} \\ &\quad + \sum_{n=1}^{\infty} [C_n \sin r_n T + D_n \sin s_n T] \cos n\pi\xi \end{aligned} \quad (16)$$

where

$$r_n, s_n = \sqrt{2/2r\eta} \sqrt{l^2 + r^2(n\pi)^2(1 + \gamma^2)} \pm \sqrt{[l^2 + r^2(n\pi)^2(1 - \gamma^2)][l^2 + r^2(n\pi)^2(1 + \gamma^2)]}$$

The constants A_n, B_n, C_n, D_n can be found from the initial conditions (14), and have the forms

$$\begin{aligned} A_n &= \frac{-2P_0 l [(n\pi)^2 - \eta^2 s_n^2] [(n\pi)^2 r^2 (\gamma^2 - \eta^2) + l^2] - l^2 (n\pi)^2}{(n\pi) \eta^2 r_n (r_n^2 - s_n^2) [(n\pi)^2 (1 - \eta^2) r^2 (\gamma^2 - \eta^2) - l^2 \eta^2]} \\ B_n &= \frac{2P_0 l [(n\pi)^2 - \eta^2 r_n^2] [(n\pi)^2 r^2 (\gamma^2 - \eta^2) + l^2] - l^2 (n\pi)^2}{(n\pi) \eta^2 s_n (r_n^2 - s_n^2) [(n\pi)^2 (1 - \eta^2) r^2 (\gamma^2 - \eta^2) - l^2 \eta^2]} \end{aligned} \quad (17)$$

and

$$\begin{aligned} C_n &= \frac{[(n\pi)^2 - r_n^2 \eta^2] A_n}{l(n\pi)} \\ D_n &= \frac{[(n\pi)^2 - s_n^2 \eta^2] B_n}{l(n\pi)} \end{aligned} \quad (18)$$

2.2 Closed Form Solutions. The functions $w_A(\xi, T)$ and $\varphi_A(\xi, T)$ are aperiodic vibrations and fulfill Eqs. (6) and (7), but, do not satisfy the initial conditions (9) and $w_S(\xi, T)$ and $\varphi_S(\xi, T)$ are free vibrations of the beam and satisfy homogeneous analogs of Eqs. (6) and (7). Now, we will present the aperiodic solutions $w_A(\xi, T)$ and $\varphi_A(\xi, T)$ given by the first series in expressions (15) and (16) in the closed forms.

Let us notice an important fact that these functions are solutions not only to the system of partial differential equations (6) and (7) but also to the system of ordinary differential equations

$$\begin{aligned} -(1 - \eta^2) w_A^{\text{II}}(\xi, T) + l \varphi_A^{\text{I}}(\xi, T) &= P_0 \delta(\xi - T) \\ l w_A^{\text{I}}(\xi, T) - l^2 \varphi_A(\xi, T) + r^2 (\gamma^2 - \eta^2) \varphi_A^{\text{II}}(\xi, T) &= 0 \end{aligned} \quad (19)$$

for the boundary conditions (8).

The variable T in Eqs. (19) is the only parameter, which describes the location of the moving force on the beam. The system of Eqs. (19) has been created from the system of the partial differential equations (6) and (7) by changing differentiation with respect to the time T to differentiation with respect to the geometrical coordinate ξ , namely,

$$\ddot{w}_A(\xi, T) \rightarrow w_A^{\text{II}}(\xi, T) \text{ and } \ddot{\varphi}(\xi, T) \rightarrow \varphi_A^{\text{II}}(\xi, T)$$

After solving Eqs. (19) using, for example, the Laplace transform, we can obtain the functions $w_A(\xi, T)$ and $\varphi_A(\xi, T)$ in the closed form instead of a series. The closed form of the solutions depends on the velocity of moving force.

In the case if $\eta < 1$ or $\eta > \gamma$ when the velocity of the force is smaller than the velocity of the shear wave ($\eta < 1$) or larger than the bar velocity ($\eta > \gamma$), the solutions have the forms

$$w_A(\xi, T) = \frac{P_0}{(1 - \eta^2) \eta^2} \frac{\sin \sigma(1 - T) \sin \sigma \xi}{\sigma \sin \sigma} - \frac{P_0}{\eta^2} (1 - T) \xi \quad \text{for } \xi \leq T \quad (20)$$

$$w_A(\xi, T) = \frac{P_0}{(1 - \eta^2) \eta^2} \frac{\sin \sigma T \sin \sigma(1 - \xi)}{\sigma \sin \sigma} - \frac{P_0}{\eta^2} T(1 - \xi) \quad \text{for } \xi \geq T$$

and

$$\varphi_A(\xi, T) = -\frac{P_0}{2l\eta^2} + \frac{P_0 \sin \sigma(1 - T) \cos \sigma \xi}{l\eta^2 \sin \sigma} \quad \text{for } \xi < T \quad (21)$$

$$\varphi_A(\xi, T) = \frac{P_0}{2l\eta^2} - \frac{P_0 \sin \sigma T \cos \sigma(1 - \xi)}{l\eta^2 \sin \sigma} \quad \text{for } \xi > T$$

where $\sigma^2 = l^2 \eta^2 / [r^2 (1 - \eta^2) (\gamma^2 - \eta^2)]$.

In the case when the velocity of the moving force is equal to the velocity of the shear wave, so $v = v_s$ since $\eta = 1$, we have

$$w_A(\xi, T) = -P_0(1 - T)\xi - P_0 r^2 (\gamma^2 - 1) \delta(\xi - T) \quad \text{for } \xi \leq T \quad (22)$$

$$w_A(\xi, T) = -P_0 T(1 - \xi) - P_0 r^2 (\gamma^2 - 1) \delta(\xi - T) \quad \text{for } \xi \geq T$$

and

$$\varphi_A(\xi, T) = -\frac{P_0}{2l} \quad \text{for } \xi < T \quad (23)$$

$$\varphi_A(\xi, T) = \frac{P_0}{2l} \quad \text{for } \xi > T$$

The solution (22) has a Dirac delta singularity and the function

(23) is discontinuous in the point $\xi=T$.

In the case when the velocity of the moving force is equal to the bar velocity $v=v_g$ ($\eta=\gamma$), the solutions have the forms

$$\begin{aligned} w_A(\xi, T) &= -\frac{P_0}{\eta^2}(1-T)\xi \quad \text{for } \xi < T \\ w_A(\xi, T) &= -\frac{P_0}{\eta^2}T(1-\xi) \quad \text{for } \xi > T \end{aligned} \quad (24)$$

and

$$\begin{aligned} \varphi_A(\xi, T) &= -\frac{P_0}{2l\eta^2} \quad \text{for } \xi < T \\ \varphi_A(\xi, T) &= \frac{P_0}{2l\eta^2} \quad \text{for } \xi > T \end{aligned} \quad (25)$$

If the velocity of the moving force is in the range $v_s < v < v_g$ ($1 < \eta < \gamma$), the solutions for the aperiodic vibration have the forms

$$w_A(\xi, T) = \frac{P_0}{(1-\eta^2)\eta^2} \frac{sh\sigma(1-T)sh\sigma\xi}{\sigma sh\sigma} - \frac{P_0}{\eta^2}(1-T)\xi \quad \text{for } \xi \leq T \quad (26)$$

$$w_A(\xi, T) = \frac{P_0}{(1-\eta^2)\eta^2} \frac{sh\sigma T sh\sigma(1-\xi)}{\sigma sh\sigma} - \frac{P_0}{\eta^2}T(1-\xi) \quad \text{for } \xi \geq T$$

and

$$\varphi_A(\xi, T) = -\frac{P_0}{2l\eta^2} + \frac{P_0}{l\eta^2} \frac{sh\sigma(1-T)ch\sigma\xi}{sh\sigma} \quad \text{for } \xi < T \quad (27)$$

$$\varphi_A(\xi, T) = \frac{P_0}{2l\eta^2} - \frac{P_0}{l\eta^2} \frac{sh\sigma T ch\sigma(1-\xi)}{sh\sigma} \quad \text{for } \xi > T$$

where $\sigma^2 = l^2 \eta^2 / [r^2(\eta^2 - 1)(\gamma^2 - \eta^2)]$.

Let us notice that when the condition

$$\frac{l^2 \eta^2}{(1-\eta^2)r^2(\gamma^2 - \eta^2)} = \pi^2 \quad (28)$$

is fulfilled, then the solutions (15) and (16), and also (20) and (21) tend to infinity. Thus, the critical velocity v_{cr} is equal to

$$v_{cr} = \frac{1}{\sqrt{2\pi r}} \sqrt{(l^2 + r^2 \pi^2)v_s^2 + r^2 \pi^2 v_g^2 - \sqrt{[(l^2 + r^2 \pi^2)v_s^2 + r^2 \pi^2 v_g^2]^2 - 4r^2 \pi^4 v_s^2 v_g^2}} \quad (29)$$

The closed form solution is particularly important for a bending moment and shear force. For example, if the velocity v of the moving force is smaller than the velocity v_s of the shear wave ($\eta < 1$), the bending moment and shear force have the forms

$$M(\xi, T) = M_A(\xi, T) - \frac{EI}{l} \sum_{n=1}^{\infty} (n\pi) (C_n \sin r_n T + D_n \sin s_n T) \sin n\pi\xi \quad (30)$$

where

$$M_A(\xi, T) = \frac{-P_0 EI \sigma \sin \sigma(1-T) \sin \sigma\xi}{l^2 \eta^2 \sin \sigma} \quad \text{for } \xi \leq T$$

$$M_A(\xi, T) = \frac{-P_0 EI \sigma \sin \sigma T \sin \sigma(1-\xi)}{l^2 \eta^2 \sin \sigma} \quad \text{for } \xi \geq T$$

and

$$\begin{aligned} Q(\xi, T) &= Q_A(\xi, T) + \frac{GA\eta^2}{l\kappa} \sum_{n=1}^{\infty} \left(\frac{r_n^2}{n\pi} A_n \sin r_n T \right. \\ &\quad \left. + \frac{s_n^2}{n\pi} B_n \sin s_n T \right) \cos n\pi\xi \end{aligned} \quad (31)$$

where

$$Q_A(\xi, T) = P \left[\frac{\sin \sigma(1-T) \cos \sigma\xi}{(1-\eta^2) \sin \sigma} - \frac{1}{\eta^2} \left(\frac{1}{2} - T \right) \right] \quad \text{for } \xi < T$$

$$Q_A(\xi, T) = -P \left[\frac{\sin \sigma T \cos \sigma(1-\xi)}{(1-\eta^2) \sin \sigma} + \frac{1}{\eta^2} \left(\frac{1}{2} - T \right) \right] \quad \text{for } \xi > T$$

3 Euler-Bernoulli Beam

Let us consider, for comparison, the vibration of the Euler-Bernoulli beam under a force moving with a constant velocity. The governing equation in terms of dimensionless variables (5) has the form

$$w^{IV}(\xi, T) + \sigma_0^2 \ddot{w}(\xi, T) = \frac{Pl^3}{EI} \delta(\xi - T) \quad (32)$$

where $\sigma_0^2 = \rho A v^2 l^2 / EI$

For a finite, simply supported beam, the solution has the form

$$\begin{aligned} w(\xi, T) &= w_A(\xi, T) + w_S(\xi, T) \\ &= \frac{Pl^3}{EI} \left\{ \sum_{n=1}^{\infty} \frac{\sin n\pi T \sin n\pi\xi}{(n\pi)^2 [(n\pi)^2 - \sigma_0^2]} \right. \\ &\quad \left. - \sigma_0 \sum_{n=1}^{\infty} \frac{\sin[(n\pi)^2/\sigma_0] T \sin n\pi\xi}{(n\pi)^3 [(n\pi)^2 - \sigma_0^2]} \right\} \end{aligned} \quad (33)$$

The first series in Eq. (33) represents an aperiodic vibration and has slower convergent than the second series, which represents the free vibration. The function $w_A(\xi, T)$ satisfies also the ordinary equation

$$w_A^{IV}(\xi, T) + \sigma_0^2 w_A^{II}(\xi, T) = \frac{Pl^3}{EI} \delta(\xi - T) \quad (34)$$

After solving Eq. (34) using, for example, Laplace transform, the function $w_A(\xi, T)$ can be obtained in the closed form

$$w_A(\xi, T) = \frac{Pl^3}{EI} \left[\frac{\sin \sigma_0(1-T) \sin \sigma_0\xi}{\sigma_0^3 \sin \sigma_0} - \frac{(1-T)\xi}{\sigma_0^2} \right] \quad \text{for } \xi \leq T \quad (35)$$

$$w_A(\xi, T) = \frac{Pl^3}{EI} \left[\frac{\sin \sigma_0 T \sin \sigma_0 (1 - \xi)}{\sigma_0^3 \sin \sigma_0} - \frac{T(1 - \xi)}{\sigma_0^2} \right] \quad \text{for } \xi \geq T$$

The solution (35) was obtained for the first time by Kączkowski [16] using the method of "superposed deflection."

The bending moment and shear force for the Euler-Bernoulli beam are equal to

$$M(\xi, T) = M_A(\xi, T) - 2Pl\sigma_0 \sum_{n=1}^{\infty} \frac{\sin[(n\pi)^2/\sigma_0]T \sin n\pi\xi}{(n\pi)[(n\pi)^2 - \sigma_0^2]} \quad (36)$$

where

$$M_A(\xi, T) = Pl \frac{\sin \sigma_0 (1 - T) \sin \sigma_0 \xi}{\sigma_0 \sin \sigma_0} \quad \text{for } \xi \leq T$$

$$M_A(\xi, T) = Pl \frac{\sin \sigma_0 T \sin \sigma_0 (1 - \xi)}{\sigma_0 \sin \sigma_0} \quad \text{for } \xi \geq T$$

and

$$Q(\xi, T) = Q_A(\xi, T) - 2P\sigma_0 \sum_{n=1}^{\infty} \frac{\sin[(n\pi)^2/\sigma_0]T \cos n\pi\xi}{(n\pi)^2 - \sigma_0^2} \quad (37)$$

where

$$Q_A(\xi, T) = P \frac{\sin \sigma_0 (1 - T) \cos \sigma_0 \xi}{\sin \sigma_0} \quad \text{for } \xi < T$$

$$Q_A(\xi, T) = -P \frac{\sin \sigma_0 T \cos \sigma_0 (1 - \xi)}{\sin \sigma_0} \quad \text{for } \xi > T$$

From Eqs. (31) and (37), it follows that the jump in the shear force Q at the point of the moving force ($\xi=T$) is equal to $P/(1 - \eta^2)$ in Timoshenko beam and P in the Euler-Bernoulli beam. When η tends to 1 ($v \rightarrow v_s$), the shear force in the Timoshenko beam tends to infinity.

4 Conclusion

The dynamic response of a finite, simply supported Timoshenko beam loaded by a force moving with a constant velocity has been considered. The classical solution for transverse displacement and the rotation of cross section has a form of a sum of two infinite series. It has been shown that one of the series (the one which represent aperiodic vibrations of the beam) can be pre-

sented in closed forms. The closed form solutions take different forms depending if the velocity v of the moving force is smaller or larger than the velocities of certain shear and bending waves. This follows from the fact that for Timoshenko beam (contrary to Euler-Bernoulli beam), shear phenomena can occur, which is seen in the closed form solution. The presented closed form solutions have an important meaning in the case when we consider the bending moment or shear force in the beam, particularly in the vicinity of the load point.

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Discussion: “Exploring Effective Methods for Simulating Damaged Structures With Geometric Variation: Toward Intelligent Failure Detection” (McAdams, D. A., Comella, D., and Tumer, I. Y., 2007, ASME J. Appl. Mech., 74, pp. 191–202)

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The paper of McAdams et al. (ASME J. Appl. Mech. 74, pp. 191–202) explored two different approaches for damage detection in vibrating beams having both manufacturing variations in geometry and crack damage. One of the approaches, however, has a significant error in its formulation. The effects of this error on the formulation and the analytical results are discussed.
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In Ref. [1], the authors pursued two approaches for damage detection in the presence of geometric variations in cross section due to manufacturing tolerances. They exploit the fact that both the damage and the manufacturing tolerances produce spatial dependence in the beam's mass density and flexural rigidity, which in turn influences the modal vibration properties of the beam, specifically its natural frequencies. The consideration of such spatial variations has recently attracted attention in the structural health monitoring community, and other researchers [2,3] have proposed related approaches for identifying damage based on this concept, albeit without consideration of manufacturing tolerances.

One of the approaches used in Ref. [1] is a finite difference scheme for approximating the governing equation for Euler–Bernoulli beam vibration that allows for spatial variation of the mass density and flexural stiffness. For free vibration, the relevant equation is

$$\rho A(x) \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left[EI(x) \frac{\partial^2 w}{\partial x^2} \right] = 0 \quad (1)$$

where $w=w(x,t)$ is the transverse vibration response and $\rho A(x)$ and $EI(x)$ are the spatially dependent mass per unit length and flexural rigidity. This equation can be expanded by performing the indicated differentiation of the second term, resulting in

$$\rho A(x) \frac{\partial^2 w}{\partial t^2} + EI''(x) \frac{\partial^2 w}{\partial x^2} + 2EI'(x) \frac{\partial^3 w}{\partial x^3} + EI(x) \frac{\partial^4 w}{\partial x^4} = 0 \quad (2)$$

where $(\cdot)'$ denotes an ordinary spatial derivative. However, Eq. (2) in Ref. [1] neglects the term $2EI'(x) \partial^3 w / \partial x^3$ due to the incorrect use of a “multiplication rule” for second derivatives. As a consequence, the authors’ finite difference approximations for the free vibration equation (most importantly, Eqs. (14), (17), (26),

and (27)) are incorrectly formulated, compromising the results of the analytical study performed by the authors that employs these equations.

A clear indication of the effect of this error can be seen in the results obtained by the authors for the shifts in natural frequencies induced by crack damage. They report in Ref. [1] that, while the first, third, and fifth vibration modes had natural frequencies that decreased (on average) in the presence of damage, the natural frequencies of the second and fourth modes increased. In light of the error made in the formulation of the finite difference equations, this issue can now be understood by considering the contributions of the missing term in Eq. (2) to the overall dynamics. Assuming that the perturbations of Eq. (2) arising from the geometric variations and the crack damage are suitably small, the mode shape $W_n(x)$ corresponding to the n th mode of vibration may be expressed as

$$W_n(x) = \sin\left(\frac{n\pi x}{L}\right) + W_{n,\text{pert}}(x) \quad (3)$$

where $W_{n,\text{pert}}(x)$ represents the (presumably small) perturbations of the mode shape away from its “ideal” value in the absence of spatial variations. Letting $A_n(t)$ denote the corresponding time-dependent amplitude of vibration for the n th mode, we can observe that the term missing from the authors’ formulation of the governing equation makes the following *approximate* contribution to the dynamics:

$$2EI'(x) \frac{\partial^3 w}{\partial x^3} \approx -2 \left(\frac{n\pi}{L} \right)^3 A(t) EI'(x) \cos\left(\frac{n\pi x}{L}\right) \quad (4)$$

(The *exact* contribution to the dynamics depends on the behavior of $W_{n,\text{pert}}'''(x)$, which may or may not be negligible but is assumed to be smaller in magnitude than $-(n\pi/L)^3 \cos(n\pi x/L)$.)

Now, in the authors’ example, the damage location is taken to be $x=\frac{1}{2}L$, the midpoint of the beam. Evaluating Eq. (4) at this location, we obtain

$$2EI' \left(x = \frac{1}{2}L \right) \frac{\partial^3 w}{\partial x^3} \approx -2 \left(\frac{n\pi}{L} \right)^3 A(t) EI' \left(x = \frac{1}{2}L \right) \cos\left(\frac{n\pi}{2}\right) \quad (5)$$

Note that, for the first, third, and fifth modes of vibration, the right-hand side of Eq. (5) is zero, showing that the term missing from the authors’ formulation of the dynamics makes very little contribution to the overall dynamics for these modes. (Of course, the term proportional to $W_{n,\text{pert}}'''(x)$ will make a nonzero contribution, but this should be relatively small.) Thus, it is not unexpected that the authors’ analysis should reproduce the anticipated behavior of the natural frequencies for these modes, as their choice of damage location effectively causes an important effect of their error to “disappear.” (It is understood that this missing term will make other important contributions to the overall dynamics of these modes at other locations; thus, the previous statement should not be construed as saying the missing term makes no important contributions.) However, when one considers Eq. (5) for the second and fourth modes, it is apparent that the missing term’s contribution to the overall dynamics of these modes will be quite noticeable; in fact, the right-hand side of Eq. (5) has its largest magnitude at the damage location. Hence, it is understandable that the natural frequencies of these modes do not shift appropriately, as a significant source of perturbation has been neglected.

It should be noted that the authors’ second approach to the damage identification problem makes no use of Eq. (2), so there is no reason to believe that the results obtained via this approach have this source of error. The authors, however, are encouraged to revisit their finite difference analysis in light of the points raised by this discussion.

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